Homework #4, EECS 556, W21. Due **Thu. Feb. 18**, by 9:00AM

Skills and Concepts

- 2D discrete-space (DS) signals and systems
- DSFT
- 2D filter design

Problems

1. [15] The response of a 2D DS system $\mathcal{S}$ to the unit step input $\text{step}_2[m,n]$ is known to be $s[m,n]$. For each of the following cases, determine the class of input images $f[m,n]$ for which we can determine the output $g[m,n]$ in terms of $s[m,n]$. For each input in the class, express the output in terms of $s[m,n]$.
   (a) [5] $\mathcal{S}$ is **linear**, but not **shift invariant** (SI)
   (b) [5] $\mathcal{S}$ is SI, but not linear
   (c) [5] $\mathcal{S}$ is LSI
   Hint: express $\delta_2[m,n]$ in terms of $\text{step}_2[m,n]$.

2. [10] The course notes give a somewhat messy expression for the **sum-pooling** layer used in some **CNN** models.
   Find a concise expression using a combination of down-sampling notation and convolution with an appropriate filter.
   Is your filter separable? Hint. The filter is quite small.

3. [15]
   (a) [10] Find a (2D DS) FT property for **up-sampling** (using insertion of zeros) by a factor of 2, *i.e.*, if $g[m,n] = f_{\uparrow 2}[m,n]$ then relate $G(\Omega_1,\Omega_2)$ and $F(\Omega_1,\Omega_2)$.
   (b) [5] Suppose the image with spectrum $F(\Omega_1,\Omega_2) = \text{rect}\left(\frac{\sqrt{p(\Omega_1)^2+p(\Omega_2)^2}}{2\pi}\right)$ is up-sampled by a factor of 2, where $p(\Omega) = (\Omega+\pi) \bmod 2\pi - \pi$ ensures that $F(\Omega_1,\Omega_2)$ is appropriately periodic.
   Carefully sketch the resulting spectrum after up-sampling.
   (c) [0] Do you see evidence of aliasing effects? If so, how would you remove them?

4. [10] In continuous space, if $g(x,y) = f(x,y) ** h(x,y) \overset{\mathcal{F}_2}{\longleftrightarrow} G(\nu_X,\nu_Y) = F(\nu_X,\nu_Y)\, H(\nu_X,\nu_Y)$, then we have the following "combined scaling/convolution property" of the 2D FT:

$$f(2x,2y) ** h(2x,2y) \overset{\mathcal{F}_2}{\longleftrightarrow} \frac{1}{4} F\left(\frac{\nu_X}{2},\frac{\nu_Y}{2}\right) \frac{1}{4} H\left(\frac{\nu_X}{2},\frac{\nu_Y}{2}\right) = \frac{1}{16} G\left(\frac{\nu_X}{2},\frac{\nu_Y}{2}\right).$$

   What about in discrete space? If $g[m,n] = f[m,n] ** h[m,n] \overset{\text{DSFT}}{\longleftrightarrow} G(\Omega_1,\Omega_2) = F(\Omega_1,\Omega_2)\, H(\Omega_1,\Omega_2)$, then is there a simple relationship between $G(\Omega_1,\Omega_2)$ and the spectrum of $y[m,n] \triangleq f_{\downarrow 2}[m,n] ** h_{\downarrow 2}[m,n] = f[2m,2n] ** h[2m,2n]$? If so, find it. If not, at least find an expression for the spectrum of $y[m,n]$ in terms of $F(\Omega_1,\Omega_2)$ and $H(\Omega_1,\Omega_2)$.
   Hint: first derive a **down-sampling** relation for the 2D DSFT, for which the expression $[1+(-1)^n]/2$ will be useful.
   Hint: arguments like $\frac{\Omega_1}{2} - \pi$ will appear as part of your answer.

5. [10] Using the **Hankel transform** pair

$$\frac{\sin(2\pi r)}{r} \overset{\mathcal{F}_2}{\longleftrightarrow} \frac{\text{rect}\left(\frac{\rho}{2}\right)}{\sqrt{1-\rho^2}},$$

   determine analytically the **impulse response** $h[m,n]$ of the digital filter having **frequency response**

$$H(\Omega_1,\Omega_2) = \begin{cases} \dfrac{1}{\sqrt{1-4\tilde{\Omega}_1^2-4\tilde{\Omega}_2^2}}, & \tilde{\Omega}_1^2+\tilde{\Omega}_2^2 \leq 1/4 \\ 0, & \text{otherwise,} \end{cases}$$

   where $\tilde{\Omega} = (\Omega+\pi) \bmod 2\pi - \pi$.
   Optional questions:
   - Is $h[m,n]$ weak-sense circularly symmetric?
   - Is $h[m,n]$ strong-sense circularly symmetric?
   - Is $H(\Omega_1,\Omega_2)$ circularly symmetric?

6. [10] Because there are many good 1D FIR filter design methods, this problem explores how one might build 2D FIR filters from 1D designs.
   (a) [10] Given a 1D symmetric filter $h[n]$ with frequency response $H(\Omega)$, one way to specify a 2D frequency response is by $H(\Omega_1, \Omega_2) = H(\Omega) \Big|_{\cos\Omega = \frac{1}{2}(-1+\cos\Omega_1+\cos\Omega_2+\cos\Omega_1\cos\Omega_2)}$.
   For the specific case $h[n] = 2\,\delta[n] + \delta[n-1] + \delta[n+1]$, determine the resulting 2D impulse response $h[m, n]$.
   (b) [0] Is the resulting filter separable?
   (c) [0] An alternative approach would be to use: $H(\Omega_1, \Omega_2) = H\left(\sqrt{\Omega_1^2 + \Omega_2^2}\right)$ with appropriate modulo 2π consideration.
   What would be the primary advantage and primary disadvantage of this alternative?

7. [80] The purpose of this problem is to explore the design of an (approximately) rotationally invariant filter, *i.e.*, a filter whose frequency response is approximately circularly symmetric.
   (a) [10] Consider the 1D filter with impulse response $h[n] = [-1/4\ 0\ \underline{1/2}\ 0\ -1/4]$.
   Determine and plot the frequency response of this 1D filter, and describe qualitatively what kind of filter it is.
   (b) [10] Specify the (circularly-symmetric) frequency response $H(\Omega_1, \Omega_2)$ of a 2D filter with equivalent behavior.
   (c) [10] Display that frequency response using `imagesc` or `jim`. (As usual, make sure DC is at the center, and remember to include a `colorbar`.)
   (d) [10] Find the impulse response $h[m, n]$ of that filter. (You may try to do this analytically, or just do it numerically following the example in the notes.) Clearly record the central 5×5 part of $h[m, n]$ for grading.
   (e) [10] Extract the central 5×5 portion of $h[m, n]$ (*i.e.*, truncate the impulse response) and compute the frequency response $H_t(\Omega_1, \Omega_2)$ of the truncated filter. Display this frequency response using `imagesc` or `jim`.
   (f) [10] Use the `contour` command (with an appropriate optional argument to show 8 contour lines at levels from 0.1 to 1) to overlay (use `hold` or `contour!`) the contours of the truncated frequency response $H_t(\Omega_1, \Omega_2)$ with those of the ideal response $H(\Omega_1, \Omega_2)$.
   (g) [20] Overlay plots of the central horizontal profiles $H(\Omega, 0)$ and $H_t(\Omega, 0)$ vs Ω.
   (h) [0] Do the contours and profiles agree? Explain why or why not.
   What do you conclude about FIR design of circularly symmetric filters?
   Experiment with other amounts of truncation, or with non-rectangular truncation.

Optional problems

8. [0] This problem is an elementary preview of the principles underlying transform coding with truncation. A typical digital image $x[m,n]$ has a spectrum that decays with increasing spatial frequency.

As a concrete model, suppose that $|X(\Omega_1, \Omega_2)| = \begin{cases} A\,\mathrm{e}^{-\alpha\sqrt{\Omega_1^2+\Omega_2^2}}, & \Omega_1^2 + \Omega_2^2 \leq \pi^2 \\ 0, & \text{otherwise.} \end{cases}$

Suppose that we truncate the tails of this spectrum by as follows: $Y(\Omega_1, \Omega_2) = \begin{cases} X(\Omega_1, \Omega_2), & \sqrt{\Omega_1^2 + \Omega_2^2} \leq \pi/10 \\ 0, & \text{otherwise,} \end{cases}$

and then reconstruct the signal $y[m,n]$ by an inverse 2D DSFT.

(a) [0] For $\alpha = 5$, evaluate the normalized root mean-squared error (NRMSE) $\sqrt{\frac{\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |y[m,n] - x[m,n]|^2}{\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |x[m,n]|^2}}$.

(b) [0] This procedure retains only the fraction $\pi(\pi/10)^2/(4\pi^2)$, or about 1%, of the frequency components. So it can be thought of as 100-fold data compression. Discuss the distortion caused by this drastic data reduction.

9. [0]

(a) [0] If $h_l[n]$ is the impulse response of a 1D FIR lowpass filter, then a simple way to design a 1D FIR highpass filter $h[n]$ is by letting $h[n] = (-1)^n\, h_l[n]$. Show that $h[n]$ is a highpass filter.

(b) [0] Suppose $h_l[m,n]$ is the impulse response of a good 2D FIR lowpass filter. A natural extension of the above method is to try designing a 2D FIR highpass filter by letting $h[m,n] = (-1)^m(-1)^n\, h_l[m,n]$.
Is this a good method of designing a 2D highpass filter?

10. [0]

(a) [0] Find the frequency response $H(\Omega_1, \Omega_2)$ and impulse response $h[m,n]$ of a FIR highpass filter whose frequency response satisfies the following:

$$H(\Omega_1, \Omega_2) = \begin{cases} 0, & \Omega_1 = \Omega_2 = 0 \\ 1, & (\Omega_1, \Omega_2) \in \{(-\pi, \pm\pi), (0, \pm\pi), (\pi, \pm\pi), (\pm\pi, 0)\} \\ ?, & \text{otherwise.} \end{cases}$$

Try to choose the "?" part of $H(\Omega_1, \Omega_2)$ so that $h[m,n]$ is as simple as possible. Hint: a 3×3 filter suffices.

(b) [0] Suppose that processing an input image $x[m,n]$ with the above filter $h[m,n]$ yields the output image $y[m,n]$. Determine $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y[m,n]$.

(c) [0] Suppose $x[m,n]$ is real and nonnegative; from your preceding answer, describe how $y[m,n]$ will appear on a display where all negative values of $y[m,n]$ appear as black (like zero).