Homework #3, EECS 556, W21. Due **Thu. Feb. 11**, by 9:00AM

Skills and Concepts

- 2D sampling, signal reconstruction from samples, lens filtering,

Problems

1. [5] We apply ideal 2D **sampling**, *i.e.*, $g[m,n] = g(m\Delta_X, n\Delta_Y)$, to the following 2D signal:

$$g(x,y) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}^2(x-5k)\operatorname{sinc}^2(3y-7)\,.$$

Determine range of values that the sampling intervals (Δ_X, Δ_Y) can assume while avoiding **aliasing**.

2. [5] A circular lens with diameter 2mm is used in an optical scanner with laser illumination at a wavelength of 800nm in a 1200dpi acquisition mode. The distance between the lens and the detector is 3mm. Can **aliasing** occur in this system?

3. [15] Generalize Nyquist-Shannon sampling theory (the sampling theory and methods developed in the notes) beyond the ideal "impulse" model to the more realistic case where we account for finite detector element size in the sampling relationship:

$$\begin{aligned} g_d[m,n] &= \frac{1}{\Delta_X \Delta_Y} \int_{(n-1/2)\Delta_Y}^{(n+1/2)\Delta_Y} \int_{(m-1/2)\Delta_X}^{(m+1/2)\Delta_X} g_a(x,y)\,dx\,dy \\ &= (h ** g_a)(m\Delta_X, n\Delta_Y), \text{ where } h(x,y) = \frac{1}{\Delta_X \Delta_Y} \operatorname{rect}_2\!\left(\frac{x}{\Delta_X}, \frac{y}{\Delta_Y}\right). \end{aligned}$$

(Even further generalizations are possible, *e.g.*, [1].)
Specifically: determine if it is ever possible to recover a band-limited $g_a(x,y)$ from $g_d[m,n]$, and if so, what conditions are there on (Δ_X, Δ_Y), and *how* do you recover it?

4. [10] For the digital **display** example given in the notes that uses a rect **interpolator** for **reconstruction** (D/A conversion), how much more finely than the **Nyquist rate** must one sample to guarantee that the amplitude of every aliased component will be attenuated down to a value that is at most 10% of its original value?
Hint: this may require a small numerical calculation for which a careful plot and/or JULIA's `Roots.find_zero` or MATLAB's `fzero` or `fsolve` functions may be useful.

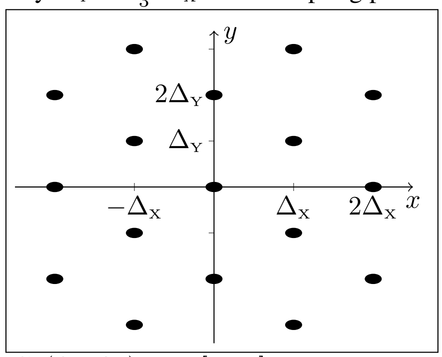
5. [0] Identify a team of 4 students with whom you would like to do the course team project. Enter the uniquenames of the other three students into this google form: https://forms.gle/9337X5xJQzf8uEAL7
Teams of 3 or 5 students will be approved by Prof. Fessler only if needed due to class size being not a multiple of 4.
The final question on the form is to list a couple topics that you personally might be interested in investigating for the team project. (You do not have to have team consensus on the topic yet.)
(Future project-related deadlines will likely not be announced by HW; see Ch. 0.)
You have until the deadline given in Ch. 0 to complete the google form.
Completing the form on time will be roughly 10% of your project proposal score.

The following problem is optional for 498 students. For any such problems, see how far you can get just for yourself. Put a note in your answer for this problem that you are a 498 student, so that the graders know!

6. [40] This problem compares **hexagonal sampling** with rectangular sampling. This is not merely a homework problem; there are commercially available hexagonal detectors used in mammography, such as the ASPIRE Cristalle FFDM System. Ideal sampling on a hexagonal grid has the following form:

$$g_{\mathrm{d}}[m,n] = \frac{1}{2}\left(1 + (-1)^{m+n}\right) g_{\mathrm{a}}(m\Delta_{\mathrm{X}}, n\Delta_{\mathrm{Y}}) = \begin{cases} g_{\mathrm{a}}(m\Delta_{\mathrm{X}}, n\Delta_{\mathrm{Y}}), & m \text{ and } n \text{ both even, or } m \text{ and } n \text{ both odd} \\ 0, & \text{otherwise,} \end{cases}$$

where the sampling intervals are related by $\Delta_{\mathrm{Y}} = \frac{\sqrt{3}}{3}\Delta_{\mathrm{X}}$. This sampling pattern is illustrated below.



Our goal here is to relate the spectrum $G_{\mathrm{d}}(\Omega_1, \Omega_2)$ of $g_{\mathrm{d}}[m,n]$ to the spectrum $G_{\mathrm{a}}(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$ of $g_{\mathrm{a}}(x,y)$ and to develop a method for recovering $g_{\mathrm{a}}(x,y)$ (by interpolation) from the samples $g_{\mathrm{d}}[m,n]$.
For analysis, it can be helpful to consider the "synthetic" signal $g_{\mathrm{s}}(x,y) \triangleq g_{\mathrm{a}}(x,y)\, p_c(x,y)$, where

$$\begin{aligned} p_c(x,y) \triangleq & \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Delta_{\mathrm{X}}\Delta_{\mathrm{Y}}\, \delta_2(x - 2m\Delta_{\mathrm{X}}, y - 2n\Delta_{\mathrm{Y}}) \\ & + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Delta_{\mathrm{X}}\Delta_{\mathrm{Y}}\, \delta_2(x - (2m+1)\Delta_{\mathrm{X}}, y - (2n+1)\Delta_{\mathrm{Y}}) \,. \end{aligned}$$

(a) [0] Describe how $p_c(x,y)$ relates to the sampling pattern.

(b) [5] Express $g_{\mathrm{s}}(x,y)$ in terms of the samples $g_{\mathrm{d}}[m,n]$.

(c) [10] Determine $P_c(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$. Hint: a simplified answer will have something like $(-1)^{k+l}$ in it.
Hint: write $p_c(x,y)$ in terms of Dirac comb(s).

(d) [0] Relate $G_{\mathrm{s}}(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$ to $G_{\mathrm{a}}(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$.

(e) [5] Sketch $G_{\mathrm{s}}(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$ when $G_{\mathrm{a}}(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}}) = \mathrm{rect}\left(\frac{\rho}{2\rho_c}\right)$.

(f) [10] Suppose $G_{\mathrm{a}}(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$ is circularly band-limited, *i.e.*, is zero for $\rho > \rho_c$. Find conditions on Δ_{X} (and hence Δ_{Y}) that ensure that $g_{\mathrm{a}}(x,y)$ can be recovered from $g_{\mathrm{d}}[m,n]$.

(g) [0] Compare the previous result with the corresponding result when one samples on a rectilinear grid. Discuss which sampling grid is more efficient for circularly band-limited images.

(h) [5] Find the frequency response $H(\nu_{\mathrm{X}}, \nu_{\mathrm{Y}})$ of a filter that we could imagine applying to $g_{\mathrm{s}}(x,y)$ so as to recover $g_{\mathrm{a}}(x,y)$ in the case that the hexagonal sampling rate is sufficient. (A graphical answer suffices.)

(i) [5] Let $h(x,y)$ denote the impulse response of that filter (you do not need to find it). Express $g_{\mathrm{a}}(x,y)$ in terms of $g_{\mathrm{d}}[m,n]$ and $h(x,y)$, *i.e.*, as an interpolation / recovery formula.

(j) [0] Find the impulse response $h(x,y)$ of the ideal interpolator analytically. (Challenging!)

[1] W. Sun and X. Zhou. "Reconstruction of band-limited signals from local averages". In: *IEEE Trans. Info. Theory* 48.11 (Nov. 2002), 2955–63.