Homework #2, EECS 556, W21. Due **Thu. Feb. 04**, by 9:00AM

Skills and Concepts

- 2D FS, 2D FT, resolution, 2D FT via lens, 2D sampling

General hint. There is a `jinc` function in the `MIRT` package.

Problems

1. [50]

(a) [10] Determine the 2D Fourier series representation of the following periodic signal:

$$g(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 5 \operatorname{rect}\left(\frac{1}{2}\sqrt{(x-3m)^2 + (y-4n)^2}\right).$$

Hint. Use the relation between Fourier series coefficients and Fourier transforms derived in the lecture notes.

(b) [40] Follow the example in the lecture notes to display images of both $g(x, y)$ and its (truncated) FS approximation

$$g_K(x, y) = \sum_{k=-K}^{K} \sum_{l=-K}^{K} c_{k,l} \, e^{\imath 2\pi(xk/T_X + yl/T_Y)}.$$

Do this for (at least) *two* levels of truncation K—one where Gibbs artifacts are quite visible, and one where the reconstruction looks reasonably close to the original signal.

2. [10] Determine the 2D FT of the following 2D image using properties and tables, *not* integration

$$g(x, y) = \begin{cases} 2, & (x-5)^2 + (y-7)^2 \leq 9 \\ 0, & \text{otherwise.} \end{cases}$$

Hint: first rewrite $g(x, y)$ in terms of $\operatorname{rect}()$. Your answer should contain a $\operatorname{jinc}()$.

3. [10] Using FT properties, determine the 2D FT of an ellipse function:

$$g(x, y) \triangleq \begin{cases} 1, & \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $a > 0$ and $b > 0$ denote the half-widths of the ellipse along the x and y directions, respectively.
Hint. Let $\alpha = 2a$, $\beta = 2b$ denote the long axes of the ellipse, and rewrite $g(x, y)$ in terms of a rect function.

4. [10] A lens with focal length F is distance F from a translucent cancer cell on a slide and distance F from a sensor. The cell is illuminated with light of wavelength λ. For simplicity, suppose the cell transparency is the sum of a disk function and an ellipse function:

$$t_0(x, y) = \operatorname{circ}\left(\frac{r}{r_0}\right) + e(x, y), \quad e(x, y) \triangleq \begin{cases} 1, & \left(\frac{x-x_0}{r_X}\right)^2 + \left(\frac{y-y_0}{r_Y}\right)^2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}.$$

(Optional: if you want to sketch this, use $r_0 = 100$ um, $r_X = 200$ um, $r_y = 100$ um, $x_0 = 300$ um, and $y_0 = 0$; it might look like the cell has recently divided.)
Determine the optical intensity recorded on the sensor. Hint. Use Pr. 3 and FT properties, not integration!
In your answer, let $d_0 = 2r_0$ denote the disk diameter, and let $d_X = 2r_X$, $d_Y = 2r_Y$ denote the long axes of the ellipse.
It suffices to find $U_F(x, y)$ and express your final answer in terms of that U_F.

5. [10] The signal $g(x, y) = 48 \operatorname{sinc}_2(6x, 8y)$ is sampled with sampling intervals (Δ_X, Δ_Y).
Sketch the spectrum $G_s(\nu_X, \nu_Y)$ of the "sample-carrying" signal $g_s(x, y)$ in eqn. (4.3) for the following two cases.
(a) $(\Delta_X, \Delta_Y) = (1/8, 1/12)$
(b) $(\Delta_X, \Delta_Y) = (1/5, 1/12)$

Optional problems

6. [0] Find the just-resolved distances (according to Rayleigh's resolution criterion given in the notes) for a system having a "square" frequency response $H_1(\nu_X, \nu_Y) = \mathrm{rect}_2\left(\frac{\nu_X}{2\nu_{\max}}, \frac{\nu_Y}{2\nu_{\max}}\right)$ and for one having a "disk" frequency response $H_2(\rho) = \mathrm{rect}\left(\frac{\rho}{2\nu_{\max}}\right)$, where $\nu_{\max} = 5\mathrm{mm}^{-1}$. Which system would you rather use for imaging and why?
Hint: MATLAB's `fzero` function may be useful here, or JULIA's `Roots.find_zero` function.

7. [0] Find an expression for the 2D FT of the circularly symmetric function $g(r) = \mathrm{comb}(r/3)$.
Hint. The answer is *not* $3\,\mathrm{comb}(3\rho)$.

8. [0] Prove Parseval's theorem for the 2D FT.

9. [0] Prove the rotation property of the 2D FT.

10. [0] Find a version of Parseval's theorem that is appropriate for circularly symmetric functions.