# Chapter A

## Ultrasound Arrays

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Originally based on [1, Ch. 10]. See also [2].

### Outline

- Image formation
  - Transmit steering
  - Receive delays
  - Sector scans
  - Using real signals
  - Baseband processing
- PSF analysis
  - Array design parameters
  - Steering
  - Sampling
- Dynamic focusing
Introduction to ultrasound with arrays

The previous treatment considered a single transducer with a single transmit pulse $p(t)$ and output signal $v(t)$. By using multiple transducer elements, each with their own input/output signals, we can have:

- electronic beam steering (better size, weight, reliability than mechanical steering),
- narrower near-field lateral PSF (by electronic focusing), yet retain diffraction-limited far-field PSF.

The drawbacks of using multiple elements are cost, complexity, and (we will see) PSF sidelobes.

Imaging array (simple mode)

An imaging array consists of a row of several small transducers. We pulse one at a time, then form a 2D image $\hat{R}(x,0,z)$. This design eliminates mechanical motion, but still there remains the tradeoff between lateral resolution and depth of field. Lateral FOV = array size, so to increase FOV will increase cost. For cardiac notch (2cm) need FOV larger than transducer. Useful for superficial structures: eyes, vessels.

Phased array

An “imaging array” barely uses the potential of a transducer array. Because each transducer element has its own electrical connection, we can do more than just use each element sequentially. The most common approach is to use all elements simultaneously while applying different delays to each element, changing the corresponding signal phases (at least for narrowband signals). Such a transducer array is called a phased array.

Design choices

Design choices for a phased array include the following.

- Array geometry: number $N$, size(s), shapes, and arrangement of transducer elements.
- What signal $p_n(t)$ to transmit on each transducer? (For $n = 1, \ldots, N$.)
  Typically $p_n(t) = \alpha_n^n p_0(t - \tau_n^n)$, giving us the design parameters:
  - $\alpha_n^n$ amplitudes, $\tau_n^n$ delays
  - $a_0(t)$ transmit envelope, $\omega_0$ carrier frequency (and hence wavelength $\lambda$)
- On receive, how do we combine the output signals $v_n(t)$ to form a reflectivity image $\hat{R}(x,y,z)$?

Image formation with phased arrays

We first focus on the basics of image formation using phased arrays. Later we consider (diffraction) analysis of the PSF.

- We build up a 2D image, called a sector scan, one angle $\theta$ at a time.
- To minimize the echoes from reflectors that are away from the line at angle $\theta$, we use transmit delays to steer the transmitted beam energy in the $\theta$ direction primarily.
- The echoes that return from reflectors along the line at angle $\theta$ will reach each transducer element at slightly different times, as illustrated in a later figure.
  On receive, we combine the signals recorded by each array element appropriately to estimate $\hat{R}(r,\theta) = \hat{R}(r\sin \theta, 0, r\cos \theta)$.
- It is also possible to do (dynamic) receive focusing. More on this topic later.
- For an image that covers a depth of 15cm, the round trip propagation time is about $30 \text{ cm} / 1500 \text{ m/s} = 0.2 \text{ ms}$. For a video display with 20 frames / sec, we have 50 ms to form each frame, so $50 \text{ ms} / 0.2 \text{ ms} = 250$ lines can be collected per frame. (These are rough numbers to illustrate that real-time imaging is feasible.)
- For display, we interpolate the polar samples $\hat{R}(r,\theta)$ into a Cartesian grid. This process is called scan conversion.
Steering and focusing via delays

Transmit steering via transmit delays

We have seen previously that to steer the transmit beam in the direction $\theta$, ideally we would like to induce a linear phase of the form $e^{i2\pi(x\sin\theta)/\lambda}$ across the transducer. How can we approximate this linear phase with a phased array system? For a given value of $x$, we can induce the appropriate phase by using a time delay $\tau$ that satisfies $\omega_0\tau = 2\pi(x\sin\theta)/\lambda$, or equivalently $\tau = (x\sin\theta)/c$. But with a phased array system we can apply only $N$ discrete time delays, not a continuum of values. Specifically, if $x_n$ denotes the $x$ position of the center of the $n$th transducer element, then we use delayed pulses of the form $p_n(t) = p_0(t - \tau_n^T)$, where the appropriate transmit delays are

$$\tau_n^T = \tau_n^T(\theta) = \left( x_n \sin \theta \right)/c.$$

We will analyze the resulting beam pattern carefully later; for now we understand that the use of such delays will steer most of the energy of the transmitted beam in the direction $\theta$. Clearly we need different delays for each direction $\theta$ of interest.

By choosing these transmit delays, the transmitted pulses arrive at a given point in the far-field along the line at angle $\theta$ at nearly the same time. Why?

In fact, because $\tau_n^T = 0$ for the center element where $x_n = 0$, for timing purposes (but not for the PSF analysis!) we can treat the transmitted pulse as if it emanated from the center element.

Example. The following figure illustrates how applying suitable transmit delays effectively steers the beam wavefront along a certain angle.
Echoes received by a phased array

After the beam is transmitted, portions of the beam will reflect or scatter and return to the phased array. Because diffraction spreads the beam widely, echoes will return from essentially everywhere.

The following figure illustrates echoes returning from a single point \((r, \theta)\) as recorded by each of the transducer elements. The distance traveled from the center of the array out to the point \((r, \theta)\) and back to the array element centered at \(x_n\) is

\[
r + d_n, \text{ where } d_n = d_n(r, \theta) = d(x_n; r, \theta) = \sqrt{(x_n - r \sin \theta)^2 + (0 - r \cos \theta)^2}.\]

Therefore, the echo occurs at time \(t_n = t_n(r, \theta) = \frac{r}{c} + d_n \frac{d(x_n; r, \theta)}{c}.

For a given direction \(\theta\), how should we use the signals \(\{v_n(t)\}\) to estimate the reflectivity at all points \((r, \theta)\)?
Here is a block diagram of the traditional approach to delay-sum beamforming.

**Delay–Sum Beamforming**

\[
\frac{1}{N} \sum_{n=1}^{N} v_n(t - \tau_n)
\]
Receive steering (and focusing) viewed as propagation delays

We can consider the problem of choosing the receive delays $\tau_n^r$ from a simple and intuitive geometric perspective. Suppose we want to examine the point $(r, \theta)$, i.e., we are trying to determine $\hat{R}(r, \theta)$. As the echoes return, we would like to “apply delays” to each $v_n(t)$ to “line up” the ultrasound echoes such that when they are summed together the echoes returning from point $(r, \theta)$ will add together coherently, whereas echoes from other locations will hopefully destructively self-interfere and contribute less to our estimate $\hat{R}$ at $(r, \theta)$.

Knowing the $x$ position of each transducer element, it is simple geometry to determine what the propagation distance is from the point $r, \theta$ to the center $x_n$ of each element. As defined in the earlier discussion of diffraction in polar coordinates, the distance is:

$$d_n \triangleq d(x_n; r, \theta) = \sqrt{(x_n - r \sin \theta)^2 + (0 - r \cos \theta)^2} \approx r - x_n \sin \theta + \frac{x_n^2 \cos^2 \theta}{2r} \approx r - x_n \sin \theta.$$  

Let $\tau^T$ denote the time required for the transmitted pulse to propagate from the transducer to the point $(r, \theta)$. If the pulse can be thought of as emanating from the transducer center, then what is $\tau^T$? Answer: $\tau^T = \tau^T(r) = r/c$.

If there is a reflector or scatterer at $(r, \theta)$, then the time required for the pulse to return from the reflector to the $n$th transducer element is $d_n/c = d(x_n; r, \theta)/c$. So the echo time is $\tau^T + d_n/c$. Thus the natural estimate of reflectivity is:

$$\hat{R}(r, \theta) = \text{gain}(r) \left| \frac{1}{N} \sum_{n=1}^{N} v_n \left( \tau^T(r) + \frac{d(x_n; r, \theta)}{c} \right) \right|.$$  

(A.1)

If we sample the signals $v_n(t)$, then the above expression really is just a form of “table lookup.” (This expression would be the essence of what is required for the project if we did RF beamforming.) Compare this to our image formation equation for a single transducer: $\hat{R}(r, 0) = \text{gain}(r) |v(2r/c)| = \text{gain}(r) |v(r/c + r/c)|$.

Why do we take the envelope after summing?

Answer: to get destructive interference of echoes returning from reflectors in other locations. All this talk about “delays” may seem unusual because the “delay” $d(x_n; r, \theta)/c$ depends on $r$, so it is not a simple signal “delay,” unless there were only a single reflector at known position $(r, \theta)$, which is unrealistic.

However, in the far field we can use the first-order approximation $d_n = d(x_n; r, \theta) \approx r - x_n \sin \theta$. Then we have

$$\tau^T(r) + \frac{d(x_n; r, \theta)}{c} = \frac{2r}{c} - \frac{x_n \sin \theta}{c}.$$

Thus we rewrite the boxed image formation equation above as follows:

$$\hat{R}(r, \theta) = \text{gain}(r) \left| v \left( \frac{2r}{c} \right) \right|,$$

where $v(t) \triangleq \frac{1}{N} \sum_{n=1}^{N} v_n(t - \tau_n^R)$,

and where the (now properly constant!) receive delays are

$$\tau_n^R \triangleq \frac{x_n \sin \theta}{c}.$$  

(A.2)

Thus the signal $v(t)$ is the average of appropriately delayed versions of the individual transducer output signals. The delays here are all relative to the signal from the center of the array ($x_n = 0$), for which $d(0; r, \theta) = r$. However, the above treatment achieves only beam steering, not dynamic receive focusing. We form images with better spatial resolution by using dynamic receive focusing, such as is provided by the preceding “table lookup” formula (A.1). We will return to this later.
Beamforming with real signals

For analyzing the PSF of ultrasound imaging, it is convenient to consider complex signals. But real-world acoustic pressures and voltages are real valued, so for implementation one must consider sinusoidal signals rather than complex exponentials. In particular, a more realistic model for the transmitted pulse is

\[ p(t) \triangleq a(t) \cos(\omega_0 t + \phi_0), \]

where \( a(t) \) is the transmitted pulse envelope and \( \phi_0 \) is a phase factor that may be unknown. Note for use below that under the narrowband assumption discussed previously:

\[ \frac{1}{2\pi c} \frac{d}{dt} p(t) \approx -\frac{1}{\lambda} a(t) \sin(\omega_0 t + \phi_0). \]

For pulse transmission, assume that each of the \( N \) transducer elements is fired by the same pulse shape \( p(t) \), but possibly its own amplitude \( \alpha_n^+ \) and with its own delay \( \tau_n^+ \). Then the applied pressure at the transducer plane is the superposition of the contribution from each element: \( i.e. \),

\[
 u(P_0, t) = \sum_{n=1}^{N} s_n(x_0, y_0) \alpha_n^+ p(t - \tau_n^+). 
\]

Also assume that transmit delays are applied to steer the beam in the desired direction \( \theta \), \( i.e. \),

\[
 \tau_n^+ = \tau_n^+ (\theta) = (x_n \sin \theta)/c. 
\]

For considering timing issues, (as opposed to analyzing the entire PSF) we treat each transducer element as a point at \( (x_n, 0, 0) \):

\[
 s_n(x, y) = \lambda^2 \delta(x - x_n) \delta(y). 
\]

Combining the above assumptions with the diffraction formula, the incident pressure field at \( P_1 = (r \sin \theta, 0, r \cos \theta) \) is:

\[
 u(P_1, t) = \int \int \frac{\cos \theta \sin \theta}{r_0^1} \frac{1}{2\pi c} \frac{d}{dt} u\left(P_0, t - \frac{r_0^1}{c}\right) dx_0 dy_0 
\]

\[
 \approx \cos \theta \int \int \frac{1}{2\pi c} \frac{d}{dt} u\left(P_0, t - \frac{r_0^1}{c}\right) dx_0 dy_0 
\]

\[
 = \cos \theta \int \int \sum_{n=1}^{N} s_n(x_0, y_0) \alpha_n^+ \frac{1}{2\pi c} \frac{d}{dt} p\left(t - \tau_n^+ - \frac{r_0^1}{c}\right) dx_0 dy_0 
\]

\[
 = \lambda^2 \cos \theta \int \int \sum_{n=1}^{N} \alpha_n^+ \frac{1}{2\pi c} \frac{d}{dt} p\left(t - \tau_n^+ - \frac{d_n}{c}\right) dx_0 dy_0 
\]

\[
 \approx -\cos \theta \int \int \sum_{n=1}^{N} \alpha_n^+ \left(t - \tau_n^+ - \frac{d_n}{c}\right) \sin(\omega_0(t - \tau_n^+ - d_n/c) + \phi_0) dx_0 dy_0 
\]

\[
 \approx -\cos \theta \int \int \sum_{n=1}^{N} \alpha_n^+ \left(t - \tau_n^+\right) \sin(\omega_0(t - \tau_n^+) + \phi_0) dx_0 dy_0. 
\]

In the final line of the above analysis of the transmit part, we used the far-field approximation \( d_n = d(x_n; r, \theta) \approx r - x_n \sin \theta \), for which \( \tau_n^+ + d_n/c \approx r/c = \tau^+ \). In fact, we even invoked that approximation within the sinusoid term. Physically, we are assuming that along the line at angle \( \theta \) in the far field, the pressures induced by all the transducer elements add coherently. Under these approximations, the far-field incident pressure is

\[
 u(P_1, t) \approx \alpha^+ \cos \theta \frac{\alpha^+}{r/\lambda} a(t - \tau^+) \sin(\omega_0(t - \tau^+) + \phi_0), 
\]

where the “total transmit amplitude” is \( \alpha^+ \triangleq -\sum_n \alpha_n^+ \).

The incident pressure is essentially a (scaled and differentiated and) delayed version of the pulse from the center element.
If the object is a single “Dirac impulse” reflector at \( P_1 \), a fraction \( R(r, \theta) \) of this incident pressure is reflected back to the transducer plane. Thus again using the diffraction formula (with reciprocity), back at the transducer plane the reflected pressure is

\[
u(x, t) = u((x, 0, 0), t) \approx R(r, \theta) \lambda^2 \frac{\cos \theta}{r} \frac{1}{2 \pi c} \frac{d}{dt} u \left( P_1, t - \frac{d(x; r, \theta)}{c} \right)
\]

\[
\approx R(r, \theta) \lambda^2 \frac{\cos \theta}{r} \frac{1}{2 \pi c} \frac{d}{dt} \left[ \alpha^t \frac{\cos \theta}{r} a \left( t - \frac{d(x; r, \theta)}{c} - \tau^t \right) \sin \left( \omega_0 \left( t - \frac{d(x; r, \theta)}{c} - \tau^t \right) + \phi_0 \right) \right]
\]

\[
\approx R(r, \theta) \lambda^2 \left( \frac{\cos \theta}{r} \right)^2 \alpha^t a \left( t - \tau^t - \frac{d_n}{c} \right) \cos \left( \omega_0 \left( t - \tau^t - \frac{d_n}{c} \right) + \phi_n \right).
\] (A.3)

The key property to note is that we transmitted an amplitude-modulated sinusoid, and what returns from a single reflector is also an amplitude-modulated sinusoid.

For an ideal point transducer element, the output voltage is (proportional to) the incident pressure:

\[
v_n(t) = u(x_n, t) * h_n(t)
\]

\[
\approx KR(r, \theta) \lambda^2 \left( \frac{\cos \theta}{r} \right)^2 \alpha^t a \left( t - \tau^t - \frac{d_n}{c} \right) \cos \left( \omega_0 \left( t - \tau^t - \frac{d_n}{c} \right) + \phi_n \right).
\] (A.3)

where \( h_n(t) \) denotes the impulse response of the \( n \)th transducer element. The factor \( K \) includes \( \alpha^t \) and the magnitude response \( |H_n(f_0)| \) of the \( n \)th transducer element, and the phase difference \( \phi_n - \phi_0 \) is induced by the corresponding phase response \( \angle H_n(f_0) \).

Now that we have the above signal equation, we can contemplate image formation.

**Image formation with the RF signals**

By convention, assume that the peak value of \( a(t) \) is at \( t = 0 \). Then based on (A.3), the natural reflectivity estimate is

\[
\hat{R}(r, \theta) \triangleq \text{gain}(r) \left| \frac{1}{N} \sum_{n=1}^{N} v_n \left( \tau^t + \frac{d(x_n; r, \theta)}{c} \right) \right|,
\] (A.4)

where \( \text{gain}(r) = \frac{r^2}{K^2 \lambda^2 \cos^2 \theta |a(0)|} \).

Note that this formation method requires buffering \( v_n(t) \) for “table lookup” because of its argument \( \tau^t + d(x_n; r, \theta) \).

However, in the far field \( \tau^t + d(x_n; r, \theta) \approx 2r/c - \tau^t_n \), where \( \tau^t_n = (x_n \sin \theta)/c \), which is an \( r \)-independent delay.

Substituting (A.3) into (A.4) yields

\[
\hat{R}(r, \theta) \approx |R(r, \theta)| \left| \frac{1}{N} \sum_{n=1}^{N} \cos(\phi_n) \right|.
\]

If each \( \phi_n \) were known to be zero, this would work well.

However, small errors in the delays \( \tau^t_n, \tau^n_n \), and the (unknown) phases \{\phi_n\} can lead to destructive interference rather than coherent summing, yielding degraded reflectivity estimates.

The basic problem here is that each signal \( v_n(t) \) oscillates at RF frequencies (\( \omega_0 \) is 1-10 MHz), making (A.4) sensitive to small timing differences.

A practical solution to this problem is to first convert the signal to baseband.