Gaussian Random Sequences, Linear Systems, and Countable Sums
EECS 501, J. Fessler, December 1, 1995

A finite sum of jointly Gaussian random variables has a Gaussian distribution. This is easily shown using characteristic functions. But what about an infinite sum, such as arises if one has a linear system with an infinite impulse response? Under certain conditions described below, if the input to a linear system is Gaussian random sequence, then the output random sequence is also Gaussian.

Theorem. If $\{X_i\}$ is a Gaussian random sequence and if

$$\sum_{i=1}^{\infty} |a_i E[X_i]| < \infty, \text{ and } \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_i a_j K_X(i,j)| < \infty, \tag{1}$$

then

$$Y = \sum_{i=1}^{\infty} a_i X_i$$

is a Gaussian random variable with finite mean and variance

$$\mu_Y = E[Y] = \sum_{i=1}^{\infty} a_i E[X_i], \text{ and } \sigma_Y^2 = \mathrm{Var}(Y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j K_Y(i,j). \tag{2}$$

Proof: First note that by the assumptions (1), the series in (2) are absolutely summable and hence unconditionally convergent, and hence $E[Y]$ and $\mathrm{Var}(Y)$ are well-defined finite values. (See handout on convergence of sums.) Now define

$$Y_n = \sum_{i=1}^{n} a_i X_i,$$

then by definition of convergent sums,

$$\mu_n = E[Y_n] \to \mu_Y \text{ and } \sigma_n^2 = \mathrm{Var}(Y_n) \to \sigma_Y^2, \text{ as } n \to \infty. \tag{3}$$

Since the Gaussian distribution is continuous, it follows from (3) that

$$F_{Y_n}(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-(x-\mu_n)^2/\sigma_n^2}\, dx \to \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-(x-\mu_Y)^2/\sigma_Y^2}\, dx \text{ as } n \to \infty.$$

So Y_n converges in distribution to a $N(\mu_Y, \sigma_Y^2)$ distribution, hence Y is Gaussian.

By similar arguments, one can show that a set of random variables defined by such countable sums is jointly Gaussian, i.e., if we let $Z_n = \sum_i h_{n,i} X_i$, then $\{Z_n\}$ is a Gaussian random sequence if $\sum_i |h_{n,i}| < \infty,\ \forall n$.

Corollary. If $\{X_i\}$ is WSS and $\sum_i |a_i| < \infty$, then (1) is satisfied. Thus, if we consider bounded-input bounded-output systems with WSS Gaussian inputs, the output will also be a Gaussian random sequence.