1. $E = [1^6 \epsilon^5] \cup [01^6 \epsilon^4] \cup [\epsilon 01^6 \epsilon^3] \cup [\epsilon^2 01^6 \epsilon^2] \cup [\epsilon^3 01^6 \epsilon] \cup [\epsilon^4 01^6]$, where $\epsilon$ means either a 1 or a 0 is OK. So $P(E) = (1 - 2^5 + 5 \cdot 2^4)/2^{11} = 7/2^7$. Thus the expected number of errors is $1024 \cdot 7/2^7 = 56$.

2. First, by taking the derivative of $F_X(x)$: $f_X(x) = (\theta - 1)x^{-\theta}u(x - 1)$. So the log-likelihood is: $l(\theta) = \sum_{i=1}^n \log f_X(x_i) = \sum_{i=1}^n (\log(\theta - 1) - \theta \log x_i) = n \log(\theta - 1) - \theta \sum_{i=1}^n \log x_i$, so $l'(\theta) = \frac{\sum_{i=1}^n x_i}{\theta - 1} - \sum_{i=1}^n \log x_i$.

Thus $\theta = 1 + \frac{\sum_{i=1}^n \log x_i}{\sum_{i=1}^n x_i}$.

$E[\hat{\alpha}] = E[\log X_i] = \int_0^\infty (\log x)(\theta - 1)x^{-\theta} \, dx = (\theta - 1)/(\theta - 1)^2 = 1/(\theta - 1)$ (using integral given with $m = -\theta$). Thus $\hat{\alpha}$ is unbiased. $\text{Var}(\hat{\alpha}) = \text{Var}(\log X_i)/n \to 0$ as $n \to \infty$, if $\text{Var}(\log X_i) < \infty$. But from integral table, $E[(\log X_i)^2] = \int_0^\infty (\log x)^2(\theta - 1)x^{-\theta} \, dx = 2/(\theta - 1)^2$ is finite, so $\text{Var}(\log X_i) = E[(\log X_i)^2] - 1/(\theta - 1)^2 = 2/(\theta - 1)^2$ is also finite, so $\hat{\alpha}$ is consistent, by the Theorem proved in class using Chebyshev's inequality.

3. $F_Y(y) = P[Y \leq y|Z = 1]0.8 + P[Y \leq y|Z = 0]0.2 = P[X \leq y|0.8 + u(y)0.2 = 0.8F_X(y) + 0.2u(y)$, so $f_Y(y) = 0.8f_X(y) + 0.2 \delta(y)$. By i.i.d.: $f_Y(y_1, \ldots, y_n) = \prod_{i=1}^n f_Y(y_i)$.

$E[\hat{\mu}_X] = E[\frac{1}{n}\sum_{i=1}^n Y_i] = E[Y_i] = \int y(0.8 f_X(y) + 0.2 \delta(y)) \, dy = 0.8 \mu_X + 0.2 \cdot 0 = 0.8 \mu_X$. Thus the bias is $E[\hat{\mu}_X] - \mu_X = -0.2 \mu_X$.

No Dirac deltas means $X$ is continuous r.v., so $P[X = 0] = 0$, so estimate $\mu_X$ using $\hat{\mu}_X = (\sum_{i=1}^n Y_i 1_{\{Y_i \neq 0\}})/(\sum_{i=1}^n 1_{\{Y_i \neq 0\}})$. Alternatively, divide the sample mean by 0.8.

Let $a = E[Y_2i + Y_{i2-1}] = 2E[Y] = 2(0.8 \cdot \mu_X + 0.2 \cdot 0) = 1.6 = 8$.

Note that $E[Y^2] = 0.8E[X^2] = 0.8(\text{Var}(X) + \mu_X^2) = 0.8(10 + 5^2) = 0.8(35) = 28$.


$b^2 = \text{Var}(Y_{2i} + Y_{i-1-2}) = \text{Var}(Y_{2i}) + \text{Var}(Y_{i2-1}) = 2\text{Var}(Y) = 2 \cdot 12 = 24$, so $b = \sqrt{24}$.

Then for $c = 1/2$, by the CLT, $W_i \overset{d}{\to} N(0, 1)$.

SSS since $Y_i$ is i.i.d., WSS since SSS.

4. We want $P[X(5 + t) > 5100|X(t) = 5200]$. (Since input is Gaussian and WSS, output is also WSS, so any $t$ will do; just use $t = 0$.) Input is a Gaussian r.p., so output is a Gaussian r.p., hence $X(5)$ and $X(0)$ are jointly Gaussian, so the conditional pdf for $X(5)$ given $X(0) = 5200$ is also Gaussian with mean $E[X(5)|X(0) = 5200] = \mu_X(5) + \text{Cov}(X(5), X(0))\text{Var}(X(0))^{-1}(5200 - \mu_X(0)) = 5000 + K_X(5)K_X(0)^{-1}(5200 - 5000) = 5000 + K_X(5)K_X(0)^{-1}200$. Since $S_N(\omega) = 100$, $R_N(\tau) = 100\delta(\tau)$, so

$$R_X(\tau) = 100\delta(\tau) \ast h(\tau) \ast h(-\tau) = \begin{cases} 1000(1 - |\tau|/10), & |\tau| \leq 10, \\
0, & \text{otherwise} \end{cases}$$

Thus $K_X(0) = 1000$ and $K_X(5) = 1000(1 - 1/2) = 500$. Thus $E[X(5)|X(0) = 5200] = 5000 + 500/1000 \cdot 200 = 5100$. Thus $P[X(5) > 5100|X(0) = 5200] = Q((5100 - 5100)/\text{Var}(X(5)|X(0))) = 1/2$.

$\text{Var}(X(5)|X(0))$ not needed, but it is $1000 - 500/1000 \cdot 500 = 750$. end