Solutions to EECS 401 Midterm 2

1.

- $P[|X-5| \le 3] = P[2 \le X \le 8] = \int_2^8 f_X(x)\ dx = 2\int_0^3 \frac{1}{2}e^{-x}\ dx = 1 - e^{-3}$ (using symmetry of f_X about 5).
  Of 92 students, 70 realized what was needed was the above integral; but only 24 of those integrated correctly.

- Since f_X is symmetric about 5: $P[X > 5] = 1/2$.
  Also $P([|X-5| \le 3] \cap [X > 5]) = P[5 < X \le 8]) = \int_0^3 \frac{1}{2}e^{-x}\ dx = \frac{1}{2}(1-e^{-3})$.
  So since $P([|X-5| \le 3] \cap [X > 5]) = P[|X-5| \le 3]P[X > 5]$, YES the two events are independent.
  Many students confused independent events with disjoint events: A and B are disjoint iff $A \cap B = \phi$.
  A and B are independent iff $P(A \cap B) = P(A)P(B)$,

- Note that $S_Y = [-3, 3]$, so $F_Y(y) = 0$ for $y < 3$, and $F_Y(y) = 1$ for $y \ge 3$.
  Find $F_Y(y) = P[Y \le y] = P[h(X) \le y]$ by equivalent event method. For $-3 \le y < 3$: $F_Y(y) = P[Y \le y]$
  $= P[X > -y/3] = \int_{-y/3}^{\infty} f_X(x)\ dx = \int_{-y/3}^{5} \frac{1}{2}e^{x-5}\ dx + \frac{1}{2} = \frac{1}{2}(1 - e^{-y/3-5}) + \frac{1}{2} = 1 - \frac{1}{2}e^{-y/3-5}$. So

$$F_Y(y) = \begin{cases} 0, & y < -3 \\ 1 - \frac{1}{2}e^{-y/3-5}, & -3 \le y < 3 \\ 1, & y \ge 3 \end{cases}$$

  Taking derivative (noting jump discontinuities at ± 3):

$$f_Y(y) = (1 - \frac{1}{2}e^{-4})\delta(y+3) + \frac{1}{2}e^{-6}\delta(y-3) + \frac{1}{6}e^{-y/3-5}(u(y+3) - u(y-3))$$

  Many students answers had x's on the right hand side. Since $f_Y(y)$ is a function of y, no x's belong on the r.h.s.!
  Also, the "plug and chug" formula does not apply to transformations having "flat" regions.
  Picture:

2.

- $P[I > 5] = P[V/R > 5] = P[R < V/5] = P[R < 3] = P[R < 3|A]P(A) + P[R < 3|B]P(B) = \frac{2}{3}\frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{11}{12}$,
  where from uniform distribution: $P[R < 3|A] = \int_1^3 \frac{1}{4-1}\ dx = 2/3$.

- $[W > 75] = [V^2/R > 75] = [R < 15^2/75] = [R < 3]$
  Bayes rule: $P(A|W > w) = P(W > w|A)P(A)/P(W > w) = P(R < 3|A)P(A)/P(R < 3) = \frac{2}{3}\frac{1}{4}/\frac{11}{12} = \frac{2}{11}$

- By total expectation:

$$E[R] = E[R|A]P(A) + E[R|B]P(B) = \frac{1+4}{2}\frac{1}{4} + \frac{1+2}{2}\frac{3}{4} = \frac{14}{8} = \frac{7}{4}$$

  If $X \sim \text{Unif}[a, b]$, then $E[X^2] = \int_a^b \frac{1}{(b-a)}x^2\ dx = \frac{1}{3(b-a)}(b^3 - a^3)$. Thus

$$E[R^2] = \frac{1}{3(4-1)}(4^3 - 1^3)\frac{1}{4} + \frac{1}{3(2-1)}(2^3 - 1^3)\frac{3}{4} = \frac{7}{4} + \frac{7}{4} = \frac{7}{2}$$

  So $\text{Var}\{R\} = E[R^2] - (E[R])^2 = 7/2 - (7/4)^2 = 7/16$
  Instead of the correct approach above, many students used a made-up formula that might be called "total variance:"
  $\text{Var}\{R\} \stackrel{?}{=} \text{Var}\{R|A\}P(A) + \text{Var}\{R|B\}P(B)$ — THIS FORMULA IS NOT CORRECT! (max of 5 pts)

3.

- $P(\text{short/long} < 1/2) = P(0 \le X < 2/3) + P(4/3 < X \le 2) = 2\int_0^{2/3} 1 - (1-x)\ dx = 2\int_0^{2/3} x\ dx = 4/9$

- $E[X(2-X)] = \int_0^2 x(2-x)f_X(x)\ dx = 2\int_0^1 x(2-x)(1-(1-x))\ dx = \int_0^1 4x^2 - 2x^3\ dx = 4x^3/3 - x^4/2\Big|_0^1 =$
  $4/3 - 1/2 = 5/6$
  For both parts the integral is simplified using symmetry.

end