Solutions to EECS 401 Final, April 29, 1996

1.

- $E[\sum_{i=1}^{100} X_i] = 100\frac{1}{2}(2+6) = 400$

- $[6 \cdot N + 2 \cdot (100 - N) > 430] = [N > 57.5]$, so P[total > 430] $= \sum_{k=58}^{100} \binom{100}{k} \frac{1}{2^{100}}$
  The central limit theorem gives an approximation, not an exact value.

- $E[X_i^2] = \frac{1}{2}(2^2 + 6^2) = 20$, so $\mathrm{Var}\{X_i\} = 20 - 4^2 = 4$, so $\sigma_X = 2$.
  By the central limit theorem: $P[\sum_{i=1}^{100} X_i > 430] = P[\frac{1}{\sqrt{100}} \sum_{i=1}^{100} \frac{X_i - \mu_X}{\sigma_X} > \frac{430 - 100 \cdot 4}{\sqrt{100} \cdot 2}] \approx Q(\frac{30}{20}) = Q(1.5) \approx 0.0668$

2.

- By independence and linearity, $E[2(X-1)^7(Y-1)+3] = E[(X-1)^7]E[Y-1] = 2E[(X-1)^7] \cdot 0 + 3 = 3$
  Several students did not recognize that Y has an exponential distribution with mean 1.

- Note that $X \geq 0$ and $Y \geq 0$. For $z \geq 0$, $F_Z(z) = P[Y/X \leq z] = P[Y \leq zX] = \int_0^\infty \int_0^{zx} f_Y(y) f_X(x)\ dy\ dx$
  $= \int_0^\infty 2e^{-2x} \int_0^{zx} e^{-y}\ dy\ dx = \int_0^\infty 2e^{-2x}(1 - e^{-zx})\ dx = 1 - \int_0^\infty 2e^{-x(z+2)}\ dx = 1 - \frac{2}{z+2}$.
  Thus $f_Z(z) = \frac{2}{(z+2)^2} u(z)$
  There is an alternative approach (not covered class but used in example 4.34) that works for this particular problem (but not most $Z = g(X,Y)$ problems) using the scale/shift property of pdfs:
  $f_Z(z) = \int_\infty^\infty f_{Z|X}(z|x) f_X(x)\ dx = \int_0^\infty x e^{-zx} 2e^{-2x}\ dx = \int_0^\infty 2xe^{-(z+2)x}\ dx = \frac{2}{(z+2)^2} u(z)$
  Almost no one who tried this approach did it correctly.
  Many students forgot the ranges ($z \geq 0$) despite the instructions.
  Many students cited "independence" and wrote the meaningless equation $f_Z(z) = f_Y(y)/f_X(x)$. Apparently these students still do not understand that $f_Z(z)$ is a 1D function of z, so the r.h.s. should have z's in it...

3.

- As for all sum processes, $C_S(n,m) = \min(n,m)\sigma_X^2$, so $C_S(50,60) = 50 \cdot 9 = 450$.

- Y_i is also i.i.d., and therefore both s.s.s. and w.s.s.
  Many students just regurgitated the definitions without actually explaining why they hold. Some students correctly verified the w.s.s. conditions, and then stated that Y_i must be also s.s.s. since "it is Gaussian," not recognizing that the square of a Gaussian r.v. is not Gaussian.
  Some students did not recognize that i is the time index for a discrete-time r.p.

- $E[S_{52}|S_{50} = 6] = E[X_{52} + X_{51} + S_{50}|S_{50} = 6] = E[X_{52}|S_{50} = 6] + E[X_{51}|S_{50} = 6] + E[S_{50}|S_{50} = 6]$
  $= E[X_{52}] + E[X_{51}] + 6 = 2 + 2 + 6 = 10$

- $P[S_{100} < 260] = P[\frac{S_{100} - 100\mu_X}{\sqrt{100\sigma_X^2}} < \frac{260 - 100 \cdot 2}{10 \cdot 3}] = 1 - Q(2) \approx 0.9772$
  Many students cited the central limit theorem to justify their answer, not realizing that since the sum of Gaussian r.v.s is Gaussian, the answer is exactly $1 - Q(2)$ (no approximation).

- $\mathrm{Cov}\{S_n, X_j\} = \mathrm{Cov}\{\sum_{i=1}^n X_i, X_j\} = \sum_{i=1}^n \mathrm{Cov}\{X_i, X_j\} = \mathrm{Cov}\{X_j, X_j\} = \sigma_X^2$
  So $\rho = \mathrm{Cov}\{S_n, X_j\}/\sqrt{\mathrm{Var}\{S_n\}\sigma_X^2} = 1/\sqrt{n} = 1/8$.
  As a general rule, it seems that when students get stuck at an expectation such as $E[S_{67}X_7]$, they simply (incorrectly) assume independence and write $E[S_{67}]E[X_7]$ and then proceed. In this problem that leads to $\rho = 0$, which is obviously incorrect: since X_7 is a part of the sum S_{64}, the correlation coefficient should be positive (although small).
  Many students also confused σ with σ^2 (variance).
  Sigh.

end