Sets

Mathematical Symbols

Symbol	Meaning
$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	Integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Symbol	how it is read
$:$	"such that"
$\exists$	"there exists"
$\forall$	"for all"
$\Leftrightarrow$	"if and only if" (iff)
$\Rightarrow$	"implies"

Definitions

A set is a collection of entities called *elements* or *members*.
Upper case letters early in the alphabet denote sets: A, B, C.
Lower case letters denote generic elements of sets: a, b, x, y, s.

The Universal Set, denoted S, contains all the elements of interest in a particular context.

The Empty Set, denoted ϕ, contains no elements.

Set Membership

- $x \in A$ means x is an element of set A. Read: x is in A, or x belongs to A, or x is a member of A.
- $x \notin A$ means x is not an element of set A

Set Notation

Often we enumerate set contents in braces: $A = \{1, 2, \ldots, 10\}$
By definition: $A = \{x \in S : x \in A\}$
An interval on the real line: $A = [2, 5) = \{x \in \mathbb{R} : 2 \leq x < 5\}$

Set Relationships

Subset $A \subseteq B$ iff $\forall x,\ x \in A \Rightarrow x \in B$ (i.e., every element of A must also be in B)
Proper Subset $A \subset B$ iff $A \subseteq B$ and $\exists x \in B$ such that $x \notin A$
Equality $A = B$ iff $(\forall x,\ x \in A \Leftrightarrow x \in B)$ iff $(A \subseteq B$ and $B \subseteq A)$
Disjoint or Mutually Exclusive Sets: no common elements: $x \in A \Rightarrow x \notin B$. Equivalently: $x \in B \Rightarrow x \notin A$

Set Operations

Union	$A \cup B = A + B = \{x \in S : x \in A \text{ or } x \in B\}$
Intersection	$A \cap B = AB = \{x \in S : x \in A \text{ and } x \in B\}$
Complement	$A^c = \overline{A} = \{x \in S : x \notin A\}$
Difference **(aka Set Reduction)**	$A - B = A \cap B^c = \{x \in A : x \notin B\}$ (i.e., the elements of A that are *not* in B)

(Venn Diagram)

Disjoint Sets or Mutually Exclusive Sets

Sets A and B are disjoint iff $A \cap B = \phi$

Set Algebra

Commutative Laws	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associative Laws	$A \cup (B \cup C) = (A \cup B) \cup C$
	$A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's Laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B} \ \star$

Proof of $\star$: $x \in \overline{A \cup B} \Leftrightarrow x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B \Leftrightarrow x \in \overline{A}$ and $x \notin \overline{B} \Leftrightarrow x \in \overline{A} \cap \overline{B}$

Because of the commutative and associative laws, the following notation is unambiguous:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \qquad \text{and} \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

One can also have countable and uncountable unions. For example, by definition:

$$A = \bigcup_{x \in A} \{x\}$$

Set Partition

The sets $A_1, A_2, \ldots, A_n$ are said to *partition* B iff

- $B = \bigcup_{i=1}^{n} A_i$, and
- $A_i \cap A_j = \phi, \ i \neq j$ (i.e., the A_i's are mutually exclusive or disjoint)

Duality Principle

In any set identity, if you replace $\cap$ by $\cup$, $\cup$ by $\cap$, S by ϕ, and ϕ by S, then the new identity is also true.
Examples: Distributive Laws, De Morgan's Laws. $A \cap S = A$ becomes $A \cup \phi = A$

Set Properties (Important for Probability)

- $\overline{\phi} = S, \ \overline{S} = \phi$
- $A \cup \overline{A} = S$
- $A \cap \overline{A} = \phi$ Thus A and $\overline{A}$ partition S
- $\overline{\overline{A}} = A$
- $\overline{A} = S - A$
- $A \cup S = S$ so from Duality Principle: ...
- etc.

Set Functions
The following definitions generalize the usual definitions of functions on the real line used in calculus.

$f : A \to B$ is a *function* from A to B iff $\forall x \in A,\ f(x) \in B$, i.e., for every element x of A, the function f assigns a value denoted $f(x)$ that is a member of B.
We also say f is a *mapping* from A *into* B.
Calculus is concerned primarily with functions on the real line: $f : \mathbb{R} \to \mathbb{R}$, such as $f(x) = x^2$.
In probability we will use more general functions. For example, in a coin tossing experiment with 2 coin tosses, the sample space is $S = \{HH, HT, TH, TT\}$, and we might be interested in the function $f : S \to \mathbb{R}$ that returns the number of tails:

$$f(s) = \begin{cases} 0, & \text{if } s = \text{HH} \\ 1, & \text{if } s = \text{HT} \\ 1, & \text{if } s = \text{TH} \\ 2, & \text{if } s = \text{TT} \end{cases} .$$

When $f : A \to B$, the set A is called the *domain* of f.
We say f is *defined on* A.
The elements $f(x)$ are called the *values* of f.
The *range* of f is denoted $\text{Range}(f) = \{f(x) : x \in A\}$ and is the set of values of f.
Note that $\text{Range}(f) \subseteq B$.

If $E \subseteq A$, then $f(E) = \{f(x) : x \in E\}$ is called the *image* of E under the mapping f.
Thus $\text{Range}(f) = f(A)$. Also $f(A) \subseteq B$.
However, if $f(A) = B$, then we say f maps A *onto* B. This is a stronger condition than "into."

If $E \subseteq B$, then $f^{-1}(E) = \{x \in A : f(x) \in E\}$ is called the *inverse image* of E under f.
In particular, if $y \in B$, then $f^{-1}(y) = f^{-1}(\{y\}) = \{x \in A : f(x) = y\}$.

If for every $y \in B$, $f^{-1}(y)$ is (at most) a single element of A, then f is called a *one-to-one* mapping of A into B.
Equivalently, f is a *one-to-one* mapping of A into B iff $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
I.e., if x_1 and x_2 are two distinct elements of A, then f must map those two elements to different values in B.
If in addition, $f(A) = B$, then f is called a one-to-one mapping of A onto B.

Examples

$$\begin{aligned} A_1 &= \{1, 2, 3\} \\ A_2 &= \{-3, -2, -1, 1, 2, 3\} \\ B_1 &= \{2, 4, 8\} \\ B_2 &= \{2, 3, 4, 8\} \end{aligned}$$

The function $f_1(x) = 2^{|x|}$ maps A_2 onto B_1, and maps A_2 into B_2, because $f_1(A_2) = \{2, 4, 8\}$.

The function $f_2(x) = 2^x$ is a one-to-one mapping of A_1 onto B_1, and is a one-to-one mapping of A_1 into B_2.
Note that $f_2^{-1}(3) = \phi$, because there are no points x in A_1 (or A_2) for which $f_2(x) = 3$.

Set "Size"

If there exists a one-to-one mapping of A onto B, then we say A and B have the same *cardinality* or the same *cardinal number*, or that A and B can be put into one-to-one correspondence. Sometimes written $A \sim B$.

Possible set "sizes" or cardinalities

- **Empty**: $A = \phi$
- **Finite**: $A \sim \{1, 2, \ldots, n\}$ for a positive integer n
- **Countably Infinite**: $A \sim \{1, 2, 3, \ldots\}$ (all positive integers)
- **Uncountable** or **Uncountably Infinite**: (if none of the above apply)

Roughly speaking, a set is countably infinite if there is a way to write down all of its elements as an (infinitely long) list. Because a list is like a mapping defined on the set of positive integers, where $f(1)$ is the first item on the list, $f(2)$ is the second item, etc.

Examples

- The set {red, green, blue, yellow} is finite.
- The set of even positive integers is countably infinite. (Let $f(n) = 2n$).
- The set of all integers is countably infinite.
  (Let $f(n) = (-1)^n$ ceil$[\frac{n-1}{2}]$, where ceil[x] denotes the smallest integer $\geq x$.)
- The set of positive fractions is countably infinite.
- The interval [0,1) on the real line is uncountably infinite!

The real numbers are uncountably infinite

Proof that $A = [0, 1)$ is an uncountable set

(Proof by contradiction, i.e., assume that A is countable, and show the assumption leads to a logical contradiction, so we must then conclude that A is uncountable.)

Suppose A is countably infinite. Then every $x \in A$ is one of the elements of the collection $\{a^1, a^2, a^3, \ldots\}$, since by definition, if A is countable, then there exists a function f from the positive integers onto A.

Now write each a^n in its decimal expansion:

$$a^n = 0.u_1^n u_2^n u_3^n \cdots = \sum_{i=1}^{\infty} u_i^n 10^{-i},$$

where $u_i^n \in \{0, 1, \ldots, 9\}$.

Now we construct a new decimal number:

$$y = 0.v_1 v_2 v_3 \cdots,$$

where we define

$$v_i = \begin{cases} 1, & \text{if } u_i^i = 0, \\ 0 & \text{if } u_i^i \neq 0 \end{cases} .$$

Clearly $y \in [0, 1) = A$ by construction. But note that by construction we have $y \neq a^n$ for all n, since y and a^n differ in the nth decimal place!

This contradicts the existence of a function f from the positive integers onto A. Thus A must be uncountable. $\square$