Signals and Systems: Summary 3
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Sampling

- Impulse train sampling: $x_s(t) = x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \stackrel{\mathcal{F}}{\longleftrightarrow} X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$.
- If $x(t)$ is bandlimited to $\pm\omega_{\max}$, then one can recover $x(t)$ from its samples $x(nT_s)$ (or equivalently from $x_s(t)$) if ω_s exceeds the Nyquist sampling rate $2\omega_{\max}$.
- A lowpass filter with cutoff frequency $\omega_{\max} < \omega_c < \omega_s - \omega_{\max}$ will recover $X(\omega)$ from $X_s(\omega)$ and hence $x(t)$ from $x_s(t)$.
- In time domain:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\omega_c T_s}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c(t - nT_s)}{\pi}\right)$$

- More generally, for a periodic signal $p(t)$ with fundamental frequency ω_0:

$$y(t) = x(t)p(t) = x(t)\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega) = \sum_{k=-\infty}^{\infty} c_k X(\omega - k\omega_s),$$

  for which the sampling requirements in general depend on c_k's
- Non-sinc interpolation:

$$x_s(t) \to \boxed{h_1(t)} \to y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega) = X_s(\omega)H_1(\omega)$$

Modulation

- Modulation property: $s(t) = m(t)\cos(\omega_c t) \stackrel{\mathcal{F}}{\longleftrightarrow} S(\omega) = \frac{1}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)]$
- Need $\omega_c > \omega_{\max}$ (maximum frequency of $M(\omega)$) to prevent overlap.
- Synchronous demodulation:

$$s(t) \to \underset{\substack{\uparrow \\ \cos(\omega_c t)}}{\bigotimes} \to \boxed{H(\omega) = 2\,\mathrm{rect}\left(\frac{\omega}{2\omega_{\max}}\right)} \to m(t)$$

Bilateral Laplace transform

- $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$, $s = \sigma + j\omega$
- ROC is subset of $\mathbb{C}$ where $x(t)e^{-\,\mathrm{real}(s)t}$ is absolutely integrable
- ROC is "strips" (including RHP or LHP or all of $\mathbb{C}$)
  - ROC never contains poles
  - ROC of bounded finite signal is $\mathbb{C}$
  - ROC of right-sided signal is a RHP
  - ROC of left-sided signal is a LHP
  - ROC of two-sided signal is a strip
  - ROC of rational LT is strip bounded by poles
- $X(\omega) = X(s)|_{s=j\omega}$ if ROC includes $j\omega$ axis
- pole-zero plots + gain + ROC describe rational LT
- For rational $H(s)$:
  - pole locations describe modes of system
  - system is stable if ROC of $H(s)$ includes $j\omega$ axis
  - system is causal if ROC is a RHP
  - a causal system is stable iff all poles within LHP
  - if $M > N$, then there are $M - N$ poles at $s = \infty$, so system is unstable and non-causal.
- PFE for inverse LT when rational
- magnitude, phase response from pole-zero by geometry (draw vectors from pole/zero to $(0, j\omega)$, angle ccw from real axis)

Properties of the Laplace Transform

	Time	Laplace	ROC (of result)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	contains $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Time shift	$f(t-\tau)$	$e^{-s\tau} F(s)$	same
Time-scaling	$f(at),\ a \neq 0$	$\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$	ROC/a
Time reversal	$f(-t)$	$F(-s)$	$-\mathrm{ROC}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$	contains $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Frequency shift	$f(t) e^{j\omega_0 t}$	$F(s - j\omega_0)$	same
Frequency shift	$f(t) e^{s_0 t}$	$F(s - s_0)$	$\mathrm{ROC} +\ \mathrm{real}(s_0)$
Time Differentiation	$\frac{d^n}{dt^n} f(t)$	$s^n F(\omega)$	contains ROC
s-domain Differentiation	$(-t)^n f(t)$	$\frac{d^n}{ds^n} F(\omega)$	same
Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau = f(t) * u(t)$	$\frac{1}{s} F(s)$	contains $\mathrm{ROC} \cap \{\ \mathrm{real}(s) > 0\}$
DC Value	$\int_{-\infty}^{\infty} f(t)\, dt = F(0)$		must contain $s = 0$

Methods for finding response $y(t)$ of LTI system due to input $x(t)$

- Convolution $x(t) \xrightarrow{\mathrm{LTI}} y(t) = x(t) * h(t)$
- LTI/convolution properties, *e.g.* if $x_1(t) \xrightarrow{\mathrm{LTI}} y_1(t)$, and $x_2(t) \xrightarrow{\mathrm{LTI}} y_2(t)$, then

$$x(t) = a_1 x_1(t - t_1) + a_2 x_2(t - t_2) \xrightarrow{\mathrm{LTI}} y(t) = a_1 y_1(t - t_1) + a_2 y_2(t - t_2).$$

- Impulse properties (for finite set of impulses):

$$x(t) = \sum_{k=1}^{n} a_k \delta(t - t_k) \xrightarrow{\mathrm{LTI}} y(t) = \sum_{k=1}^{n} a_k h(t - t_k)$$

- Eigenfunctions $x(t) = e^{s_0 t} \xrightarrow{\mathrm{LTI}} y(t) = H(s_0) e^{s_0 t}$
- $x(t) = e^{j\omega t} \xrightarrow{\mathrm{LTI}} y(t) = H(j\omega) e^{j\omega t} = |H(j\omega)| e^{j(\omega t + \angle H(j\omega))}$
- $x(t) = \sum_k c_k e^{j\omega_k t} \xrightarrow{\mathrm{LTI}} y(t) = \sum_k c_k H(j\omega_k) e^{j\omega_k t}$
- If $h(t)$ is real, then $x(t) = \cos(\omega t + \phi) \xrightarrow{\mathrm{LTI}} y(t) = |H(j\omega)| \cos(\omega t + \phi + \angle H(j\omega))$
- $x(t) = \sum_k A_k \cos(\omega_k t + \phi_k) \to \boxed{\text{LTI } h(t)} \to y(t) = \sum_k A_k |H(j\omega_k)| \cos(\omega_k t + \phi_k + \angle H(j\omega_k))$
- Fourier series $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \xrightarrow{\mathrm{LTI}} y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t}$
- Fourier transform convolution property $X(\omega) \xrightarrow{\mathrm{LTI}} Y(\omega) = X(\omega) H(\omega)$. (Useful whenever $Y(\omega) = X(\omega) H(\omega)$ easily inverted.)
- Laplace transform convolution property $X(s) \xrightarrow{\mathrm{LTI}} Y(s) = X(s) H(s)$. (Best for rational $X(s)$ and $H(s)$.)

LTI system properties

	Time	Fourier	Laplace (rational)
Invertible	$h(t) * h_i(t) = \delta(t)$, for some $h_i(t)$	$\forall \omega,\ H(\omega) \neq 0$	no zeros on $j\omega$ axis
Causal	$h(t) = 0$ for $t < 0$	?	ROC is a RHP
Stable	$\int_{-\infty}^{\infty} \lvert h(t)\rvert\, dt < \infty$	?	ROC includes $j\omega$ axis
Memory	$h(t) \neq \delta(t)$	$H(\omega)$ not constant	$H(s)$ not constant

Table of Laplace transform pairs (algebraic component)

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$t^n u(t)$	$\frac{1}{s^n}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$

$f(t)$	$F(s)$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Table of Fourier transform pairs

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega) = \delta\left(\frac{\omega}{2\pi}\right)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin\omega_0 t$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
e^{-bt^2}	$\sqrt{\pi/b}e^{-\omega^2/(4b)}$

$f(t)$	$F(\omega)$
$\frac{1}{b^2+t^2}$	$\frac{\pi}{b}e^{-b\lvert\omega\rvert}$
$e^{-b\lvert t\rvert}$	$\frac{2b}{b^2+\omega^2}$
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}\left(T\frac{\omega}{2\pi}\right)$
$\mathrm{tri}(t)$	$\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\frac{\omega_0}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_0}{2\pi}t\right)$	$\mathrm{rect}\left(\frac{\omega}{\omega_0}\right)$
$\mathrm{sinc}^2(t)$	$\mathrm{tri}\left(\frac{\omega}{2\pi}\right)$
$e^{-at}u(t)$	$\frac{1}{j\omega+a}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(j\omega+a)^n}$

b is a real positive number throughout. a is a real or complex number throughout, with positive real part.