Signals and Systems: Summary 2

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Fourier Series

- Analysis equation: $c_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t}\, dt,\ k = 0, \pm 1, \pm 2, \ldots.$
- DC value: $c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt$.
- Synthesis equation: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$
- Combined trigonometric form: $x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k| \cos(k\omega_0 t + \angle c_k)$, if $x(t)$ is real.
- Trigonometric form: $x(t) = c_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t)$, where $A_k = 2\,\mathrm{real}(c_k)$ and $B_k = 2\mathrm{Imag}(c_k)$

Convergence properties

- Error signal energy $E_N = \int_{T_0} |x(t) - x_N(t)|^2\ dt$, where $x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}$ is minimized when c_k's chosen according to above formula
- $\lim_{N\to\infty} E_N = 0$, provided Dirichlet conditions hold
- If $x(t)$ has a jump discontinuity at t_0, then $\lim_{N\to\infty} x_N(t_0) = \frac{x(t_0^+) + x(t_0^-)}{2}$.
- Near jumps, persistent overshoot called **Gibbs phenomenon** will occur.
- If $x(t)$ has a right and left derivative (*e.g.* continuous but possibly a corner) at t_0, then $\lim_{N\to\infty} x_N(t_0) = x(t_0)$.
- If $x(t)$ is continuous everywhere, then for any $\delta > 0$, there exists an N such that $\max_t |x_N(t) - x(t)| < \delta$.
- If $x(t)$ is a finite sum of harmonic sinusoids, then only a finite number of the c_k's will be nonzero.
- For sufficiently large k, the kth Fourier coefficient will decrease in magnitude at least as fast as $1/k$.

One-signal properties (Fourier series transformations)

- Amplitude transformation: $ax(t) + b \leftrightarrow \begin{cases} b + ac_0, & k = 0 \\ ac_k, & k \neq 0. \end{cases}$
- Time transformation: $x(at + b) \leftrightarrow \begin{cases} c_k e^{jk\omega_0 b}, & a > 0 \\ c_{-k} e^{jk\omega_0 b}, & a < 0. \end{cases} \quad \omega_1 = |a|\omega_0$
- Time shift: $x(t - t_0) \leftrightarrow c_k e^{-jk\omega_0 t_0}$
- Conjugation: $[x(t)]^\star \leftrightarrow c_{-k}^\star$
- Complex modulation (frequency shift): $x(t) e^{j\omega_0 t N} \leftrightarrow c_{k-N}$
- Differentiation: $y(t) = \frac{d}{dt} x(t) \leftrightarrow jk\omega_0\, c_k$

Properties

- If $x(t)$ is real, then $c_{-k} = c_k^\star$.
- Linearity (add coefficients if same period T_0)
- Multiplication $c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$. (discrete convolution)
- Filtering: see below
- Circular convolution: skip
- Total harmonic distortion: THD = $(1 - 2|c_1|^2/P) \cdot 100\%$
- Power of $ce^{jk\omega_0 t}$ is $|c|^2$
- Power of $A\cos(\omega t + \phi)$ is $A^2/2$
- Parseval's theorem: $P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\ dt = \sum_{k=-\infty}^{\infty} |c_k|^2$
- Power density spectrum: $|c_k|^2$
- Magnitude spectrum: $|c_k|$. Phase spectrum: $\angle c_k$

Foundations of Filtering

- $x(t) = e^{st} \xrightarrow{\text{LTI}} y(t) = H(s) e^{st}$
- Laplace transform of $h(t)$, aka system function: $H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt$
- $x(t) = e^{j\omega t} \xrightarrow{\text{LTI}} y(t) = H(j\omega) e^{j\omega t} = |H(j\omega)| e^{j(\omega t + \angle H(j\omega))}$
- Fourier transform of $h(t)$, aka frequency response: $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt = H(s)|_{s=j\omega} = |H(j\omega)| e^{j\angle H(j\omega)}$
- $x(t) = \sum_k c_k e^{j\omega_k t} \xrightarrow{\text{LTI}} y(t) = \sum_k c_k H(j\omega_k) e^{j\omega_k t}$
- If $h(t)$ is real, then $H^\star(s) = H(s^\star)$ and $H(-j\omega) = H^\star(j\omega)$. (Hermitian symmetry)
- If $h(t)$ is real, $x(t) = \cos(\omega t + \phi) \xrightarrow{\text{LTI}} y(t) = |H(j\omega)| \cos(\omega t + \phi + \angle H(j\omega))$
- $x(t) = \sum_k A_k \cos(\omega_k t + \phi_k) \to \boxed{\text{LTI } h(t)} \to y(t) = \sum_k A_k |H(j\omega_k)| \cos(\omega_k t + \phi_k + \angle H(j\omega_k))$

Fourier Transform

- Fourier transform (analysis): $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt$.
- Inverse Fourier transform (synthesis): $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$
- For signals satisfying the Dirichlet conditions, if there is a discontinuity at t_0, then $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t_0}\, d\omega = \frac{f(t_0^+)+f(t_0^-)}{2}$, the midpoint of the jump.
- For periodic signals: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} c_k 2\pi \delta(\omega - k\omega_0)$.
- $x(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 G(k\omega_0) \delta(\omega - k\omega_0)$
- Energy density spectrum: $|X(\omega)|^2$
- Energy over a spectral band: $E_B = \frac{1}{2\pi} \int_B |X(\omega)|^2\, d\omega$
- Symmetry properties $$\begin{array}{ccccccccc} f(t) & = & f_R^e(t) & + & jf_I^e(t) & + & f_R^o(t) & + & jf_I^o(t) \\ & & \updownarrow & & \updownarrow & & \times & & \\ F(\omega) & = & F_R^e(\omega) & + & jF_I^e(\omega) & + & F_R^o(\omega) & + & jF_I^o(\omega) \end{array}$$

Filtering

- Convolution property: $x(t) \overset{\text{LTI}}{\longrightarrow} y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = H(\omega) X(\omega)$
- $e^{j\omega_0 t} \to \boxed{h(t)} \to y(t) = h(t) * e^{j\omega_0 t} = H(\omega_0) e^{j\omega_0 t}$
- Partial fraction expansion: $X(s) = \dfrac{1}{(s+a)(s+b)} = \dfrac{r_1}{s+a} + \dfrac{r_2}{s+b}$, where $r_1 = (s+a)X(s)|_{s=-a} = \dfrac{1}{-a+b}$.
- Frequency response of diffeq system $\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$, corresponds to $H(\omega) = \dfrac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$.
- MATLAB: `freqs`, `impulse`, `tf`, `lsim`, `residue`

Properties of the Continuous-Time Fourier Transform

Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Time transformation	$f(at+b),\ a \neq 0$	$\frac{1}{\lvert a\rvert} e^{j\omega b/a} F(\omega/a)$
Time shift	$f(t-\tau)$	$F(\omega) e^{-j\omega\tau}$
Time reversal	$f(-t)$	$F(-\omega)$
Time-scaling	$f(at),\ a \neq 0$	$\dfrac{1}{\lvert a\rvert} F\left(\dfrac{\omega}{a}\right)$
Convolution	$f_1(t) * f_2(t)$	$F_1(\omega) \cdot F_2(\omega)$
Time-domain Multiplication	$f_1(t) \cdot f_2(t)$	$\dfrac{1}{2\pi} F_1(\omega) * F_2(\omega)$
Frequency shift	$f(t) e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Modulation (cosine)	$f(t) \cos(\omega_0 t)$	$\dfrac{F(\omega-\omega_0) + F(\omega+\omega_0)}{2}$
Time. Differentiation	$\dfrac{d^n}{dt^n} f(t)$	$(j\omega)^n F(\omega)$
Freq. Differentiation	$(-jt)^n f(t)$	$\dfrac{d^n}{d\omega^n} F(\omega)$
Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau = f(t) * u(t)$	$\dfrac{1}{j\omega} F(\omega) + \pi F(0) \delta(\omega)$
Conjugation	$f^\star(t)$	$F^\star(-\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Parseval/Rayleigh Theorem	$E = \int_{-\infty}^{\infty} \lvert f(t)\rvert^2\, dt = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2\, d\omega$	
DC Value	$\int_{-\infty}^{\infty} f(t)\, dt = F(0)$	