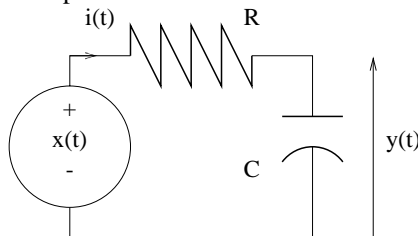


Analysis of Passive Circuits by FT Methods

1st-order example

The following RC circuit is a classic 1st-order lowpass filter.



This document describes, by example, how to analyze passive circuits using Fourier transform methods, both “by hand” and by using MATLAB. The methods apply *generally* to diffeq systems.

This 1st-order example is simple enough that both the time-domain and frequency-domain approaches have comparable effort. But for more complicated (higher-order) systems, the frequency-domain method is often easier than solving diffeqs and/or convolution. Therefore, this 1st-order example serves to illustrate the methods in the simplest possible setting.

The key FT properties used are: **convolution property**, **linearity**, and (time-domain) **differentiation property**.

- Find diffeq from circuit (210).
- Find frequency response $H(\omega)$ from circuit.
- Find frequency response $H(\omega)$ from diffeq.
- Find diffeq from frequency response $H(\omega)$.
- Find impulse response $h(t)$ from frequency response $H(\omega)$.
- Find step response from $H(\omega)$.
- Find response to eternal sinusoid.
- Find response to causal sinusoid.
- Compute and plot all of the above using MATLAB.

Find diffeq from circuit

By KCL and KVL: $i(t) = C \frac{d}{dt} y(t)$. $x(t) = v_R(t) + y(t) = Ri(t) + y(t) = RC \frac{d}{dt} y(t) + y(t)$.
So we have derived the following diffeq by time-domain circuit analysis:

$$y(t) + RC \frac{d}{dt} y(t) = x(t). \quad (\text{RC-1})$$

Find frequency response $H(\omega)$ from circuit

Basic idea: Since $X(\omega) \rightarrow \boxed{\text{LTI } H(\omega)} \rightarrow Y(\omega) = H(\omega)X(\omega)$, we can rearrange above formula to get $H(\omega) = Y(\omega)/X(\omega)$.

Treat the above circuit as a **voltage divider** using complex impedances. Using KCL:

$$I(\omega) = \frac{Y(\omega)}{Z_C(\omega)} = \frac{X(\omega)}{R + Z_C(\omega)}$$

so cross dividing:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{Z_C(\omega)}{R + Z_C(\omega)} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}. \quad (\text{RC-2})$$

Find frequency response $H(\omega)$ from diffeq

Taking the FT of both sides of the diffeq above using linearity and differentiation property yields: $Y(\omega) + RCj\omega Y(\omega) = X(\omega)$. Thus $[1 + j\omega RC]Y(\omega) = X(\omega)$, so cross dividing yields

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC}.$$

Find diffeq from frequency response $H(\omega)$

Cross multiplying the above expression for $H(\omega) = Y(\omega)/X(\omega)$ yields $Y(\omega) + RCj\omega Y(\omega) = X(\omega)$. Now take the *inverse* FT of both sides using linearity and differentiation property to yield the diffeq:

$$y(t) + RC \frac{d}{dt} y(t) = x(t).$$

Find impulse response $h(t)$ from frequency response $H(\omega)$

Approaches to finding $h(t)$ from $H(\omega)$:

- Use inverse FT integral: $h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega$.
- Use FT properties and FT tables.
- For *rational* $H(\omega)$ (ratios of polynomials in $j\omega$) use **partial fraction expansions**. (See Appendix of book.)

General idea. First note that

$$H(\omega) = \frac{1}{1 + j\omega RC} = \frac{1/RC}{j\omega + 1/RC}.$$

Recall that $e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega + a}$ if $\text{real}(a) > 0$. So $H(\omega)$ is already in a form that can use that result. So by “table lookup:”

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t). \quad (\text{RC-3})$$

For higher-order systems, one usually needs PFE to simplify $H(\omega)$ before taking the inverse FT to find $h(t)$.

Find step response**Time-domain convolution approach**

Useful if impulse response $h(t)$ is known and easily integrated.

$$u(t) \rightarrow \boxed{h(t)} \rightarrow s(t) = h(t) * u(t) = \int_{-\infty}^t h(\tau) d\tau$$

Frequency-domain approach

Since $u(t) \xleftrightarrow{\mathcal{F}} U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$, by the convolution property of the FT and the sampling property of $\delta()$, the FT of the step response is

$$\begin{aligned} S(\omega) &= H(\omega)U(\omega) = H(\omega) \left[\pi\delta(\omega) + \frac{1}{j\omega} \right] = \pi H(0)\delta(\omega) + \frac{1}{j\omega} H(\omega) \\ &= H(0) \left[\pi\delta(\omega) + \frac{1}{j\omega} \right] + \frac{1}{j\omega} [H(\omega) - H(0)] = H(0)U(\omega) + \frac{1}{j\omega} [H(\omega) - H(0)], \end{aligned}$$

where the second line is small simplifying trick. Taking the inverse FT we see that the general form of the step response is

$$s(t) = H(0)u(t) + z(t), \quad \text{where } Z(\omega) = \frac{1}{j\omega} [H(\omega) - H(0)].$$

To complete the analysis, we must find $z(t)$ from $Z(\omega)$.

For our example, we have $H(0) = 1$, so

$$Z(\omega) = \frac{1}{s} \left(\frac{1/RC}{s + 1/RC} - 1 \right) \Big|_{s=j\omega}.$$

Simplifying:

$$\frac{1}{s} \left(\frac{1/RC}{s + 1/RC} - 1 \right) = \frac{1}{s} \left(\frac{1/RC - (s + 1/RC)}{s + 1/RC} \right) = \frac{-1}{s + 1/RC}.$$

(This step requires PFE for higher-order systems.) Thus

$$Z(\omega) = \frac{-1}{j\omega + 1/RC},$$

so taking the inverse FT (by table lookup) we get the following (transient) response:

$$z(t) = -e^{-t/RC}u(t).$$

Thus the system step response is:

$$s(t) = H(0)u(t) + z(t) = u(t) - e^{-t/RC}u(t). \quad (\text{RC-4})$$

To check this answer, we should be able to differentiate the step response and get back the impulse response. Applying the product rule and sampling property:

$$\begin{aligned} h(t) &= \frac{d}{dt}s(t) = \delta(t) - \left[e^{-t/RC}\delta(t) - \frac{1}{RC}e^{-t/RC}u(t) \right] \\ &= \delta(t) - e^0\delta(t) + \frac{1}{RC}e^{-t/RC}u(t) = \frac{1}{RC}e^{-t/RC}u(t). \end{aligned}$$

Find response to eternal sinusoid

Time domain approach: tedious convolution.

Frequency domain approach (derived using Fourier series):

$$x(t) = \cos(\omega_0 t + \theta) \rightarrow \boxed{\text{LTI } H(\omega)} \rightarrow |H(\omega_0)| \cos(\omega_0 t + \theta + \angle H(\omega_0)),$$

so in this particular example the response to $x(t) = \cos(\omega_0 t + \theta)$ is

$$y(t) = \left| \frac{1}{1 + j\omega_0 RC} \right| \cos\left(\omega_0 t + \theta + \angle \frac{1}{1 + j\omega_0 RC}\right) = \frac{1}{\sqrt{1 + (\omega_0 RC)^2}} \cos(\omega_0 t + \theta - \tan^{-1} \omega_0 RC), \quad (\text{RC-5})$$

since

$$\angle \frac{1}{1 + j\omega_0 RC} = \angle \frac{1 - j\omega_0 RC}{(1 + j\omega_0 RC)(1 - j\omega_0 RC)} = \angle \frac{1 - j\omega_0 RC}{1 + (\omega_0 RC)^2} = \angle 1 - j\omega_0 RC = -\tan^{-1} \omega_0 RC.$$

Find response to causal sinusoid

Time domain approach: *very* tedious convolution (at least for higher-order systems).

Frequency domain approach

The response to an *eternal* sinusoid is the **steady state** portion of the response to a causal sinusoid, *i.e.* $x(t) = \cos(\omega_0 t)u(t)$. To find the complete response to a *causal* sinusoid, we can work in the frequency domain, which usually by applying the PFE method.

Since

$$\cos(\omega_0 t + \phi)u(t) \xleftrightarrow{\mathcal{F}} \frac{\pi}{2} [e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)] + \frac{j\omega \cos \phi - \omega_0 \sin \phi}{(j\omega)^2 + \omega_0^2},$$

the response of an LTI system to a causal sinusoid $x(t) = \cos(\omega_0 t)u(t)$ is

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) = \left[\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{(j\omega)^2 + \omega_0^2} \right] H(\omega) \\ &= \frac{\pi}{2} [H(j\omega_0)\delta(\omega - \omega_0) + H(-j\omega_0)\delta(\omega + \omega_0)] + \frac{j\omega}{(j\omega)^2 + \omega_0^2} H(\omega) \\ &= |H(\omega_0)| \frac{\pi}{2} [e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)] + \frac{j\omega}{(j\omega)^2 + \omega_0^2} H(\omega) \\ &= |H(\omega_0)| \left[\frac{\pi}{2} [e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)] + \frac{j\omega \cos \phi - \omega_0 \sin \phi}{(j\omega)^2 + \omega_0^2} \right] + Z(\omega) \\ Z(\omega) &= \frac{j\omega}{(j\omega)^2 + \omega_0^2} H(\omega) - |H(\omega_0)| \frac{j\omega \cos \phi - \omega_0 \sin \phi}{(j\omega)^2 + \omega_0^2} = \frac{j\omega H(\omega) - |H(\omega_0)| [j\omega \cos \phi - \omega_0 \sin \phi]}{(j\omega)^2 + \omega_0^2}, \end{aligned}$$

where $H(j\omega_0) = |H(\omega_0)|e^{j\phi}$, and the last step is a simplifying trick. Thus

$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \phi) + z(t),$$

where we must find $z(t)$ from the inverse FT of $Z(\omega)$.

For our RC circuit, $H(\omega) = \frac{1}{1+j\omega RC}$, so $|H(\omega_0)| = \frac{1}{\sqrt{1+q^2}}$ where $q = \omega_0 RC$, and $\phi = \angle H(j\omega_0) = -\tan^{-1} q$ so $\cos \phi = \frac{1}{\sqrt{1+q^2}} = |H(\omega_0)|$ and so $\sin \phi = -\frac{q}{\sqrt{1+q^2}} = -|H(\omega_0)|q$. Thus

$$\begin{aligned} Z(\omega) &= \frac{1}{(j\omega)^2 + \omega_0^2} \left[\frac{j\omega}{1+j\omega RC} - \frac{j\omega + \omega_0 q}{1+q^2} \right] \\ &= \frac{1}{(j\omega)^2 + \omega_0^2} \left(\frac{j\omega[1+q^2] - (1+j\omega RC)(j\omega + \omega_0 q)}{(1+j\omega RC)[1+q^2]} \right) \\ &= \frac{1}{(j\omega)^2 + \omega_0^2} \left(\frac{j\omega + j\omega q^2 - j\omega - \omega_0 q - (j\omega)^2 RC - j\omega RC \omega_0 q}{(1+j\omega RC)[1+q^2]} \right) \\ &= \frac{1}{(j\omega)^2 + \omega_0^2} \left(\frac{-RC[\omega_0^2 + (j\omega)^2]}{(1+j\omega RC)[1+(\omega_0 RC)^2]} \right) \\ &= \frac{-RC}{(1+j\omega RC)[1+(\omega_0 RC)^2]} = \left(\frac{-1}{1+(\omega_0 RC)^2} \right) \frac{1}{j\omega + 1/RC}. \end{aligned}$$

(PFE is required here for higher-order systems.) Thus

$$z(t) = \frac{-1}{1+(\omega_0 RC)^2} e^{-t/RC} u(t),$$

so

$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \phi) u(t) - \frac{1}{1+(\omega_0 RC)^2} e^{-t/RC} u(t). \quad (\text{RC-6})$$

The first part is the **steady state** response, and the second part is the **transient** response, which decays away at a rate associated with the time-constant of the RC circuit.

Using MATLAB for diffeq analysis

A diffeq system (such as an RLC circuit) is *completely* specified by the coefficients of the diffeq. For such systems, MATLAB has built-in commands for the following calculations.

- `freqs(b,a)`
Plot the frequency response $H(\omega)$ (magnitude and phase)
- `sys = tf(b,a); h = impulse(sys, t)`
Determine the impulse response $h(t)$.
- `sys = tf(b,a); y = lsim(sys, x, t)`
Determine the response $y(t)$ to an input signal $x(t)$.
- `residue(b,a)`
Perform partial fraction expansion of the ratio of two polynomials $B(s)/A(s)$.

We illustrate each of these examples with complete self-contained MATLAB commands, assuming that $RC = 0.5$ sec throughout. For this choice, the diffeq (RC-1) can be written:

$$a_0 y(t) + a_1 \frac{d}{dt} y(t) = b_0 x(t),$$

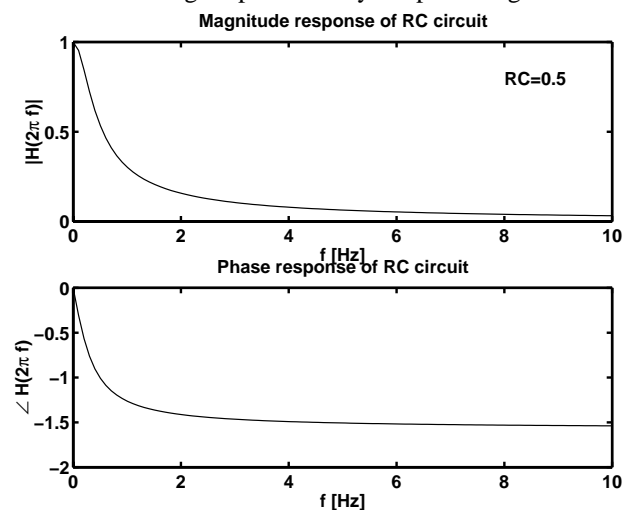
where $a_1 = 0.5$, $a_0 = 1$, and $b_0 = 1$. To use these values in MATLAB, we construct coefficient arrays in *descending* order of derivatives, *i.e.* $[a_N \ a_{N-1} \ \dots \ a_0]$ and $[b_M \ b_{M-1} \ \dots \ b_0]$. If any of the coefficients are zero, then one must include those zeros in the coefficient array in MATLAB.

MATLAB plot of the frequency response $H(\omega)$

Here is how to plot magnitude and phase response over the range 0 to 10 Hz (on a linear scale) using MATLAB's `freqs` command. Note that the `freqs` command expects radians/sec, not Hz, as the units of the 3rd argument.

```
a = [0.5 1];
b = [1];
f = linspace(0, 10, 100);
H = freqs(b, a, 2*pi*f);
clf, subplot(211), plot(f, abs(H))
xlabel('f [Hz]'), ylabel('|H(2\pi f)|')
title('Magnitude response of RC circuit')
text(8, 0.8, sprintf('RC=%g', a(1)))
subplot(212)
plot(f, angle(H))
xlabel('f [Hz]'), ylabel('\angle H(2\pi f)')
title('Phase response of RC circuit')
print('fig_rc_mag', '-deps')
```

Here is the figure produced by the preceding code.

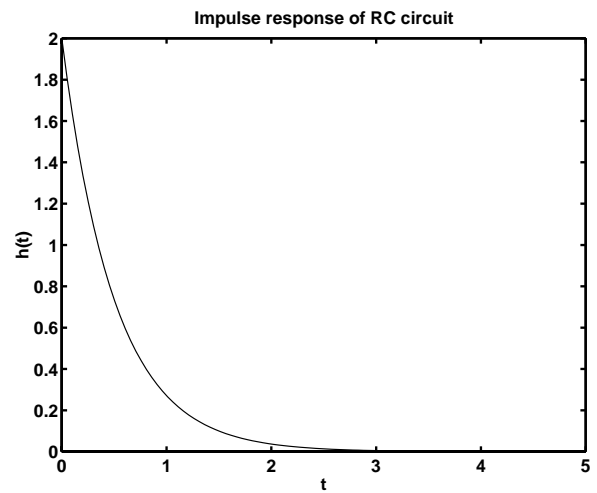


MATLAB plot of the impulse response $h(t)$

The following code plots the impulse response $h(t)$ over the time interval 0 to 5 seconds. All diffeq systems are causal (provided $M \leq N$, which always occurs for real systems like RLC circuits), so we only need to plot $h(t)$ for $t > 0$.

```
a = [0.5 1];  
b = [1];  
t = linspace(0, 5, 100);  
sys = tf(b,a);  
h = impulse(sys, t);  
clf, plot(t, h)  
xlabel('t'), ylabel('h(t)')  
title('Impulse response of RC circuit')  
print('fig_rc_h', '-deps')
```

Here is the figure produced by the preceding code.



MATLAB plot of the response $y(t)$ to a causal signal $x(t)$

The `lsim` command simulates the `diffEq` to compute the response $y(t)$ to a user-supplied (causal) input signal $x(t)$. Since MATLAB can only handle arrays (not true continuous-time signals), the user must supply matlab with an array τ of time sample points, $\tau = [t_0, t_1, \dots]$, and a corresponding array of input signal values $\mathbf{x} = [x(t_0), x(t_1), \dots]$. To determine the output signal, MATLAB must interpolate between the supplied input signal samples, so the sampling interval should be small enough to capture the details of the input signal.

The following example uses the `lsim` command to compute the response $y(t)$ to the input signal $x(t) = \cos(3t)u(t)$ over the interval 0 to 4 seconds. Three signals are plotted: the analytical output signal (RC-6), MATLAB's numerical result, and finally the steady-state response (RC-5).

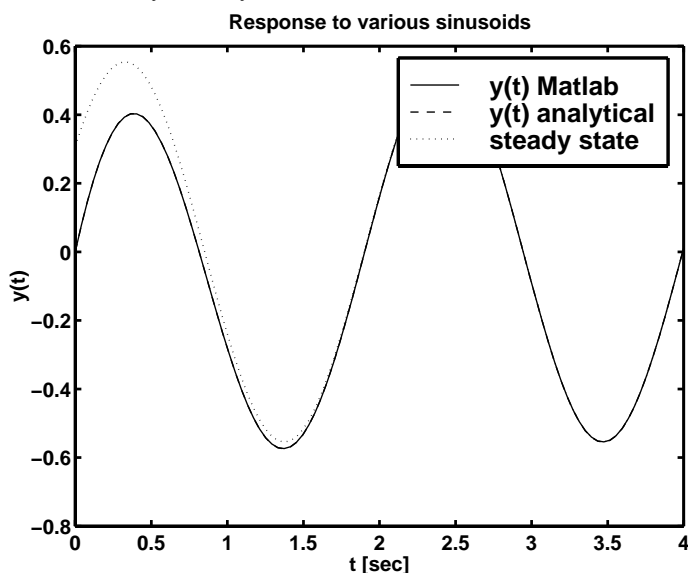
```
rc = 0.5;
a = [rc 1];
b = [1];
t = linspace(0, 4, 100);
om0 = 3;
x = cos(om0 * t);
sys = tf(b,a);
ymat = lsim(sys, x, t); % numerical convolution

% analytical formula
H0 = 1 / (1 + j*om0*rc);
yt = abs(H0) * cos(om0*t + angle(H0)) - 1/(1 + (om0*rc)^2) * exp(-t/rc);

% steady state
yss = abs(H0) * cos(om0*t + angle(H0));

clf, plot(t, ymat, '-', t, yt, '--', t, yss, ':')
xlabel('t [sec]'), ylabel('y(t)')
legend('y(t) Matlab', 'y(t) analytical', 'steady state')
title('Response to various sinusoids')
print('fig_rc_cos', '-deps')
```

Here is the figure produced by the preceding code. The analytical result (RC-6) and the simulated output of `lsim` are indistinguishable. The response $y(t)$ settles into steady state by about 2 seconds, *i.e.* about 4 time constants.



Response to input signal with rational FT

Suppose we want to find the response to the input signal $x(t) = e^{-3} \cos(5t)u(t)$, which has FT $X(\omega) = \frac{j\omega + 3}{(j\omega + 3)^2 + 5^2}$. Then by the convolution property of the FT:

$$Y(\omega) = H(\omega)X(\omega) = \left(\frac{1/RC}{s + 1/RC} \right) \frac{s + 3}{(s + 3)^2 + 5^2} \Big|_{s=j\omega}.$$

To find $y(t)$, we must first expand $Y(\omega)$ using PFE. We do this for $RC=0.5$:

$$\left(\frac{2}{s + 2} \right) \frac{s + 3}{(s + 3)^2 + 5^2} = \frac{2s + 6}{(s + 2)[(s + 3)^2 + 5^2]} = \frac{2s + 6}{s^3 + 8s^2 + 46s + 68}.$$

MATLAB's `residue` command will automatically find the poles (roots of denominator) and residues (numerators) for this polynomial. The following commands:

```
b = [2 6]
a = [1 8 46 68]
[r,p,k] = residue(b,a)
```

returns the following output:

```
r =
-0.0385 - 0.1923i
-0.0385 + 0.1923i
0.0769
p =
-3.0000 + 5.0000i
-3.0000 - 5.0000i
-2.0000
k =
[]
```

The p 's and r 's come in pairs, each corresponding to a single term in the PFE as follows:

$$Y(\omega) = \frac{-0.0385 - j0.1923}{s - (-3.0000 + j5)} + \frac{-0.0385 + j0.1923}{s - (-3.0000 - j5)} + \frac{0.0769}{s - (-2)} \Big|_{s=j\omega}.$$

Typing `abs(r(1))`, `angle(r(1))` yields `0.1961`, `-1.7682`, so $r_1 = 0.1961e^{-j1.7682}$. Taking the inverse FT yields:

$$\begin{aligned} y(t) &= 0.1961e^{-j1.7682}e^{(-3+j5)t}u(t) + 0.1961e^{j1.7682}e^{(-3-j5)t}u(t) + 0.0769e^{-2t}u(t) \\ &= 0.3922e^{-3t} \cos(5t - 1.7682)u(t) + 0.0769e^{-2t}u(t). \end{aligned}$$

The first term is the **forced response**, and the second term is the **transient response**.