Eng. 100: Music Signal Processing
DSP Lecture 4
Lab 3: Signal Spectra

Curiosity (musical style modulation)

- http://musicmachinery.com/2010/05/21/the-swinger

Announcements:

- HW2 online, due Thu. Oct. 8 in class.
- No labs this week: focus on Project 1 and HW2.
- Project 1 presentations in discussion sections this week.
- Start reading Lab 3 for next week — it is longer!

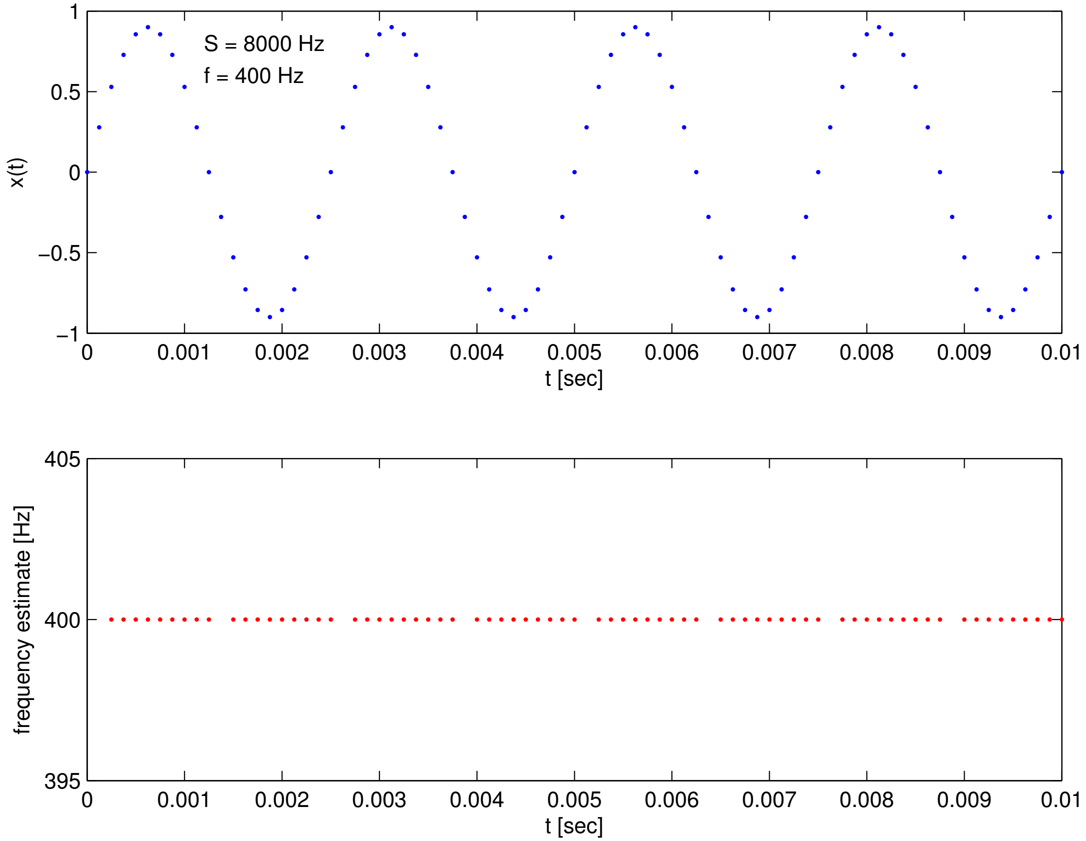
Outline

- The spectrum of a signal (first lecture)
  - Part 1. Why we need spectra
  - Part 2. Periodic signals
  - Part 3. Band-limited signals
- Methods for computing spectra (second lecture)
  - By hand by solving systems of equations
  - Using general Fourier series solution
  - Using fast Fourier transform (FFT), *e.g.*, in `Matlab`
- Using a signal's spectrum (third lecture)
  - to determine note frequencies
  - to remove unwanted noise
  - to visualize frequency content (spectrogram)
  - Lab 3

Part 1. Why we need spectra

Project 1 Transcriber for ideal sinusoid

Sinusoidal signal:



play

Method:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1]+x[n-1]}{2x[n]}\right).$$

Project 1: Transcriber limitations

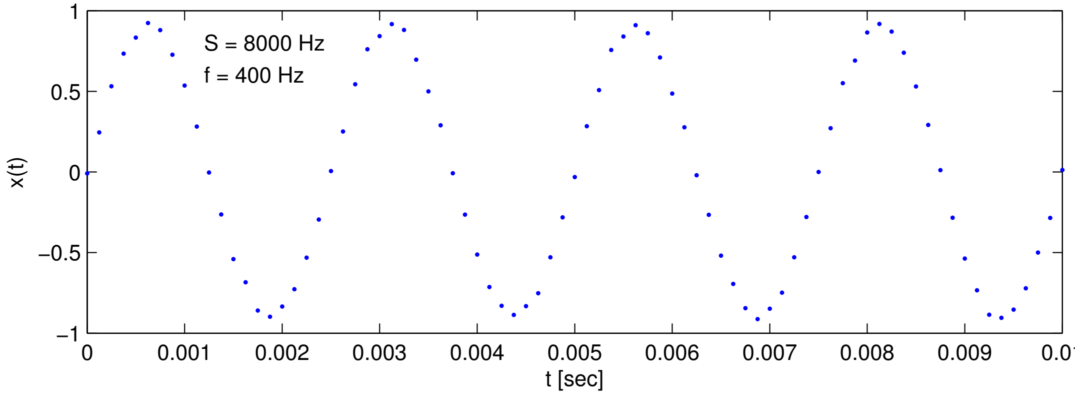
What are limitations of transcribers implemented in Project 1?

??

We need more sophisticated method(s)
for finding the (*fundamental*) frequency of a music signal.

Project 1 Transcriber for noisy sinusoid

Sinusoidal signal with *noise* (*e.g.*, in any audio recording):



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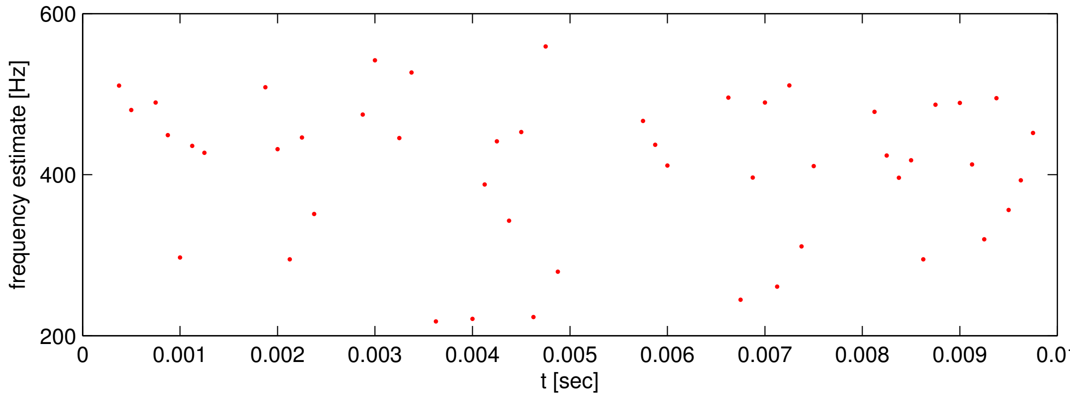
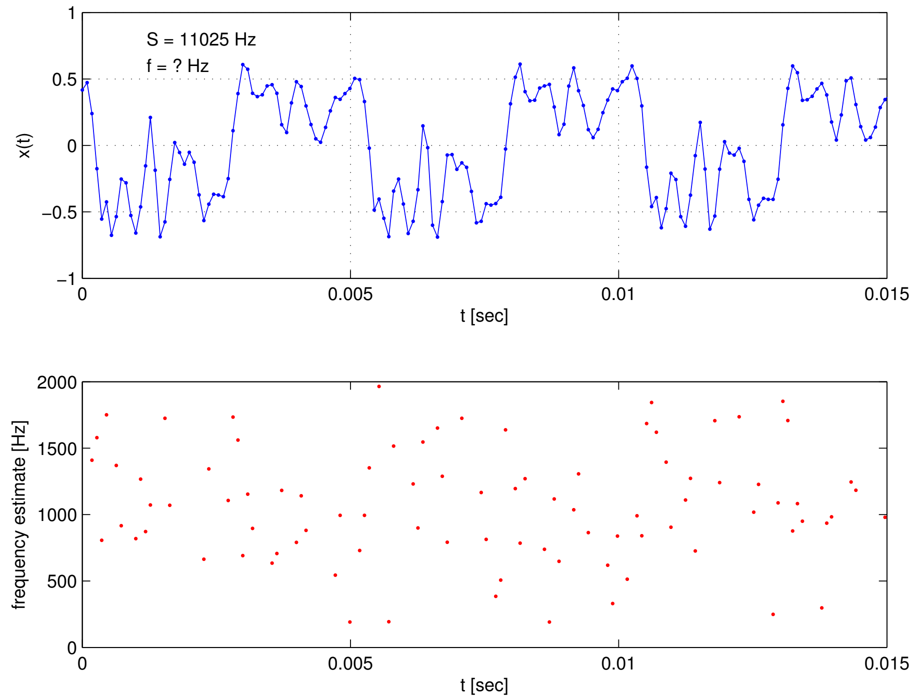


Method:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1] + x[n-1]}{2x[n]} \right).$$

Project 1 Transcriber for clarinet

Clarinet signal (roughly G below middle C): play

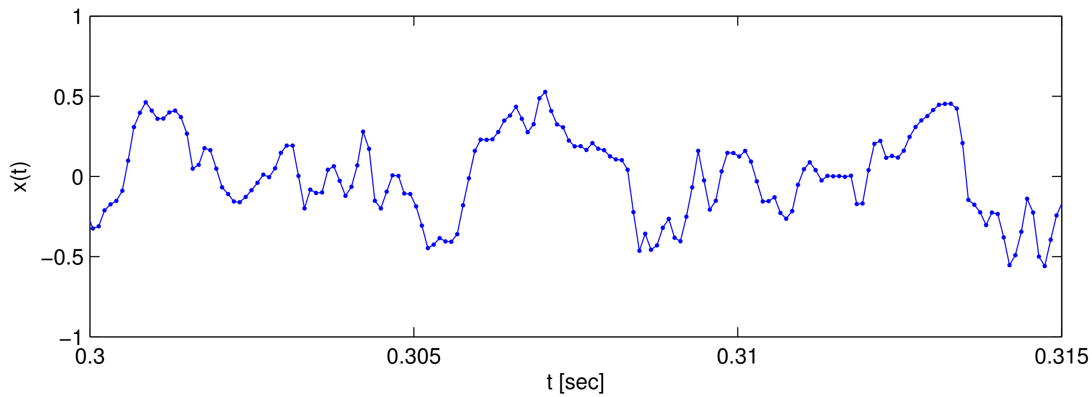


Method:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1]+x[n-1]}{2x[n]}\right).$$

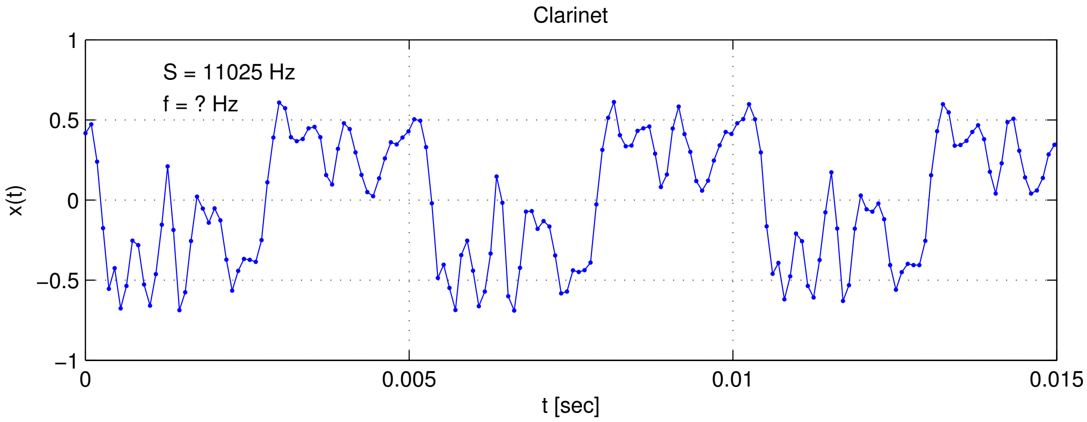
Even more challenging case

Clarinet and guitar duet: play



- Methods that use the "time domain" are very unlikely to work when two instruments play simultaneously.
- We need to use the "frequency domain"
  aka the *spectrum* of a signal. (Human ears work this way!)
- The concept of *spectra* is used widely in engineering.

Divide and conquer



This signal $x(t)$ looks complicated.
(Bamboo reed vibrations are approximately a *square wave*.)

Engineering strategy:

- Make complicated things by combining simpler things.
- Use tools from mathematics (and physics) as needed.

Mathematics provides us with a perfect tool in this case:
Fourier series.

Part 2. Spectra of periodic signals

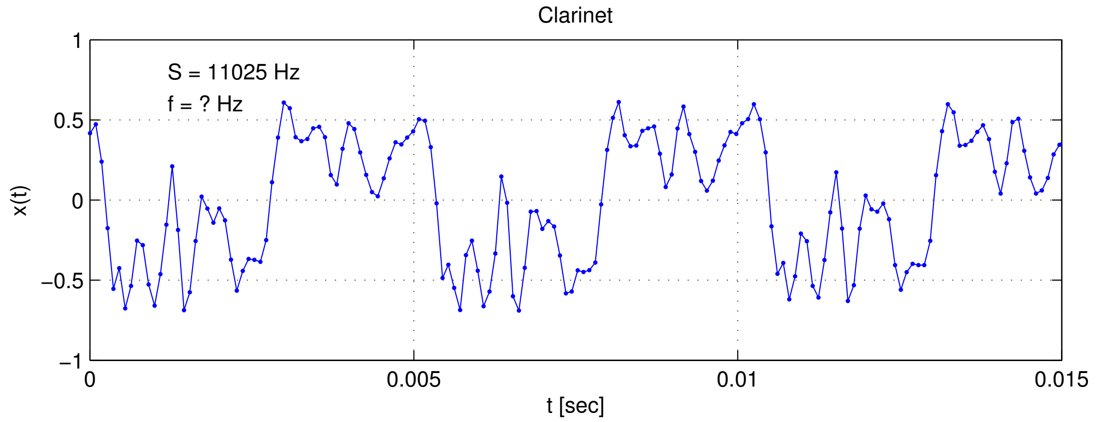
Joseph Fourier

Joseph Fourier, 1768-1830

He died after falling down the stairs at his home. [wiki]

- Fourier series theory developed circa 1807
  (Modern compared to our trigonometry method.)
- Motivating application: heat propagation in metal plates.

Periodic signals



Key property of musical signals (over short time intervals): *periodicity.*

A *periodic signal* (aka repeating signal) with *period*$=T$ satisfies

$$x(t) = x(t+T) = x(t+2T) = \cdots \text{ for all } t.$$

Example: $x(t) = \cos(2\pi 9t)$ is periodic with period $T = 1/9$ sec.
It is also periodic with period $T = 1/3$ sec.
The smallest period ($T = 1/9$ sec here) is the *fundamental period*.

Exercise

What is the (approximate) period of the clarinet signal shown on the previous slide? ??

Fourier series of periodic signals

Amazing fact #1 (discovered by Joseph Fourier 200+ years ago):

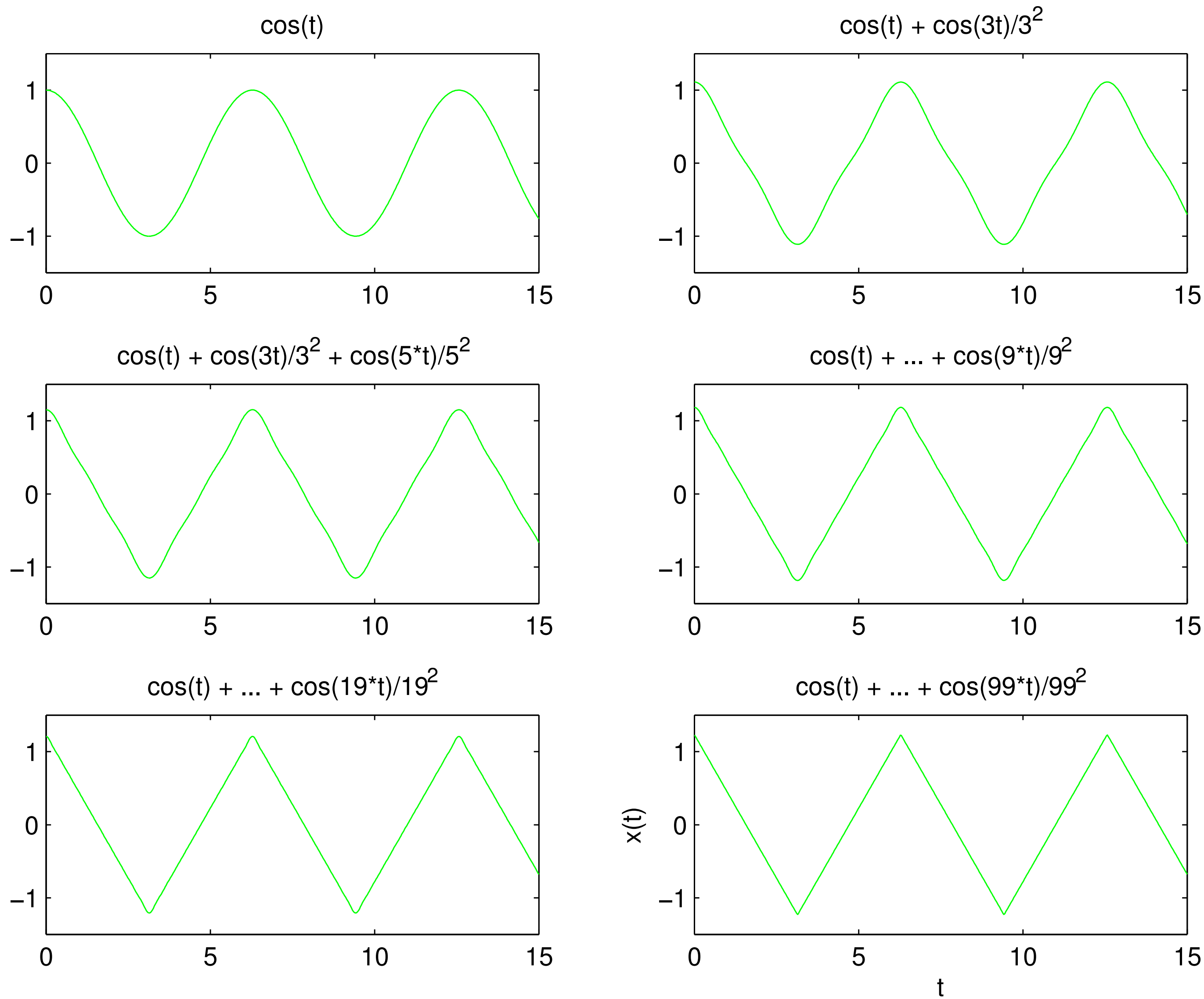
Any real-world periodic signal with period $= T$ can be *expanded* (*i.e.*, expressed mathematically as a sum) as follows:

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} c_k \cos\left(2\pi \frac{k}{T} t - \theta_k\right) \\ &= \underbrace{c_0}_{\substack{\text{DC term} \\ \text{DC value} \\ \text{DC constant}}} + \underbrace{c_1 \cos\left(2\pi \frac{1}{T} t - \theta_1\right)}_{\substack{\text{fundamental} \\ \text{period} = T \\ \text{frequency} = 1/T}} + \underbrace{c_2 \cos\left(2\pi \frac{2}{T} t - \theta_2\right)}_{\substack{\text{(first) harmonic} \\ \text{period} = T/2 \\ \text{frequency} = 2/T}} + \cdots \end{aligned}$$

- $\{c_k\}$ called *amplitudes*
- $\{k/T\}$ called *frequencies*
- $\{\theta_k\}$ called *phases*

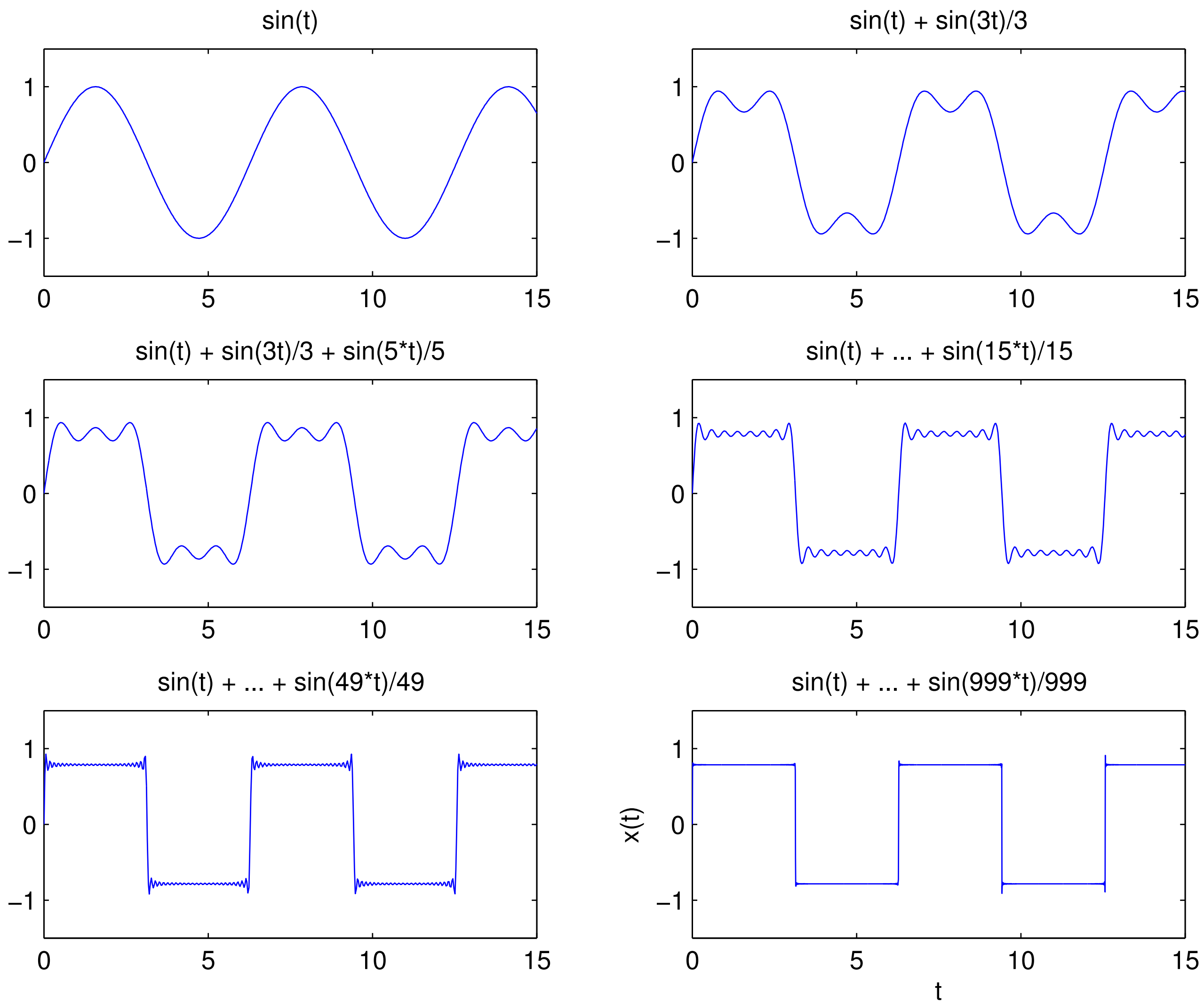
We can write even a "complicated" clarinet or guitar signal using such a "simple" sum of sinusoidal signals.

Example: Triangle wave



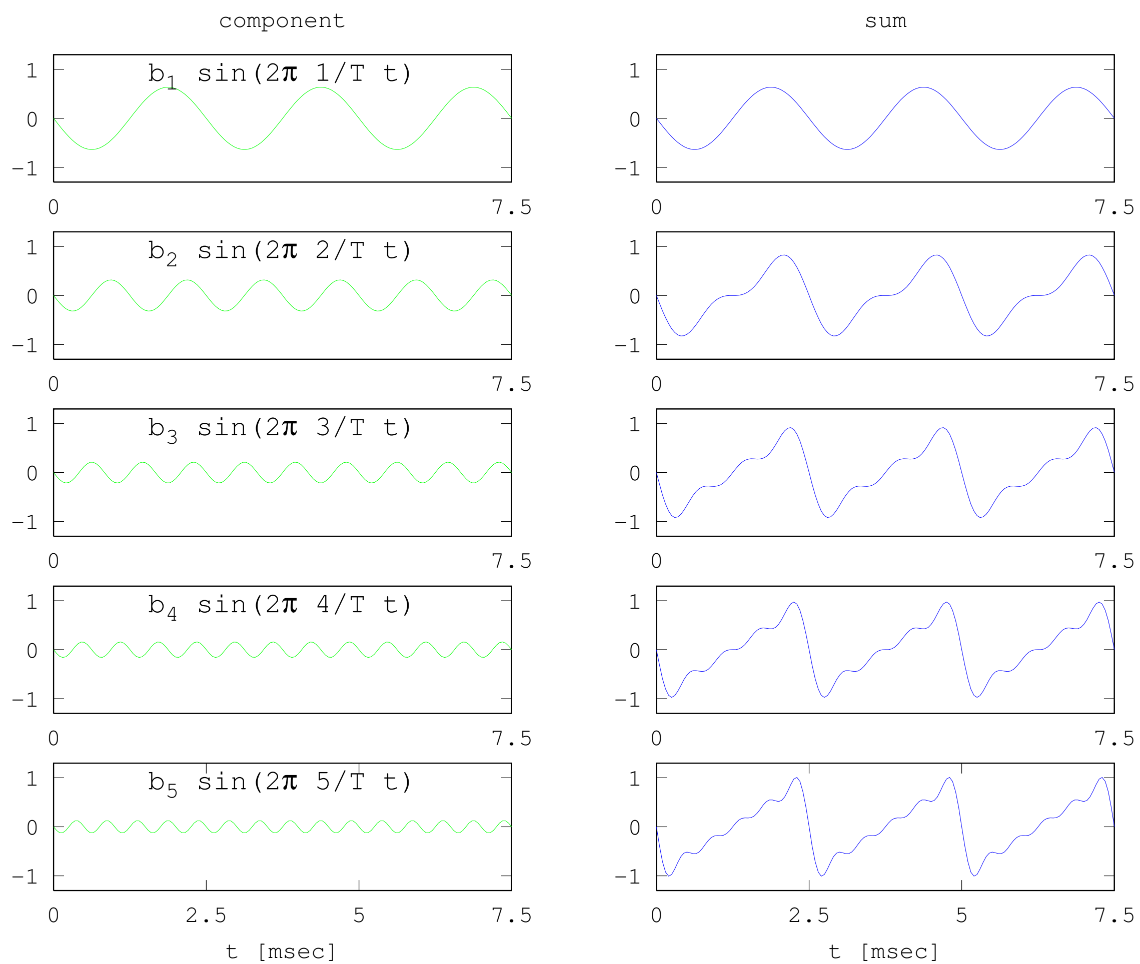
More terms in sum $\Longrightarrow$ closer approximation to triangle wave.

Example: Square wave



Sums of sinusoids can make "interesting" signals.
What is T in this example? ??

Example: Sawtooth wave



play

play

play

play

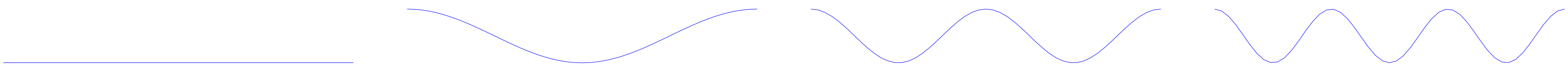
play

Does the *pitch* of the sound change? ??
Fundamental frequency? ??

The spectrum of a periodic signal

Every periodic signal can be written in the same form!

$$x(t) = c_0 + c_1 \cos\left(2\pi \frac{1}{T} t - \theta_1\right) + c_2 \cos\left(2\pi \frac{2}{T} t - \theta_2\right) + \cdots$$



So how do electric guitar and clarinet signals differ? ??

The spectrum of a periodic signal

Every periodic signal can be written in the same form!

$$x(t) = c_0 + c_1 \cos\left(2\pi \tfrac{1}{T} t - \theta_1\right) + c_2 \cos\left(2\pi \tfrac{2}{T} t - \theta_2\right) + \cdots$$

So how do electric guitar and clarinet signals differ? ??

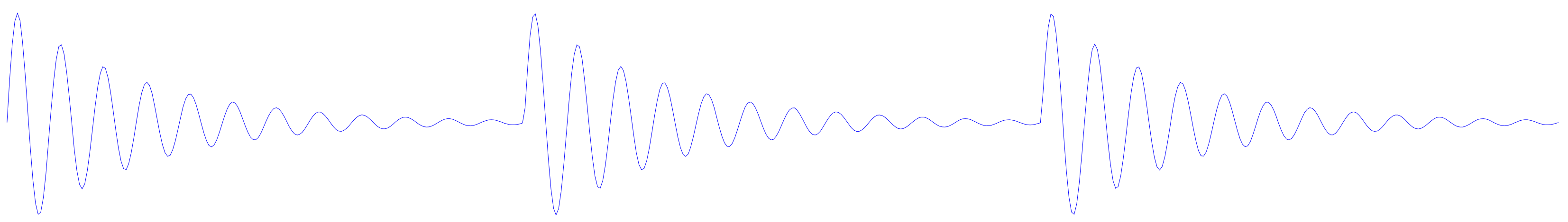
Definition. The *spectrum* of a signal $x(t)$ is just a stem plot of the amplitudes $\{c_k\}$ versus the frequencies $\{k/T\}$ in Hertz.

- ◦ The phases θ_k are unimportant for monophonic music.
- ◦ The DC term c_0 cannot be produced or heard either.
- ◦ Coefficients $\{c_k\}$ define *timbre* (TAM-ber) of sound

Spectra of periodic signals

In Eng. 100, we define spectra only of periodic signals. Why?

- Musical instruments produce approximately periodic signals.
- Definition and computation are much easier.
- Real-world non-periodic signals can be viewed as part of a periodic signal with a *very* long period.



Example: AM Radio Signal

Two Michigan AM Radio stations are:
- ∘ WSDS, 1480 kHz, 3800W, Salem Township
- ∘ WABJ, 1490 kHz, 1000W, Adrian, MI

If WSDS broadcasts a 3000 Hz sinusoidal test tone
and WABJ broadcasts a 2000 Hz sinusoidal test tone, then
(you can learn in EECS 216) that an antenna in Saline
that can pick up both stations would receive this signal:

$$\begin{aligned} x(t) \;=\; & 40\cos(2\pi 1480000t) + 20\cos(2\pi 1483000t) + 20\cos(2\pi 1477000t) \\ & + 10\cos(2\pi 1490000t) + 5\cos(2\pi 1492000t) + 5\cos(2\pi 1488000t)\,. \end{aligned}$$

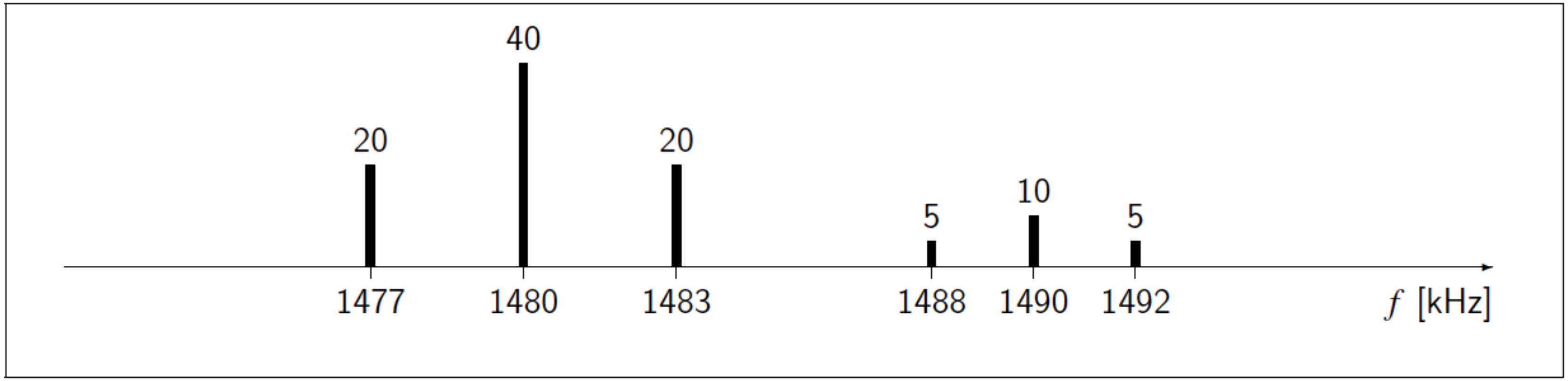
What is the *spectrum* of this signal?

Example: AM Radio Signal Spectrum

AM radio signal (expressed as mathematical formula):

$$x(t) = 40\cos(2\pi 1480000t) + 20\cos(2\pi 1483000t) + 20\cos(2\pi 1477000t) + 10\cos(2\pi 1490000t) + 5\cos(2\pi 1492000t) + 5\cos(2\pi 1488000t)\,.$$

Spectrum of this signal $x(t)$:



So the *spectrum* of a signal is a *graphical* representation.

Graphical representations are often desirable.

Exercise

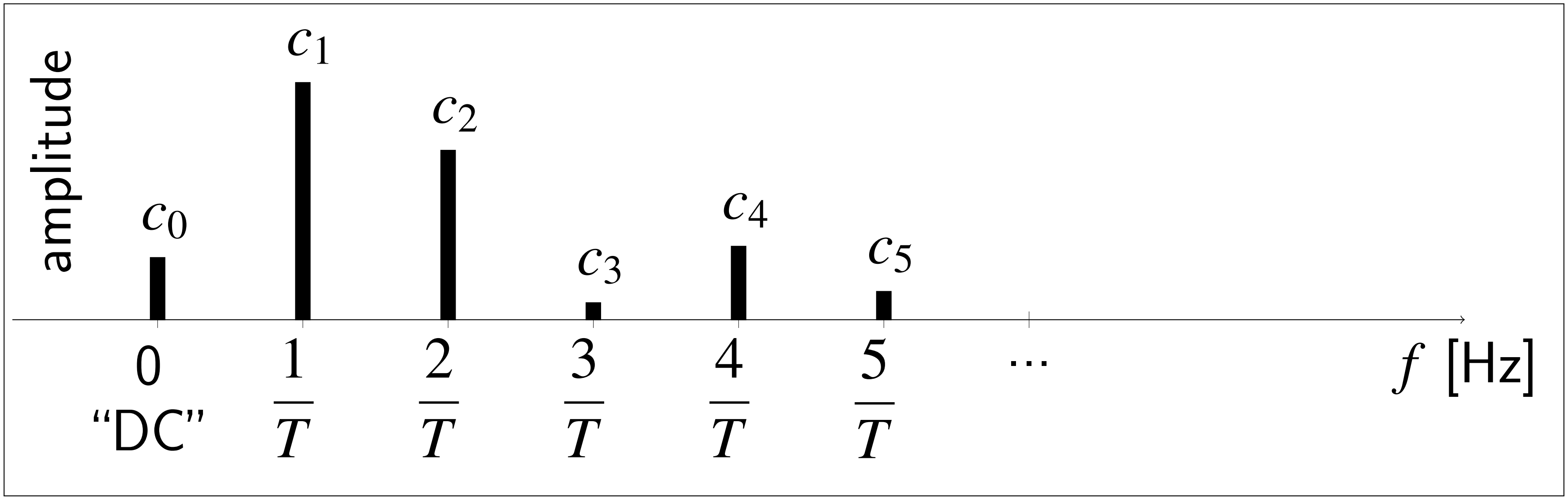
Find a *formula* for the signal that has the following spectrum.

Could an audio signal (music) have this spectrum? play

(Working forwards and backwards...)

Spectrum of a general periodic signal

A periodic signal with period T has a spectrum that looks like:

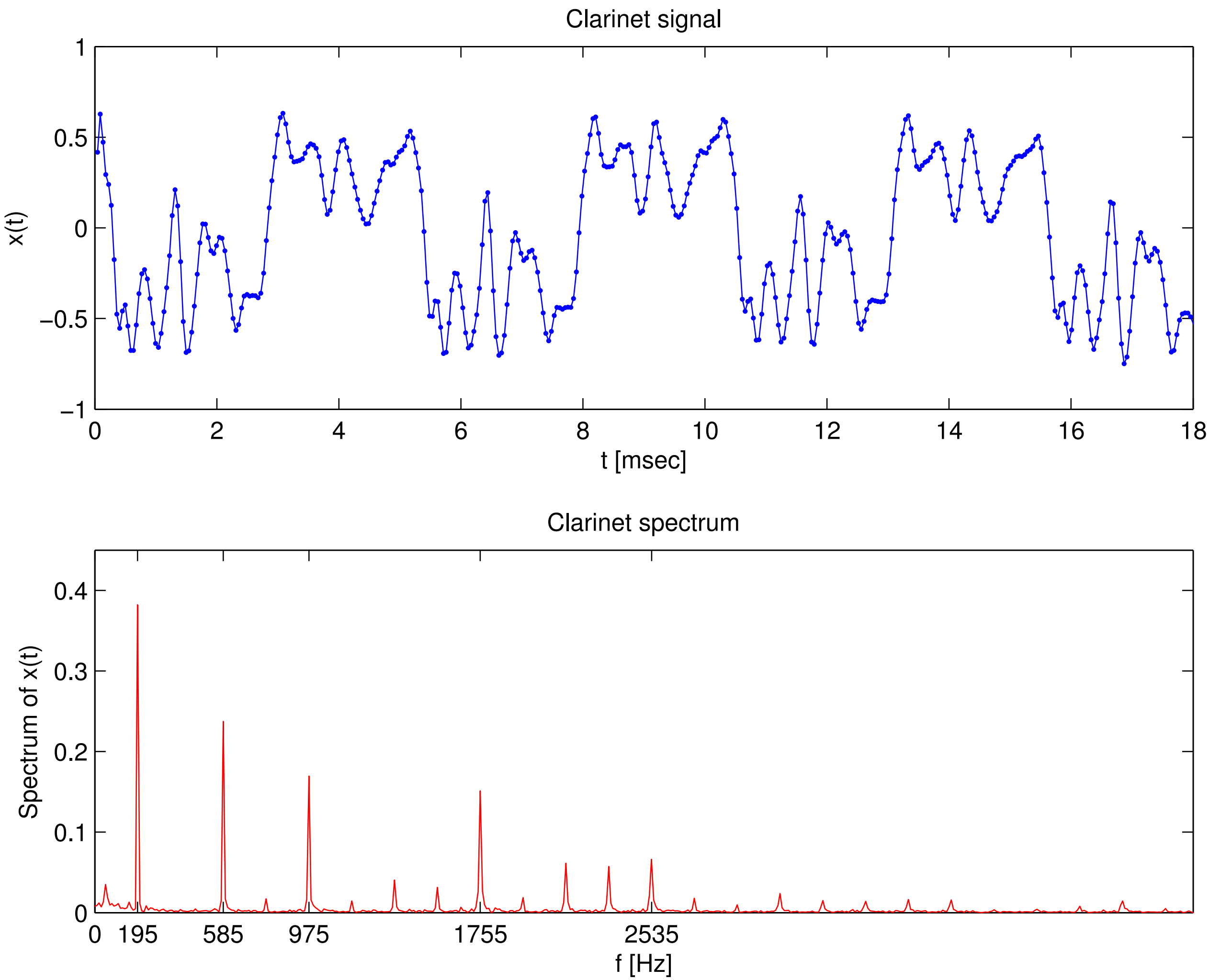


- The frequency components are $0,\ 1/T,\ 2/T,\ \ldots$
- The height of each line in the spectrum is an amplitude c_k

Ignoring phase: $x(t) = c_0 + c_1 \cos\left(2\pi \frac{1}{T} t\right) + c_2 \cos\left(2\pi \frac{2}{T} t\right) + \cdots$

What units are along the horizontal axis for a spectrum? ??

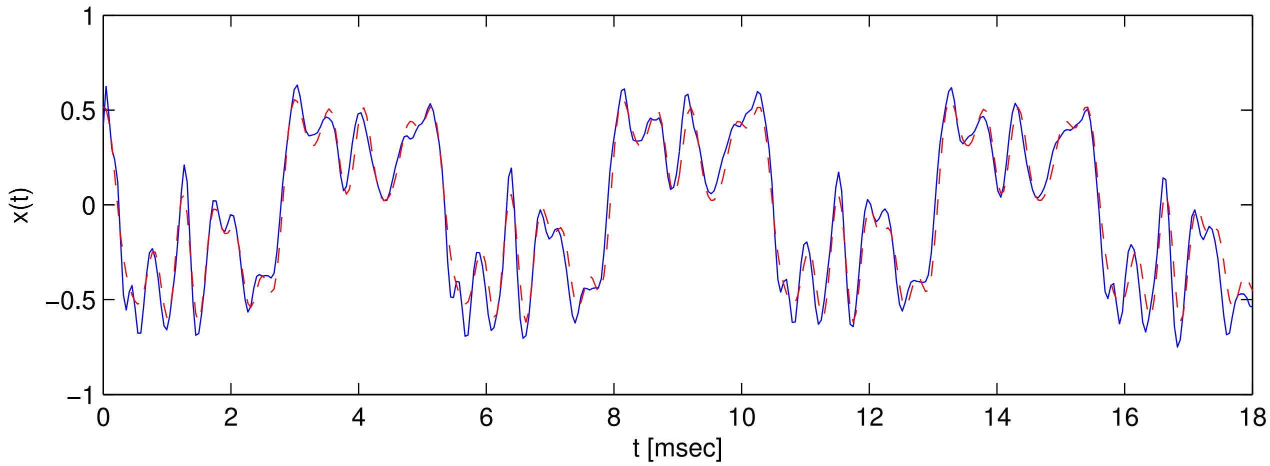
Example: Clarinet spectrum

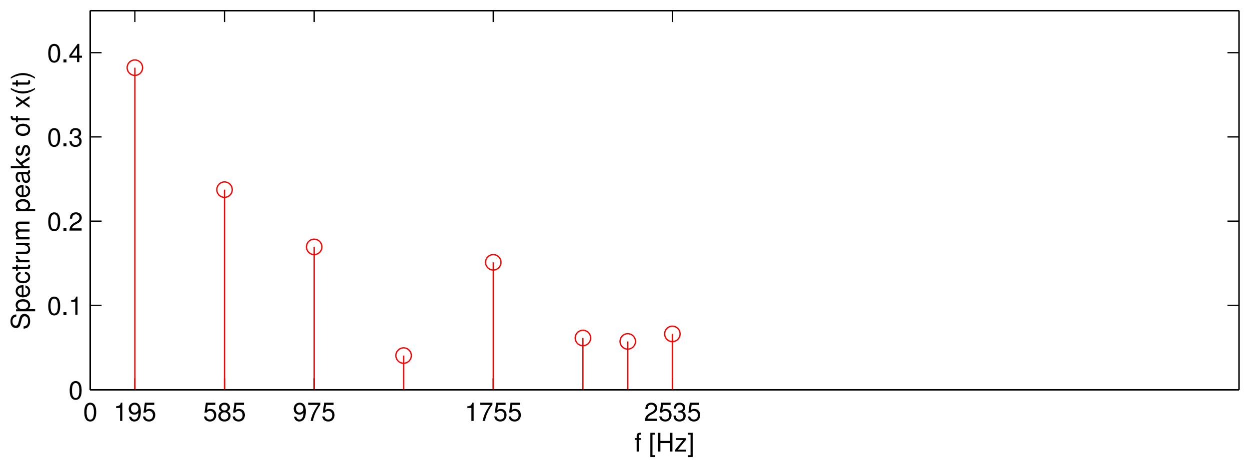


play

First significant peak: fundamental frequency $= 1/T \approx 195$ Hz
(Perfectly) periodic signals have (perfect) *line spectra*.

Example: Clarinet synthesized





play

Synthesized using 8 largest peaks in spectrum.
Sounds more interesting than Project 1 synthesizer? Why?

Example: Clarinet Fourier series

Expressing a complicated signal in terms of simple signals:

$$
\begin{aligned}
x(t) \approx\ & 0.382\cos(2\pi\ 195.0t + 1.35) + 0.237\cos(2\pi\ 584.9t + 0.48) + \\
& 0.169\cos(2\pi\ 974.8t + 0.30) + 0.151\cos(2\pi 1754.6t - 1.35) + \\
& 0.066\cos(2\pi 2534.5t - 1.41) + 0.061\cos(2\pi 2144.6t + 2.40) + \\
& 0.057\cos(2\pi 2339.5t + 0.40) + 0.041\cos(2\pi 1364.7t + 1.32)
\end{aligned}
$$

- Guitar signal would have different amplitudes and phases, even if playing the same note.
- MP3 audio coding exploits the "line" nature of music spectra.
- But how did I make the spectrum plot on previous slide?
- And how did I get all the numbers above?

Part 3: Band-limited signals: towards computing a signal's spectrum

Fourier Series: Trigonometric form

Fourier Series: *Sinusoidal form*:

$$x(t) = c_0 + c_1 \cos\left(2\pi \frac{1}{T} t - \theta_1\right) + c_2 \cos\left(2\pi \frac{2}{T} t - \theta_2\right) + \cdots$$

Fourier Series: *Trigonometric form*:

$$\begin{aligned} x(t) = a_0 \ &+ \ a_1 \cos\left(2\pi \frac{1}{T} t\right) + a_2 \cos\left(2\pi \frac{2}{T} t\right) + \cdots \\ &+ \ b_1 \sin\left(2\pi \frac{1}{T} t\right) + b_2 \sin\left(2\pi \frac{2}{T} t\right) + \cdots \end{aligned}$$

Coefficients in these two forms are related by:

$$\begin{aligned} a_0 &= c_0 \\ a_k &= c_k \cos \theta_k \\ b_k &= c_k \sin \theta_k \\ c_k &= \sqrt{a_k^2 + b_k^2} \quad \text{(need this to plot spectra)} \\ \tan \theta_k &= b_k / a_k \end{aligned}$$

because (Lab 1): $\cos(t - \theta) = \cos(\theta)\cos(t) + \sin(\theta)\sin(t)$

We will focus on finding the a_k and b_k values for music signals.

About those dots: $\cdots$

Example.
If $x(t)$ has period $= T = 0.01$ seconds then $x(t)$ has *expansion*:

$$
\begin{array}{rcllll}
x(t) & = & a_0 & + \, a_1 \cos(2\pi 100 t) & + \, a_2 \cos(2\pi 200 t) & + \, a_3 \cos(2\pi 300 t) + \cdots \\
 & & & + \, b_1 \sin(2\pi 100 t) & + \, b_2 \sin(2\pi 200 t) & + \, b_3 \sin(2\pi 300 t) + \cdots \\
 & & & \text{fundamental} & \text{first harmonic} & \text{second harmonic} \\
 & & \text{(DC)} & \text{(100 Hz)} & \text{(200 Hz)} & \text{(300 Hz)}
\end{array}
$$

- Mathematical perspective: What does "$\cdots$" mean? ??
- Engineering perspective:
  Practical signals are, or can be made to be, *band limited*.
  - Physical limits
  - Perception limits
  - Anti-alias filters in A/D converters

Band-limited signals

Definition. A signal is band-limited to B Hz if it has no frequency components *higher* than B Hz.

Example.
If $x(t)$ has period $= T = 0.01$ seconds *and* is band-limited to 800 Hz then $x(t)$ has (finite!) Fourier series expansion:

$$\begin{array}{rllll} x(t) = & a_0 & + a_1\cos(2\pi 100t) & + a_2\cos(2\pi 200t) & + \cdots + a_8\cos(2\pi 800t) \\ & & + b_1\sin(2\pi 100t) & + b_2\sin(2\pi 200t) & + \cdots + b_8\sin(2\pi 800t) \\ & & \text{fundamental} & \text{first harmonic} & \text{highest harm.} \\ & \text{(DC)} & \text{(100 Hz)} & \text{(200 Hz)} & \text{(800 Hz)} \end{array}$$

Finite sum: $x(t) = a_0 + \sum_{k=1}^{8} a_k \cos(2\pi 100kt) + b_k \sin(2\pi 100kt)$

This periodic, band-limited signal is “characterized completely” by the frequency (100 Hz) and just 17 other numbers:

$$\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}.$$

How do we find those values, called *coefficients*?

Exercise

A periodic signal $x(t)$ has period $T = 0.02$ seconds and is known to be band-limited to 200 Hz.

How many (possibly nonzero) Fourier series coefficients does it have (in trigonometric form), including the DC coefficient?

??

Recall:

$$x(t) = a_0 \; + \; a_1 \cos\left(2\pi \frac{1}{T} t\right) + a_2 \cos\left(2\pi \frac{2}{T} t\right) + \cdots$$
$$+ \; b_1 \sin\left(2\pi \frac{1}{T} t\right) + b_2 \sin\left(2\pi \frac{2}{T} t\right) + \cdots$$

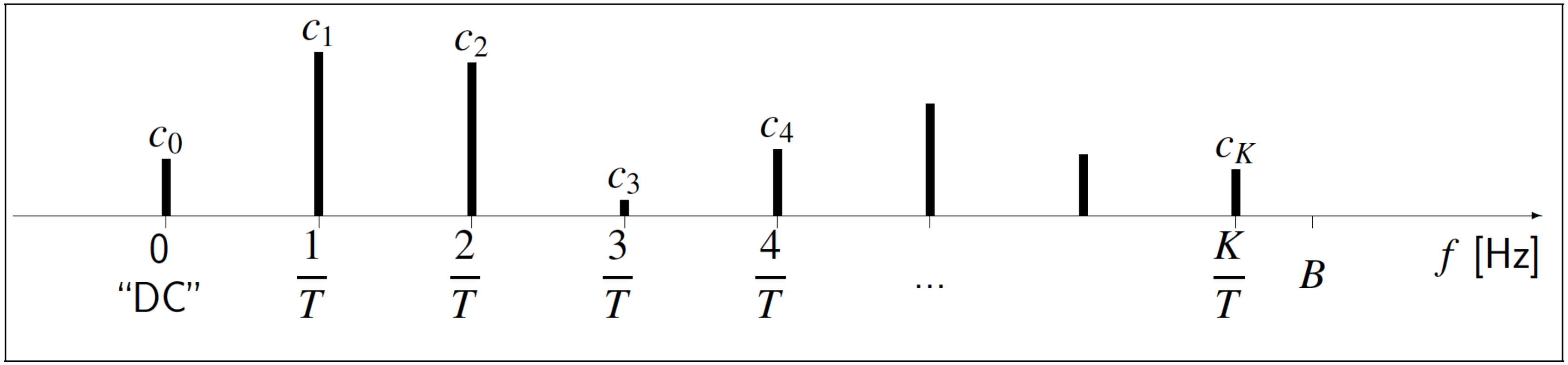
How many coefficients?

In general, if a signal has period T and is band-limited to B Hz, *how many* Fourier series coefficients $\{a_0, a_1, b_1, a_2, b_2, \ldots\}$ are needed?

??

Spectrum of a band-limited periodic signal

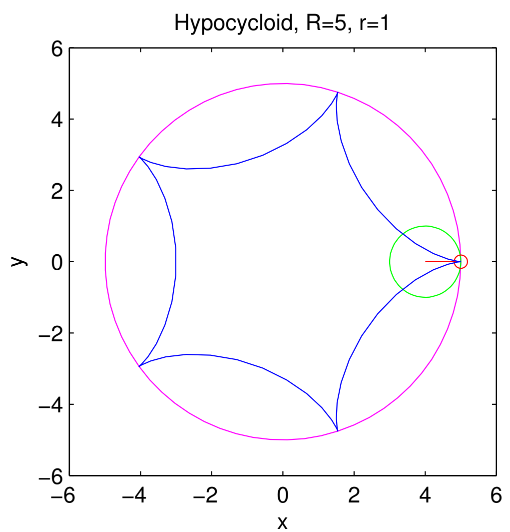
A periodic signal with period T, that is band-limited to B Hz, has a spectrum that looks like:

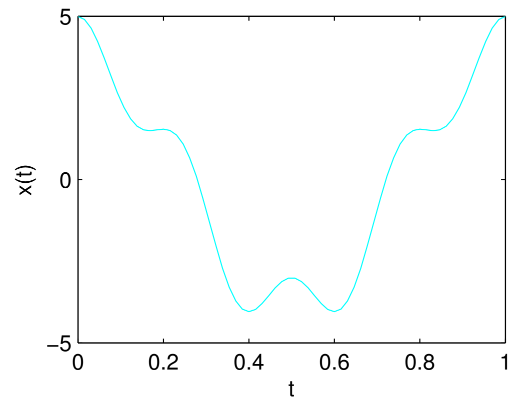


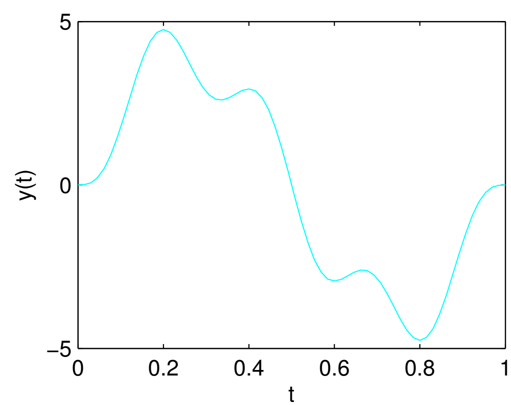
- No frequency components above B Hz
- No lines in spectrum past B Hz
- $K = BT$ if it is an integer (units of BT?)
- $K = \lfloor BT \rfloor$ more generally, $\lfloor x \rfloor$ = largest integer that is $\leq x$
  $\lfloor x \rfloor$ called `floor` function (below)
- Example: $T = 0.01$ s and $B = 360$ Hz $\Longrightarrow K = \lfloor 3.6 \rfloor = 3$

Spectrum review: A non-music example

The following figure / demo illustrates a hypocycloid
that is one special case of a spirograph.







demo: `fig_spirograph1.m`

Formula:
$x(t) = (R-r)\cos(2\pi t) + r\cos\left(2\pi \frac{R-r}{r} t\right)$
$y(t) = (R-r)\sin(2\pi t) - r\sin\left(2\pi \frac{R-r}{r} t\right)$
Here, $R = 5$ (outer circle)
and $r = 1$ (inner circle).

Recall: a signal is any time-varying quantity...

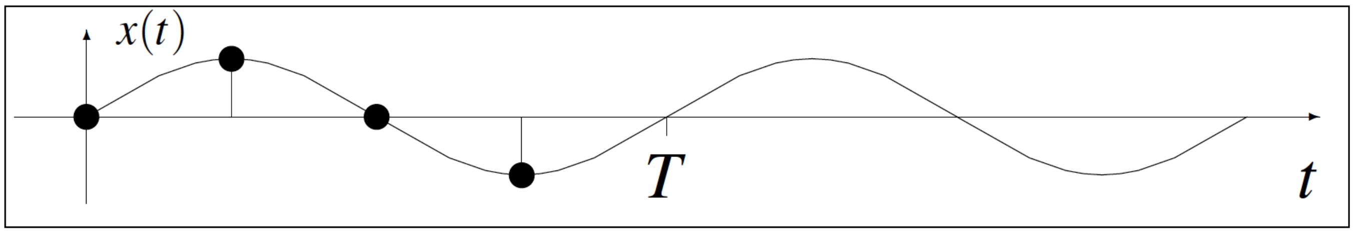
Exercise.
Sketch spectrum of $x(t)$ (or $y(t)$). ??
Is $x(t)$ band-limited? ??
What is the band-limit B? ??

Computing a signal's spectrum

Signal sampling

Idea. To determine $2BT+1$ Fourier coefficients of a signal with period T that is band-limited to B Hz, we try taking at least $N \geq 2BT+1$ *samples* of the signal over one period, *e.g.*, $[0,T]$.

In other words, take $N > 2BT$ samples: $x[0], x[1], \ldots, x[N-1]$.



What will be the sampling interval? $\Delta = \frac{T}{N} < \frac{T}{2BT} = \frac{1}{2B}$.

The sampling rate is: $S = \frac{\#\text{ of samples}}{\text{time interval}} = \frac{N}{T} > \frac{2BT}{T} = 2B$.

Sample *faster* than twice the maximum frequency: $\boxed{S > 2B.}$

2B or not 2B

The formula $\boxed{S > 2B}$ is one of the most important in DSP.
It is the foundation for all digital audio and video and more.
CD players use a sampling rate of 44.1 kHz. Why? ??

Where did T go?
The period T need not affect the sampling rate!
We can choose T arbitrarily large.

Amazing fact #2 (discovered by Claude Shannon 60+ years ago):

Sampling theorem:

If we sample a band-limited signal $x(t)$ at a rate $S > 2B$, then
we can recover the signal from its samples $x[n] = x(n/S)$. (EECS 216)

Conversely:
sampling too slowly can cause bad effects called *aliasing*.
Example: wagon wheels in Western movies.

Claude Shannon: Father of information theory

http://www.computerhistory.org/collections/accession/102665758 circa 1980

1916-2001
Born in Petosky, raised in Gaylord [wiki]
UM EE Class of 1936.
Bust outside EECS.

cf. finite element models used by, *e.g.*, mechanical and aero engineers