# Eng. 100: Music Signal Processing

# DSP Lecture 2: Lab 2 overview

Curiosity: `http://www.youtube.com/watch?v=qybUFnY7Y8w`

Announcements:

- Lab 1: Finish (revisions) by start of this week's lab. Questions?
- Read Lab 2 before lab!
  Complete reading questions 24 hours before lab (last reminder)
- We are working towards separating lab sections in Canvas.
- Lab instructor office hours:
  - Sydney Williams: Tue. 12-1PM, EECS atrium CAEN lab
  - Izzy Salley: Tue. 1-2PM, Shapiro undergrad library 1st floor
- Faculty office hours:
  - J. Fessler, Thu 9:30-10:30AM, 4431 EECS
  - P. Kominsky, Thu 12-1PM, 311 GFL

(lots of `Matlab` help!)

# Outline

- Previous class summary (TC, A/D, array, matlab)
- Part 0. Lab 1 questions?
- Part 1: Terminology

## Lab 2: Computing and visualizing the frequencies of musical tones

- Part 2. Sampling signals, especially sinusoids
- Part 3: Computing the frequency of a sampled sinusoid
  (with a computer, rather than by hand and eye like in previous class)
- Part 4: Visualizing, modeling and interpreting data
  using semi-log and log-log plots
- Part 5: Basic dimension analysis (units)

# Part 1: Terminology

# Terminology

Help me remember to define each new term!

(First overview class may have been rushed but not now...)

Course title: Music Signal Processing
What is a *signal*?

Wikipedia (electronics) 2012:
*a signal is any time-varying or spatial-varying quantity*

Common use? ??

# Terminology

Help me remember to define each new term!

(First overview class may have been rushed but not now...)

Course title: Music Signal Processing
What is a *signal*?

Wikipedia (electronics) 2012:
*a signal is any time-varying or spatial-varying quantity*

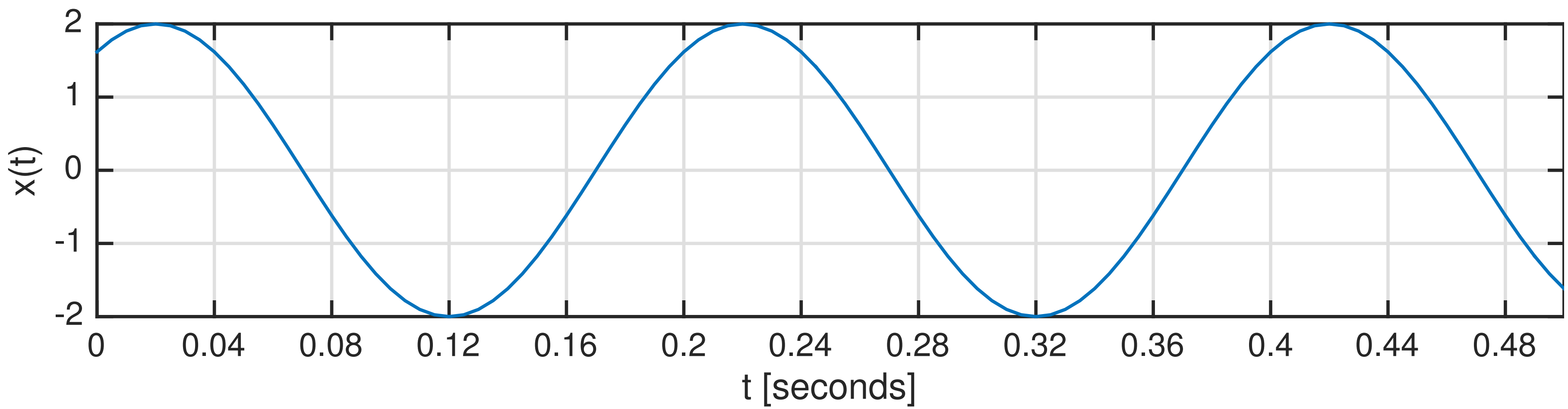
Common use? ??

smoke signal, traffic signal, railroad signal, hand signal,
maritime flag signals, telegraph signal, ...

Wikipedia (electrical engineering) 2014 [wiki]:
*a function that conveys information about the behavior or attributes of some phenomenon*

# Part 2:
# Sampling analog signals, especially sinusoidal signals

# Sinusoidal “pure” tones




From Lab 1:

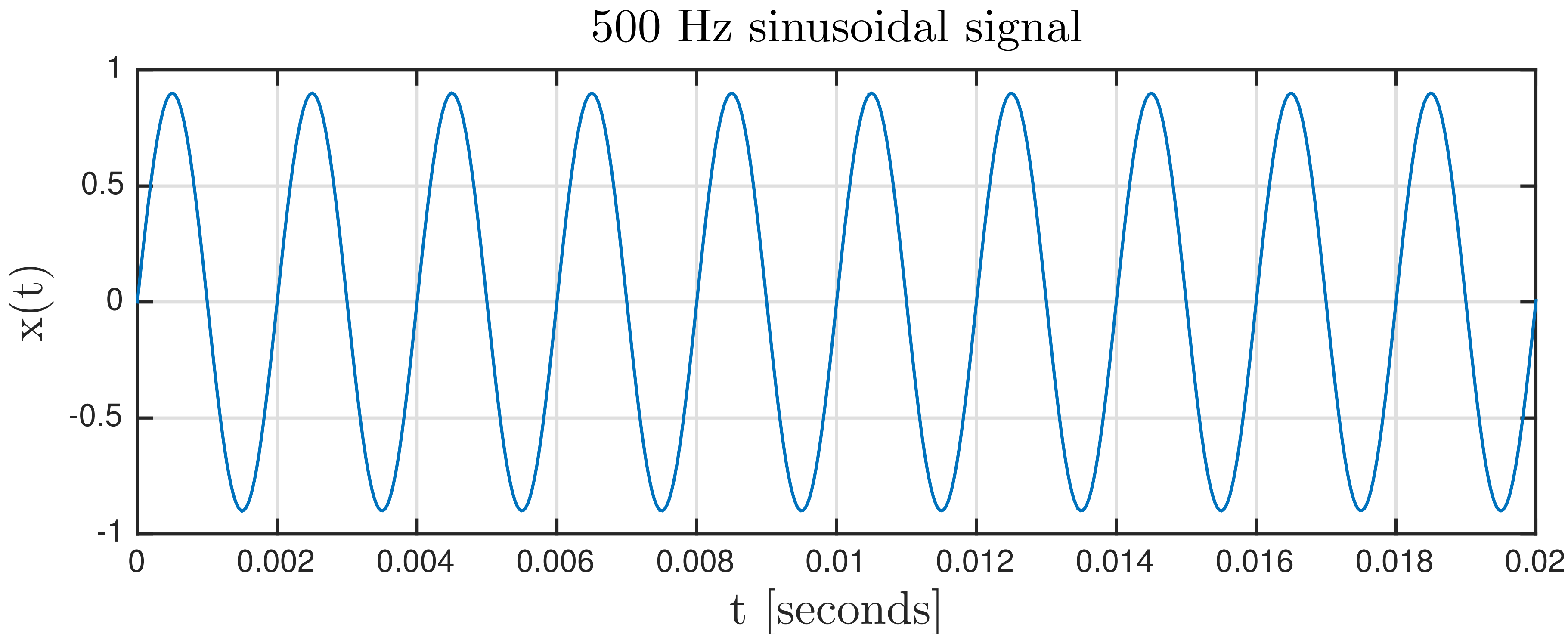
$$x(t) = 2\cos(2\pi 5\,(t - 0.02)) = 2\cos(2\pi 5 t - \pi/5)$$

- Amplitude: $A = 2$
- Frequency: $f = 5$ Hz (cycles per second)
- Period: $T = 1/f = 0.2$ sec
- Phase: $\theta = -\pi/5$ radians

“musical?” (*cf.* instruments, *cf.* hearing range)

# Sinusoidal signal at 500 Hz

$$x(t) = 0.9\cos(2\pi 500 t - \pi/2)$$




Period = ?? play

# Sampling an analog signal

Analog signal (continuous-time signal): $x(t)$,
where $t$ can be any real number. Units of "$t$" are seconds.

$$\underbrace{x(t)}_{\text{analog}} \rightarrow \boxed{\text{A/D converter}} \rightarrow \underbrace{x[n]}_{\text{digital}}$$

Crucial quantities for an A/D converter (*i.e.*, for sampling):

- *Sampling rate*: $S$. Units: $\frac{\text{Sample}}{\text{Second}}$ or Hz
- *Sampling interval*: $\boxed{\Delta = 1/S.}$ Units: seconds

On paper, or in software, sampling means: substitute $t = n/S$.

Digital signal (discrete-time signal):

$$\boxed{x[n] = x(n/S) = x(n\Delta)}$$

where $n$ can be any *integer*. Units of "$n$" ?? Units of "$n/S$" ??

Analog signal $x(t)$ and digital signal $x[n]$ are related but quite different!

# Example: Sampled sinusoidal signal

Analog signal:

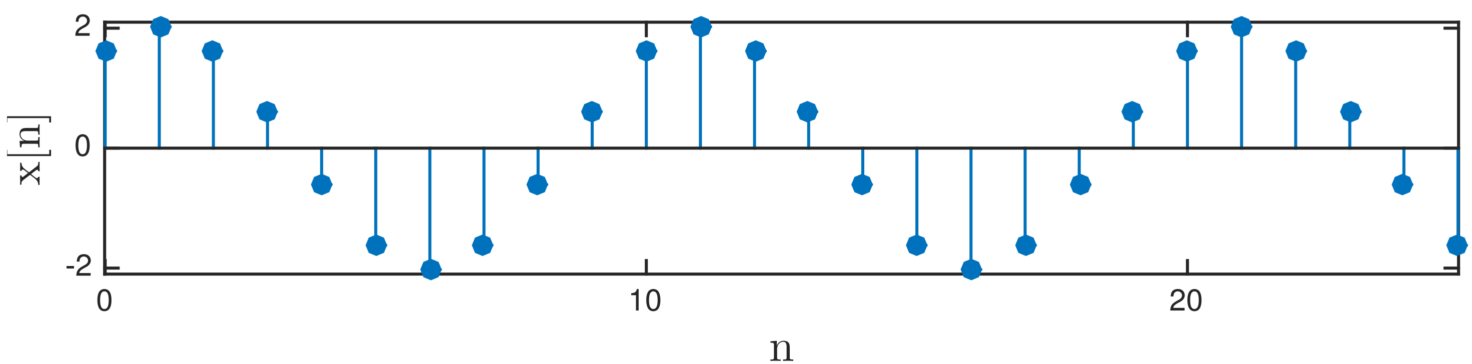
$$x(t) = 2\cos(2\pi 5t - \pi/5)$$

Choose sampling rate: $S = 50$ Hz. What sampling interval $\Delta$? ??

Substitute $t = n/S = 0.02n$ to define:

Digital signal:

$$x[n] = x(0.02n) = 2\cos(0.2\pi n - \pi/5)$$

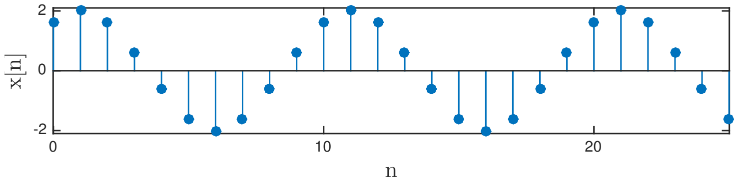



Terminology: Common use of term *sample* or *sampling*? ??

# Example: Sampled sinusoidal signal (continued)

Digital signal as a formula: $x[n] = 2\cos(0.2\pi n - \pi/5)$

Digital signal as a plot:




But in a computer (or DSP chip) it is just a list of numbers:

| $n$ | $x[n]$ (value) | $x[n]$ (formula) |
|---|---|---|
| 0 | 1.62 | $2\cos(0.2\pi 0 - \pi/5)$ |
| 1 | 2.00 | $2\cos(0.2\pi 1 - \pi/5)$ |
| 2 | 1.62 | $2\cos(0.2\pi 2 - \pi/5)$ |
| 3 | 0.62 | $\vdots$ |
| 4 | $-0.62$ | |
| 5 | $-1.62$ | |
| $\vdots$ | $\vdots$ | |

(Actually stored in binary (base 2) not in decimal.)

# Sampling a sinusoid in general

Given a pure sinusoidal (analog) signal

$$x(t) = A\cos(2\pi f t + \theta).$$

Units of the product $ft$? ?? (not feet)

If we sample it at $S \frac{\text{Sample}}{\text{Second}}$ (by substituting $t = n/S$),
we get a digital sinusoidal signal (formula):

$$x[n] = A\cos(2\pi f n/S + \theta) = A\cos\left(2\pi \frac{f}{S} n + \theta\right)$$

Units of $f/S$? ??

The quantity $\boxed{\omega = 2\pi f/S}$ is called the *digital frequency* in DSP.

A computer sound card does this (sampling) to a microphone input signal. Applications like Skype use digital audio signals.

# Sampling a sinusoid - summary

| Analog sinusoidal signal $x(t) = A\cos(2\pi f t + \theta)$ (frequency $f$) | $\rightarrow$ | Sampling A/D rate $S$ | $\rightarrow$ | Digital sinusoidal signal $x[n] = A\cos\left(2\pi\frac{f}{S}n + \theta\right)$ (list of numbers |
|---|---|---|---|---|

Basic music *transcription* requires that we *reverse* this process!

| Digital sinusoidal signal x[n] | $\rightarrow$ | DSP magic | $\rightarrow$ | frequency (pitch) $f$ |
|---|---|---|---|---|

# Part 3:
# Computing the frequency of a sampled sinusoidal signal

# Reconstructing a sinusoid from its samples

Given samples of a digital sinusoid in a computer:

$$x[n] = A\cos(\omega n + \theta), \quad n = 1, 2, \ldots, N.$$

(Stored as a list of $N$ numbers, not as a formula, for known rate $S$.)

How can we find the frequency $f$ of the original analog sinusoidal signal?

- Step 1. Determine the "digital frequency" $\omega$
- Step 2. Relate $\omega$ to the original (analog) frequency.

  This step is easy because $\omega = 2\pi \frac{f}{S}$ so rearranging: $\boxed{f = \frac{\omega}{2\pi} S.}$

Example. $x[n] = 3\cos(0.0632n + 5)$ with $S = 8192$ Hz.
Original frequency is $f = \frac{0.0632}{2\pi} 8192 = 82.4$ Hz (low E)

But what if we are given an array of signal values instead of a formula?

# Finding a digital frequency

# (The computer or DSP chip perspective)

Given:

- Signal values

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| $x[n]$ | 1.62 | 2.00 | 1.62 | 0.62 | $-0.62$ | $-1.62$ | ... |

- Sinusoidal assumption (model): $x[n] = A\cos(\omega n + \theta)$

Goal: Determine the digital frequency $\omega$.

This problem arises in many applications, including music DSP.
EE types have proposed many solutions.
Lab 2 uses an elegantly simple method based on trigonometry.

## Key trigonometric identities

Angle sum and difference formulas [wiki]

$$\begin{aligned}\cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \cos(a-b) &= \cos a \cos b + \sin a \sin b\end{aligned}$$

Product-to-sum identities:

$$\begin{aligned}2\cos a \cos b &= \cos(a-b) + \cos(a+b) \\ 2\sin a \sin b &= \cos(a-b) - \cos(a+b)\end{aligned}$$

Wikipedia says these identities date from 10th century Persia. [wiki]

We will use these (ancient) identities repeatedly!

# Practical example: Tuning a piano

Rewriting product-to-sum identity:

$$\cos(a+b)+\cos(a-b) = 2\cos a \cos b.$$

Substitute $a = 2\pi 442t$ and $b = 2\pi 2t$:

$$\cos(2\pi 444t)+\cos(2\pi 440t) = 2\cos(2\pi 442t)\cos(2\pi 2t).$$

The sum of two sinusoids having close frequencies is a sinusoid at the average of the frequencies with a (sinusoidally) time-varying amplitude!

| | | | | | |
|---|---|---|---|---|---|
| 440: | play | 444: | play | 440&444: | play |
| | | 441: | play | 440&441: | play |

The combined (sum) signal gets louder and softer, "beating" with period = 0.25 sec. Why do we consider the sum? ??
The slower the period, the closer the 2 frequencies.

Why is this relevant to tuning a piano (or guitar or ...)? ??

# Back to finding a digital frequency

Repeating product-to-sum identity:

$$\cos(a+b)+\cos(a-b) = 2\cos a \cos b.$$

As another practical application of this identity, some clever DSP expert suggested substituting $a = \omega n + \theta$ and $b = \omega$:

$$\cos(\omega(n+1)+\theta)+\cos(\omega(n-1)+\theta) = 2\cos(\omega)\cos(\omega n+\theta)$$

Now use our sinusoidal assumption: $x[n] = A\cos(\omega n+\theta)$, yielding:

$$x[n+1]+x[n-1] = 2\cos(\omega)\,x[n].$$

Rearranging yields $\cos(\omega) = \dfrac{x[n+1]+x[n-1]}{2x[n]}$, or equivalently:

$$\boxed{\omega = \arccos\left(\frac{x[n+1]+x[n-1]}{2x[n]}\right).}$$

## Example of finding a sinusoid's frequency

Recall from "Step 2" that $\boxed{f = \frac{\omega}{2\pi} S}$

Combining Step 1 and Step 2:

$$\boxed{f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1] + x[n-1]}{2x[n]}\right).}$$

Use this formula in Lab 2 to compute the frequency from a digital signal corresponding to samples of a sinusoid.

How many signal samples do we need to find $\omega$? ??

Example: $S = 1500$ Hz and $x[n] = (\ldots, ?, ?, ?, 3, 7, 4, ?, ?, ?, \ldots)$
The signal values denoted "?" are, say, lost or garbled.
Solution: $f = \frac{1500}{2\pi} \arccos\left(\frac{3+4}{2 \cdot 7}\right) = \frac{1500}{2\pi} \arccos\left(\frac{1}{2}\right) = \frac{1500}{2\pi} \frac{\pi}{3} = 250$ Hz.

# Exercise

Suppose $S = 8000$ Hz and the signal samples are
$x[n] = (\ldots, 0, 5, 0, -5, 0, 5, 0, -5, 0, \ldots)$

What is the frequency $f$ of the sinusoid? ??

Here is the arccos formula repeated for convenience:

$$\boxed{f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1] + x[n-1]}{2x[n]}\right)}.$$

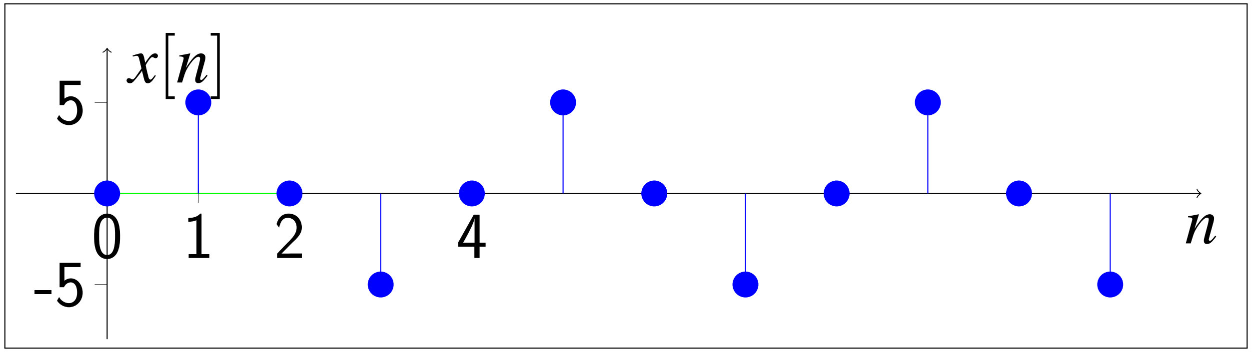
# Exercise

Suppose $S = 8000$ Hz and the signal samples are
$x[n] = (\ldots, 0, 5, 0, -5, 0, 5, 0, -5, 0, \ldots)$

What is the frequency $f$ of the sinusoid? ??

Here is the arccos formula repeated for convenience:

$$f = \frac{S}{2\pi} \arccos\left(\frac{x[n+1] + x[n-1]}{2x[n]}\right).$$




Historical note: this approach is a simplification of Prony's method from (!) 1795.

# (Dis)Advantages of this method

Many frequency estimation methods have been proposed.
Apparently not everyone just uses this one; why not?

Advantages of this method:

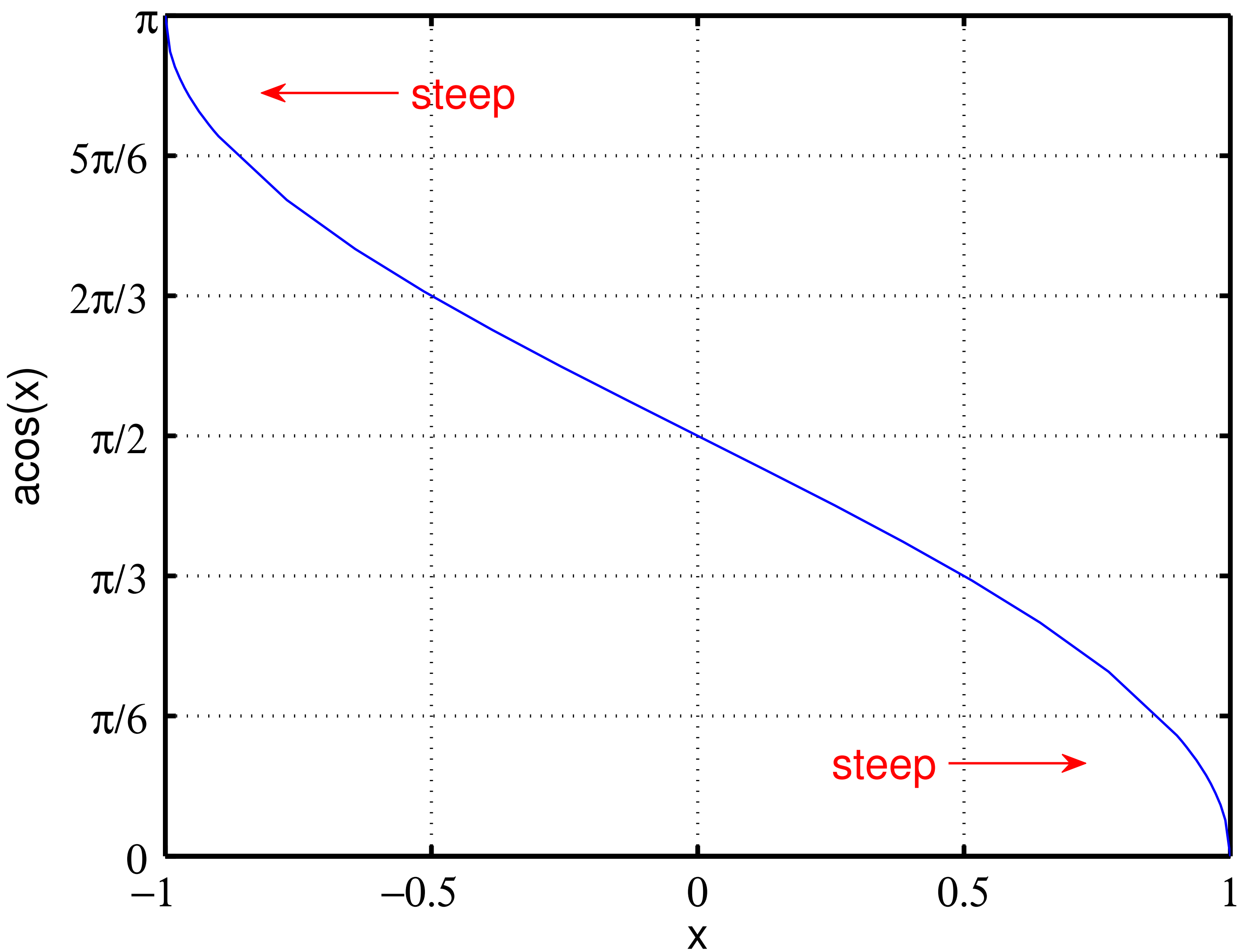
- Very simple to implement, can use simple DSP chip.
- Fast tracking of sudden frequency changes.
- Can use outliers (weird values) to segment long signals.
- Requires knowledge of trigonometry only.

Disadvantages of this method

- Very sensitive to additive noise in the data $x[n]$.
- What if $x[n] = 0$ for some $n$? Divide by 0!
- Arc-cosine function is sensitive to small changes.

A useful starting point for music DSP...

# Arc-cosine



# Part 4:
# Visualizing, modeling, and interpreting data using semi-log and log-log plots

# Data visualization and modeling

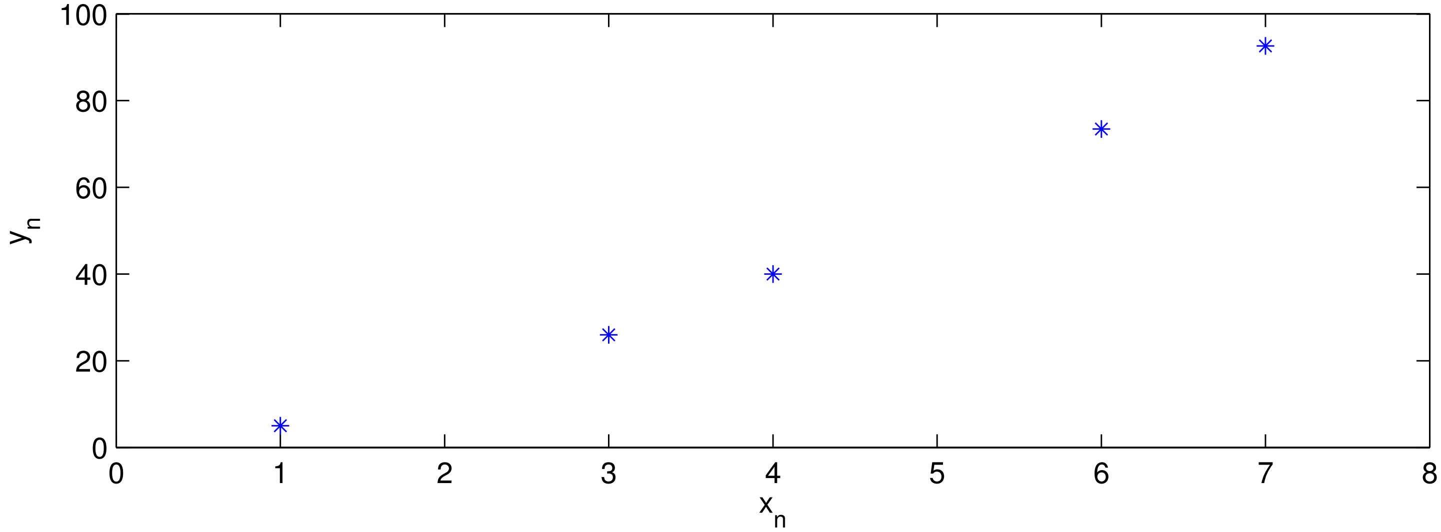
Given: $N$ pairs of data values: $(x_1, y_1), (x_2, y_2), \ldots (x_N, y_N)$

Goal: Find a relationship between the values (a model):
- ○ $y = g(x)$
- ○ $y_n = g(x_n)$, $n = 1, \ldots, N$

- Example (Physics 140):
  x = height above ground a ball is released
  y = velocity on impact with ground.
- Example (Physics 240):
  x = electrical current through a light bulb
  y = energy released in the form of heat by the bulb
- Example (Engin 100-300):
  x = piano key "number" (1 to 88)
  y = frequency of note played by a key

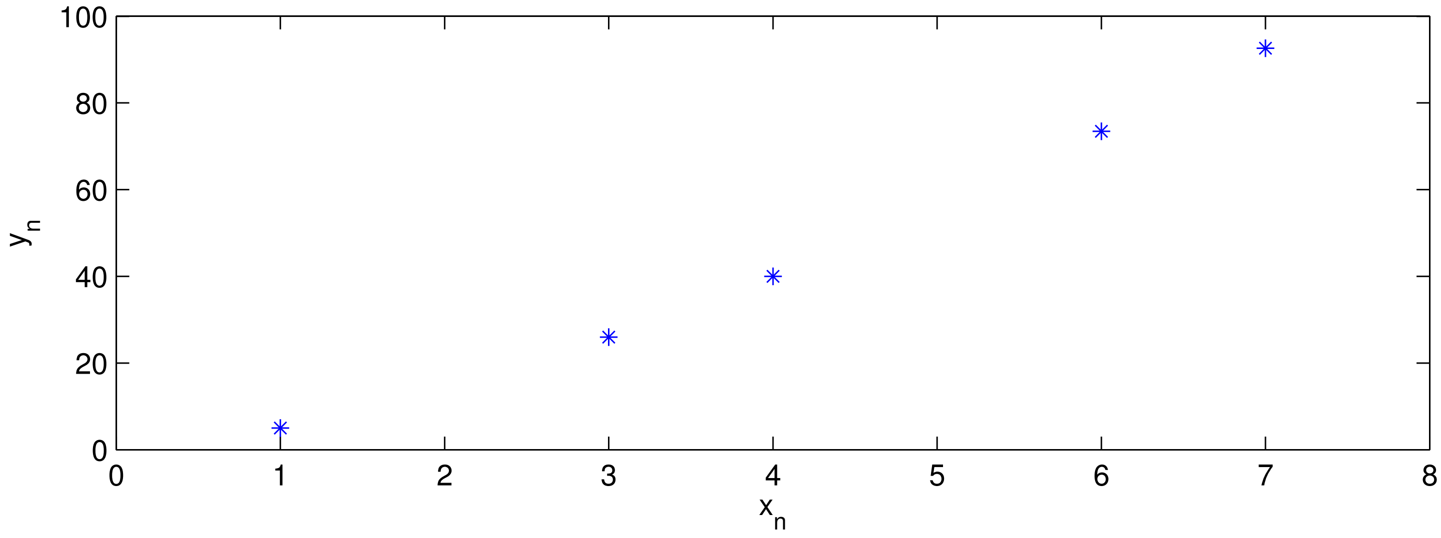
# Visualizing data using scatter plots

Example (*scatter plot* of pairs of data values):




Do your eyes try to "connect the dots?"
Your brain is trying to build a model!

# Making a scatter plot in Matlab




```
x = [1 3 4 6 7];
y = [5.00 25.98 40.00 73.48 92.60];
plot(x, y, '*')
xlabel x_n
ylabel y_n
```

# Common mathematical models

Linear model: $\boxed{y = ax}$
one parameter: *slope* $a$

Affine model: $\boxed{y = ax + b}$
two parameters: *slope* $a$ and *intercept* $b$

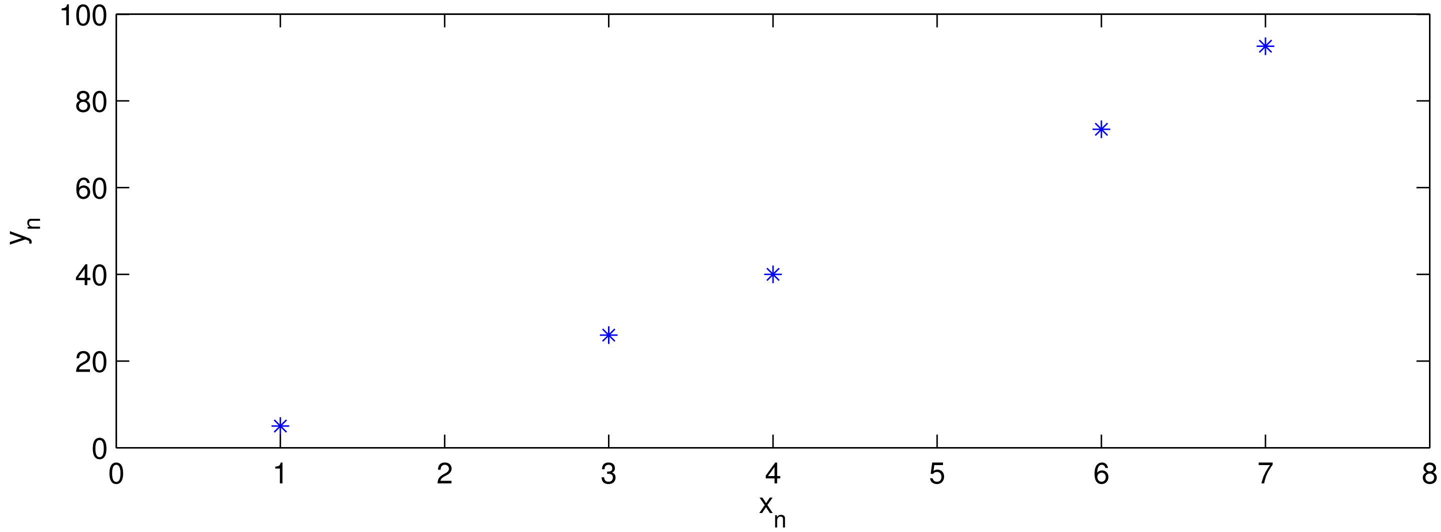
Quadratic model: $\boxed{y = ax^2 + bx + c}$

Simple "power" model: $\boxed{y = bx^p}$
*power* parameter $p$, *scale factor* $b$

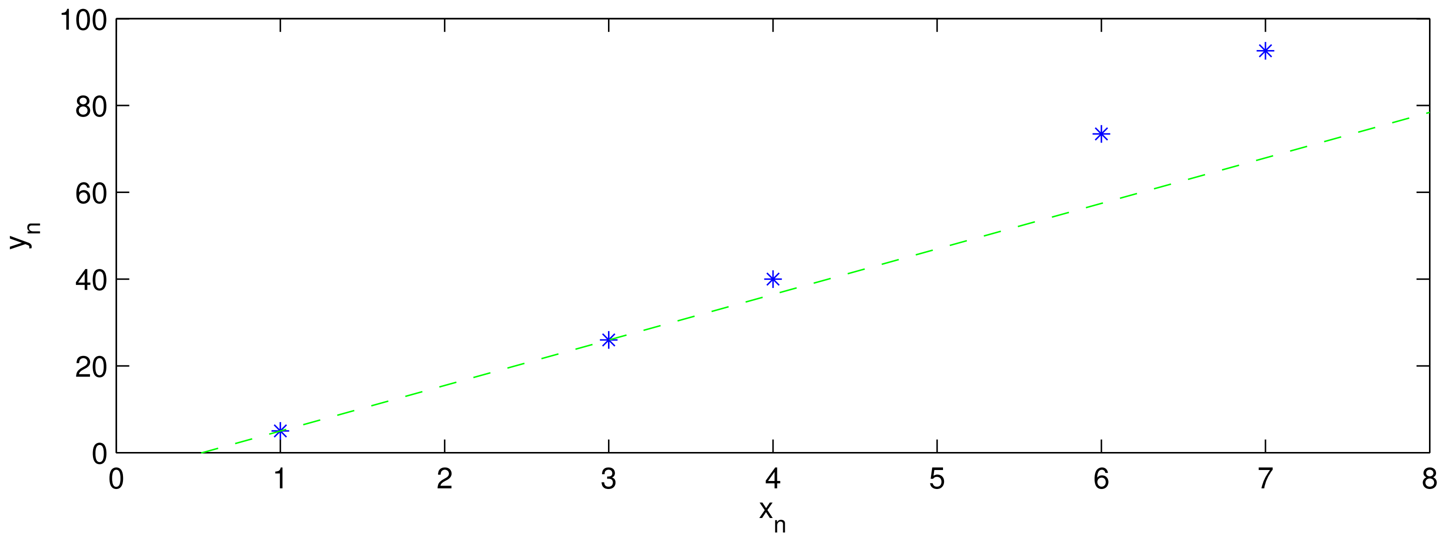
Simple "exponential" model: $\boxed{y = ba^x}$
Note that the *independent variable* $x$ is in the *exponent* here.

Which model is appropriate for height/velocity example? ??
How do we choose among these models *given data*?

# Example: Linear or affine model?



# Example: Linear or affine model?




Any two points determine the equation for a line.
Does a line *fit* this data? ??

So we rule out both linear and affine models by *visualization*.

# Review of logarithms

- In `Matlab` and in this class, `log` means *natural log*, (base $e$), which might be `ln` on your calculator.
- For base-10 logarithm use `log10` in `Matlab`, and write $\log_{10}$ on paper.
- Properties of logarithms (for any base $b > 0$):
  - $\boxed{b^{\log_b(x)} = x}$ if $x > 0$ (the defining property)
  - $\boxed{\log_b(xy) = \log_b(x) + \log_b(y)}$ if $x > 0$ and $y > 0$ log of product
  - $\boxed{\log_b(x^p) = p \log_b(x)}$ if $x > 0$ log of power
- Other related properties:
  - $\boxed{e^{\log(x)} = x}$ and $\boxed{10^{\log_{10}(x)} = x}$ if $x > 0$
  - $\boxed{\log(e) = 1}$ and $\boxed{\log_{10}(10) = 1}$
  - $\boxed{e^{a+b} = e^a e^b}$

Exercise: Simplify $e^{c \log(z)}$

# Simple "exponential" model

$$\boxed{y = ba^x}$$

Take the logarithm (any base) of both sides:

$$\log(y) = \log(ba^x) = \log(b) + \log(a^x) = \log(b) + x\log(a)$$

$$\boxed{\log(y) = \log(a)\,x + \log(b)}$$

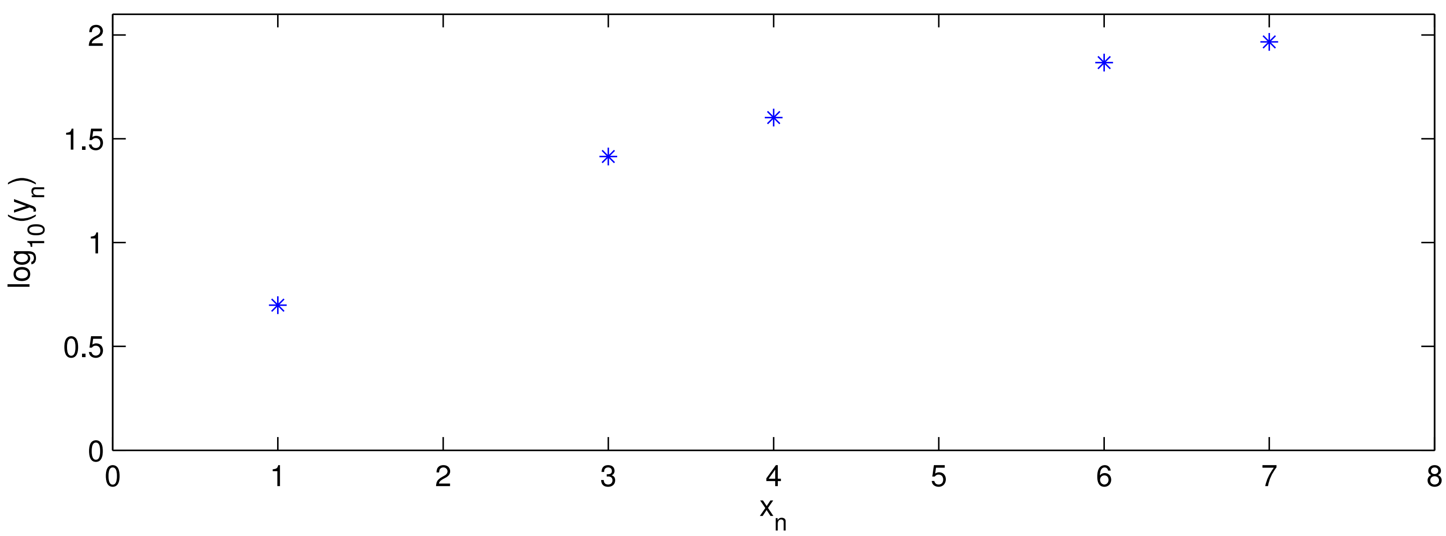
This is the equation of a line on a *log* scale:

$$\underbrace{\log(y)}_{\tilde{y}} = \underbrace{\log(a)}_{\text{slope}}\,x + \underbrace{\log(b)}_{\text{intercept}}$$

To see if the exponential model fits some data, make a scatter plot of $\boxed{\log(y_n) \text{ versus } x_n}$ and see if it looks like a straight line.

This is called a *semi-log plot*.

# Making a semi-log plot in `Matlab`




```
x = [1 3 4 6 7]; % anything after % is a "comment"
y = [5.00 25.98 40.00 73.48 92.60];
plot(x, log10(y), '*')
xlabel 'x_n' % underscore for single subscript
ylabel 'log_{10}(y_n)' % note braces
```

Is the exponential model a good fit for this data?

# Simple "power" model

$$\boxed{y = bx^p}$$

Take the logarithm (any base) of both sides:

$$\log(y) = \log(bx^p) = \log(b) + \log(x^p) = \log(b) + p\log(x)$$

$$\boxed{\log(y) = p\log(x) + \log(b)}$$

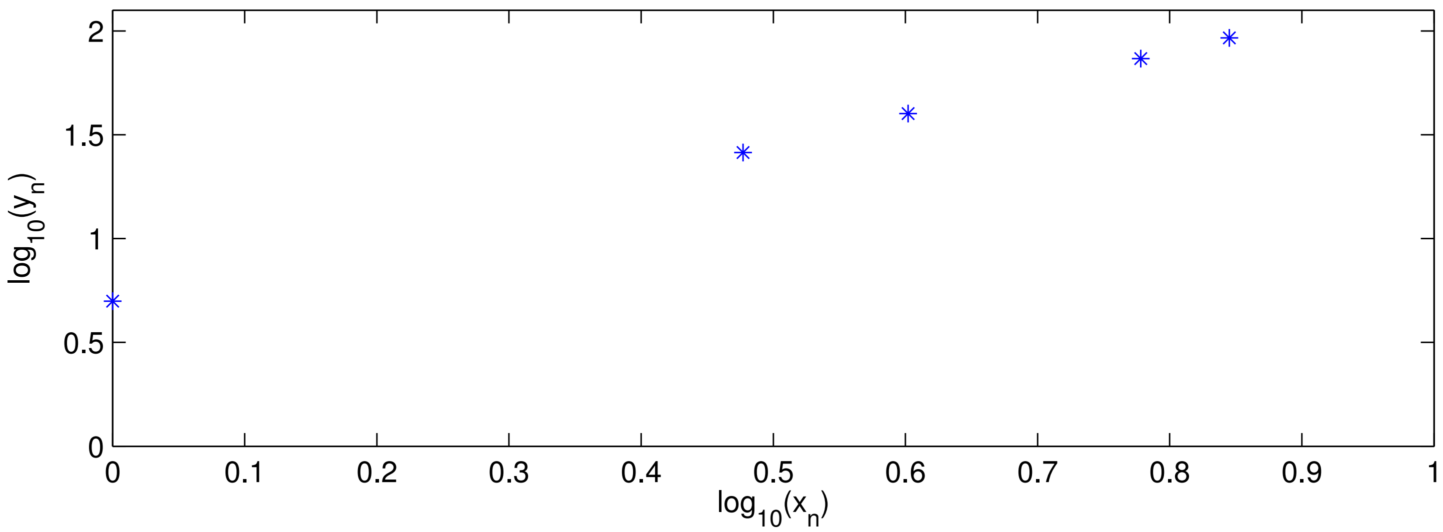
This is the equation of a line on a *log-log* scale:

$$\underbrace{\log(y)}_{\tilde{y}} = \underbrace{p}_{\text{slope}} \underbrace{\log(x)}_{\tilde{x}} + \underbrace{\log(b)}_{\text{intercept}}$$

To see if the power model fits some data, make a scatter plot of $\boxed{\log(y_n) \text{ versus } \log(x_n)}$ and see if it looks like a straight line.

What do you suppose this type of plot is called? ??

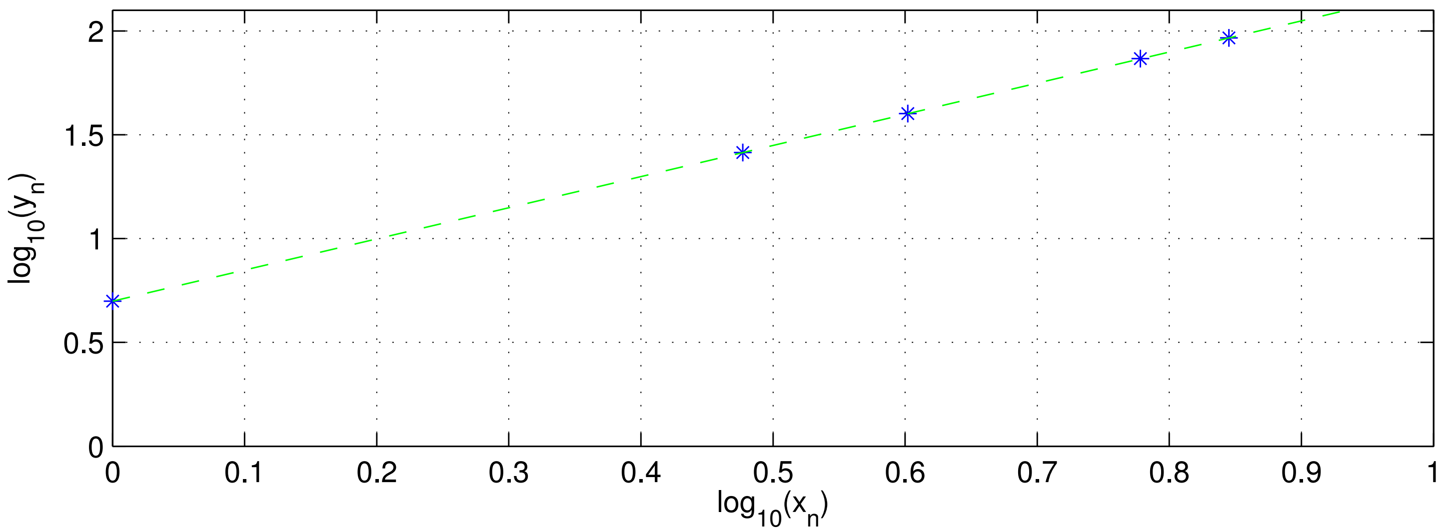
# Making a log-log plot in `Matlab`




```
x = [1 3 4 6 7];
y = [5.00 25.98 40.00 73.48 92.60];
plot(log10(x), log10(y), '*')
xlabel 'log_{10}(x_n)'
ylabel 'log_{10}(y_n)'
```

Is the power model a good fit for this data?

# Checking a log-log scatter plot in `Matlab`




`lsline` : adds "least squares best-fit line" to scatter plot (in Statistics Toolbox)

`grid` : adds grid lines

Yes! log-log plot lies along a line $\Longrightarrow$ power model is a good fit.
Now we just need the parameters to write our equation model.

- intercept $\approx 0.7 = \log_{10}(b) \Longrightarrow b = 10^{0.7} \approx 5.0$
- slope $= \frac{\Delta\tilde{y}}{\Delta\tilde{x}} \approx \frac{1.6-0.7}{0.6-0.0} = 1.5 \Longrightarrow p = 1.5$

Model: $y = bx^p = 5x^{1.5}$

# Checking a model

Recall given data:

```
x = [1 3 4 6 7];
y = [5.00 25.98 40.00 73.48 92.60];
```

Model found on previous slide: $y = 5x^{1.5}$

To check model in `Matlab`:

```
5 * (x .^ 1.5)
```

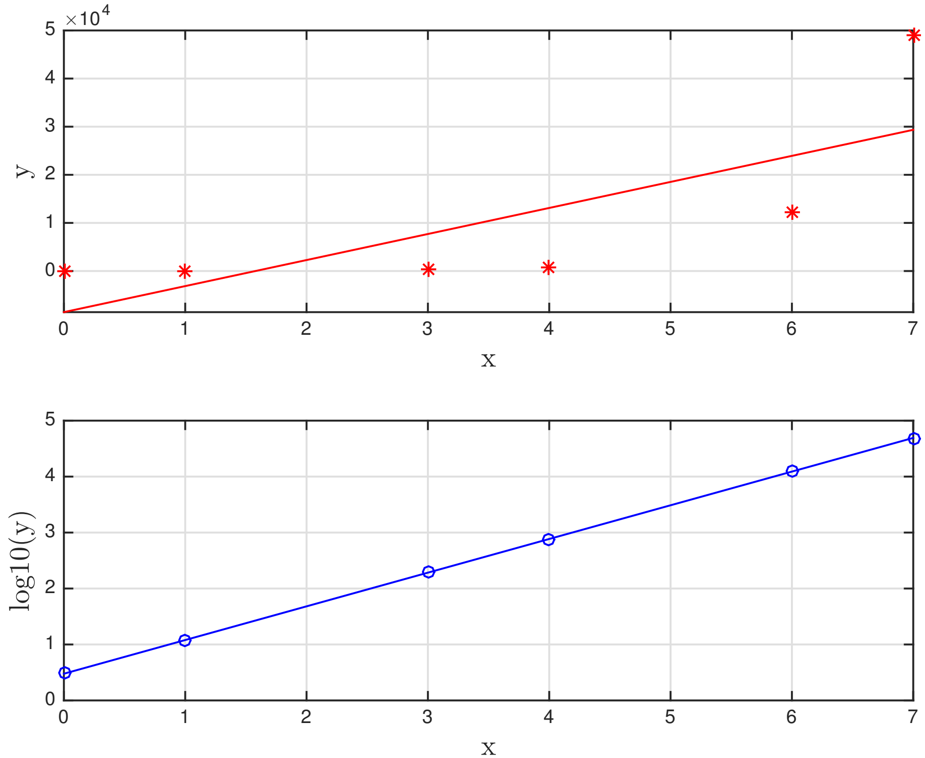
Output is:

```
5.0000 25.9808 40.0000 73.4847 92.6013
```

So our power model fits the given data very well.

# A semi-log example

```
x = [0 1 3 4 6 7];
y = [3 12 192 768 12288 49152];
subplot(211), plot(x, y, 'r*'), lsline, grid
subplot(212), plot(x, log10(y), 'bo'), lsline, grid
```




Exercise: determine a model for this data. ??

## Summary of two important models

- Simple "exponential" model: $\boxed{y = ba^x}$

  Use semi-log plot:

$$\underbrace{\log(y)}_{\tilde{y}} = \underbrace{\log(a)}_{\text{slope}} x + \underbrace{\log(b)}_{\text{intercept}}$$

- Simple "power" model: $\boxed{y = bx^p}$

  Use log-log plot:

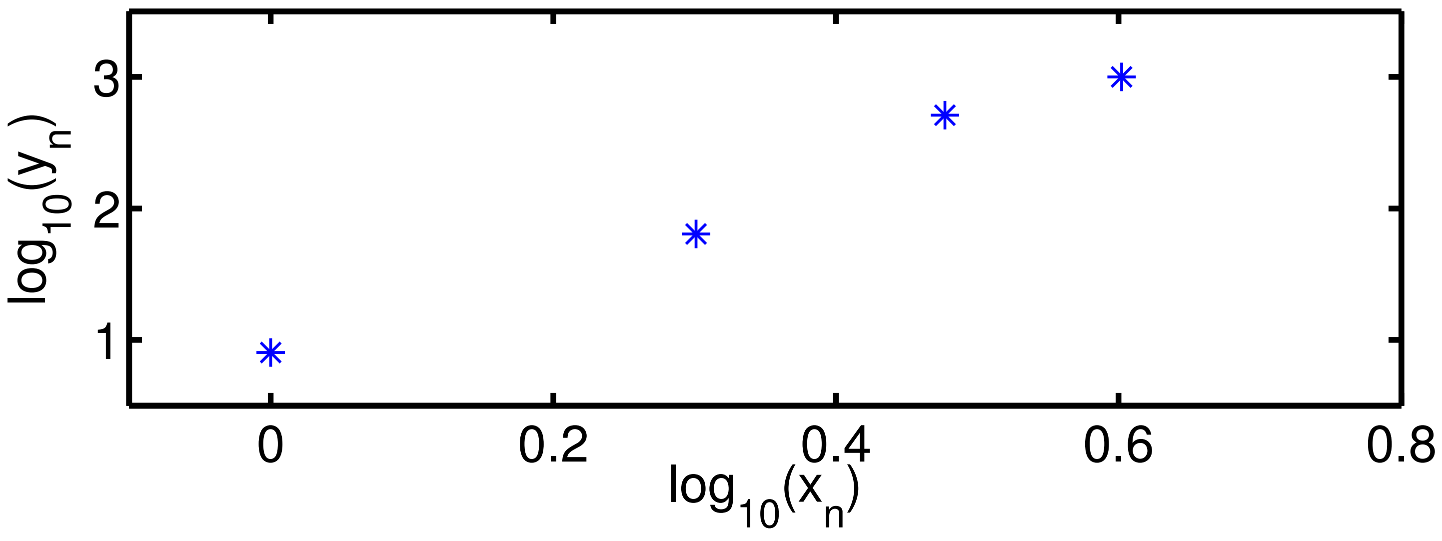
$$\underbrace{\log(y)}_{\tilde{y}} = \underbrace{p}_{\text{slope}} \underbrace{\log(x)}_{\tilde{x}} + \underbrace{\log(b)}_{\text{intercept}}$$

## Models with missing data (read yourself)

Musical context: song without all 88 notes

Given: `y = [8 64 512 1000]`
Given: `x` is *four values* out of the set [1 2 3 4 5]
*i.e.*: [1 2 3 4] or [1 2 3 5] or [1 2 4 5] or [1 3 4 5] or [2 3 4 5]

First try: `x = 1:4; plot(log10(x), log10(y), '*')`

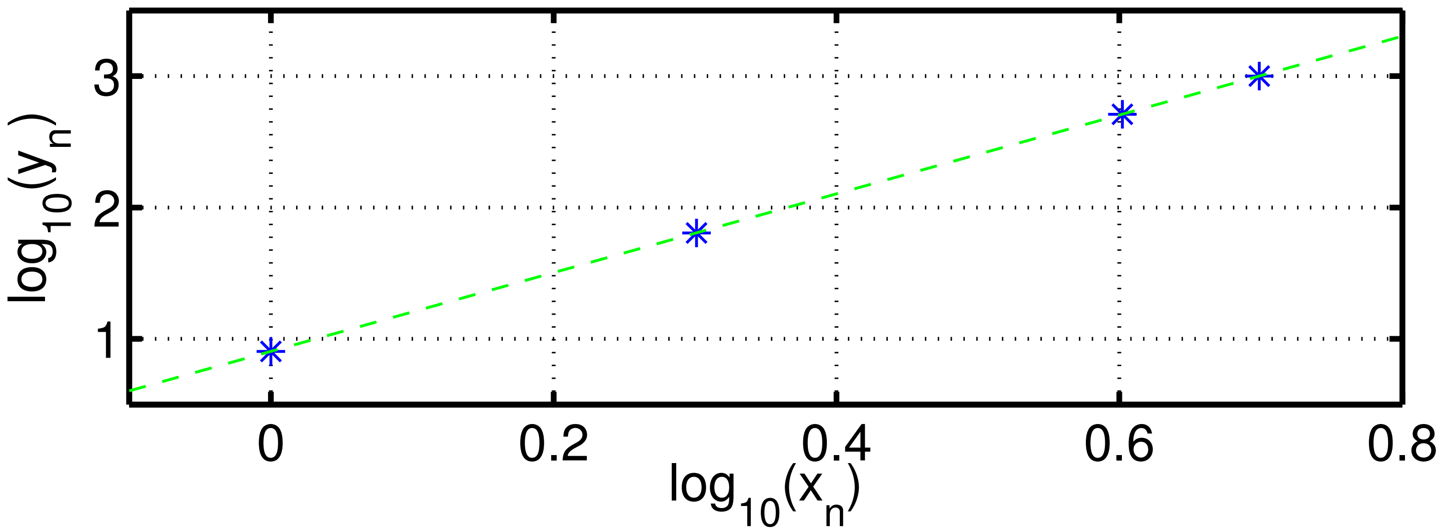



Looks like a "gap" or "jump" at 3rd value

# Example with missing data continued

Second try:

```
y = [8 64 512 1000];
x = [1 2 4 5];
plot(log10(x), log10(y), '*')
```

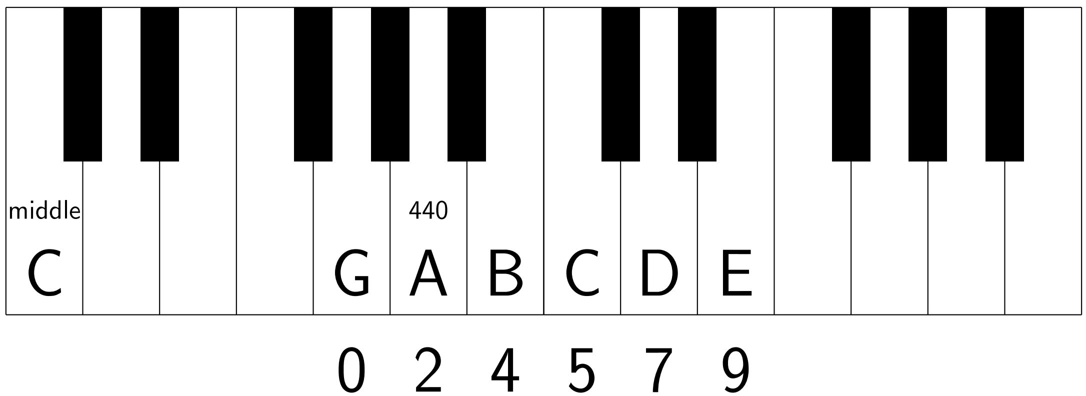



slope $= (3-0.9)/(0.7-0) = 3 = p$

intercept $= 0.9 = \log_{10}(b) \implies b = 10^{0.9} = 8$

Model: $y = 8x^3$

# Missing frequencies in “The Victors”

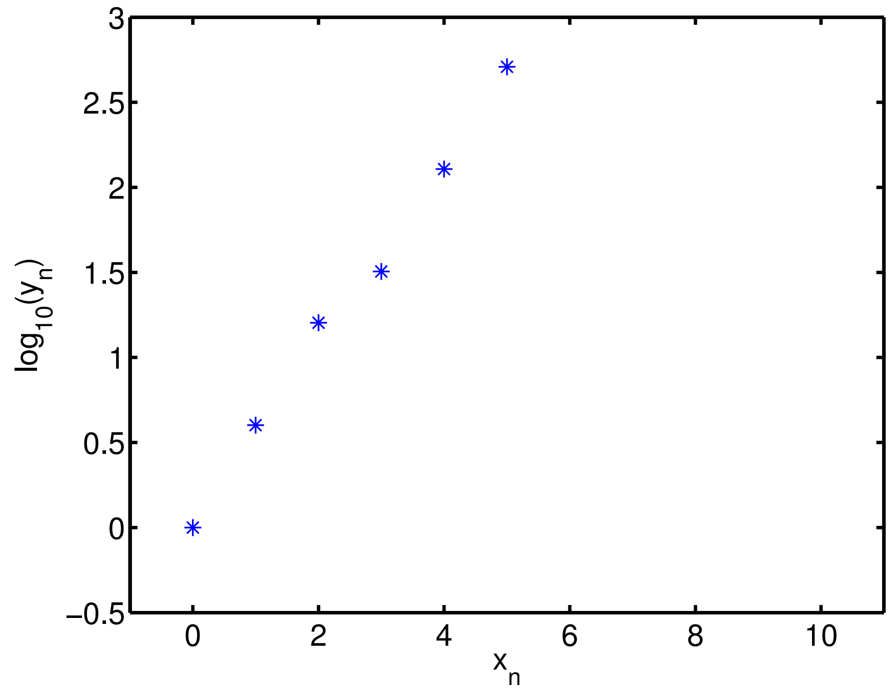



Missing: 1 3 6 8 10 11 12 13 ... -1 -2 -3 ...

# Lab 2 has lots of missing data!

"The Victors" only uses a few of the 88 keys on a piano.

Example: `y = [1 4 16 32 128 512]`
where each $x_n$ is one of the numbers in the set $\{0, 2, \ldots, 87\}$

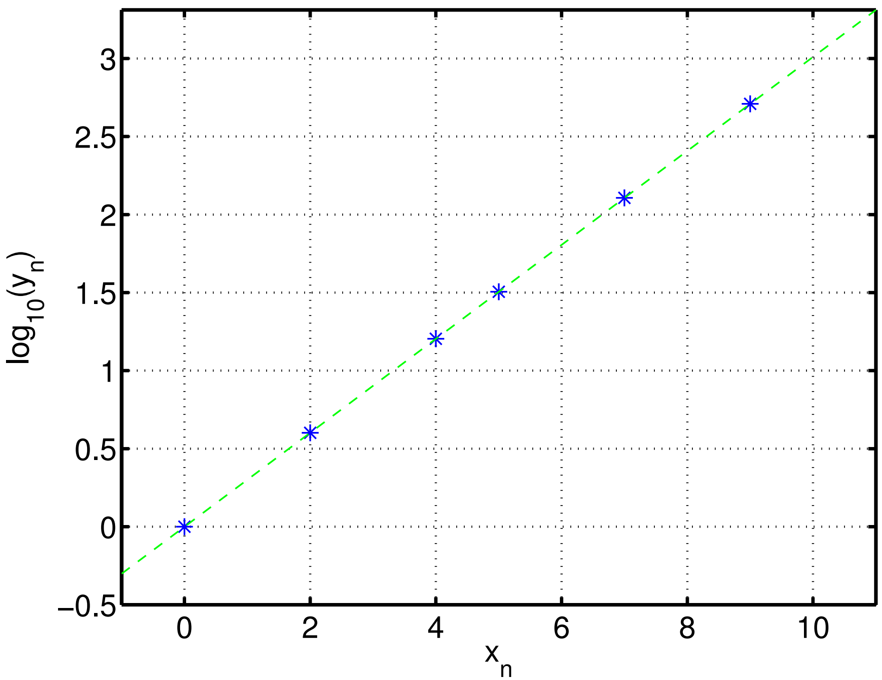



First try:

```
x = 0:5;
plot(x, log10(y), '*')
```

Lots of jumps!

# Lots of missing data continued




Second try:

```
y = [1 4 16 32 128 512]
x = [0 2 4 5 7 9];
plot(x, log10(y), '*')
```

slope $= (2.7-0)/(9-0) = 0.3 = \log_{10}(a) \implies a = 10^{0.3} = 2$
intercept $= 0 = \log_{10}(b) \implies b = 10^{0} = 1$

Model: $y = 2^x$

You will see a similar situation in Lab 2 (with different data).

## What you will do in Lab 2

- Download a sampled signal from Canvas site:
  A tonal version of the chorus of "The Victors."
- Load into `Matlab`; segment (chop up) into notes.
- Apply arccos formula to compute frequency of each note.
- Make log-log and semi-log plots of frequencies.
- Determine the formula relating frequencies of notes.
- Note: "Accidentals" are all missing; but you can infer their existence & frequencies from your plot!

# Part 5:
# Basic dimension analysis
# by example

**(read yourself if time runs out in class)**

# Dimensional Analysis Example 1

- Goal: Determine formula for the period of a swinging pendulum, without any physics!
- Find ingredients: mass, length, gravity
- Model: Period $=$ (mass)$^a$ (length)$^b$ $g^c$
  where $g =$ acceleration of gravity $9.80665\mathrm{m/s}^2$
  a,b,c are unknown constants to be found
- Approach: Find exponents using dimensional analysis:

$$\text{time} = (\text{mass})^a\ (\text{length})^b\ (\text{length}/\text{time}^2)^c$$

- ◦ No "mass" on LHS so $a = 0$
- ◦ No "length" on LHS so $0 = b + c$
- ◦ For time: $1 = -2c \implies c = -1/2$, so $b = 1/2$

Model: period $=$ length$^{1/2} g^{-1/2} =$ (length $/g)^{1/2}$
From physics: period $= 2\pi$ (length $/g)^{1/2}$

($2\pi$ is a unitless constant; cannot be found from dimension analysis.)

# Dimensional Analysis Example 2

Prof. Yagle asks:

- If 1.5 people can build 1.5 cars in 1.5 days,
  how many cars can 9 people build in 9 days?
- Do problems like this give you a headache?
- Would you like to solve problems like this with minimal thinking?

# Dimensional Analysis Example 2

Prof. Yagle asks:

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Given:
(1.5 cars) / (1.5 days) / (1.5 people) $=$ 2/3 cars / day / people

Now match units:
(2/3 cars / day / people) (9 days) (9 people) $=$ 54 cars

Simply matching the units suffices.

# Summary

- Sampling: a computer can determine frequency of a pure sinusoid from 3 consecutive samples.
- Semi-log plot of $y = ba^x$ is a straight line.
- Log-log plot of $y = bx^p$ is a straight line.
- Dimensional analysis often gives you correct answers.

Assignment: Read Lab 2 before lab!