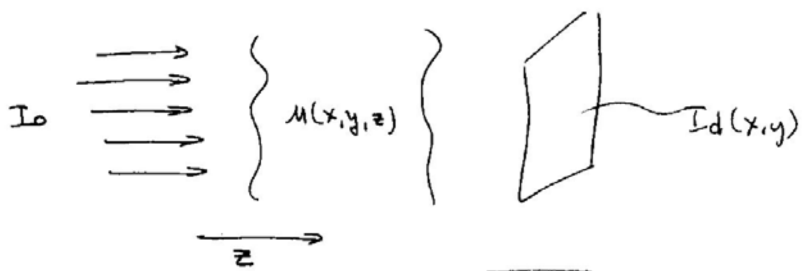
X-Ray Notes, Part II

Source Issues

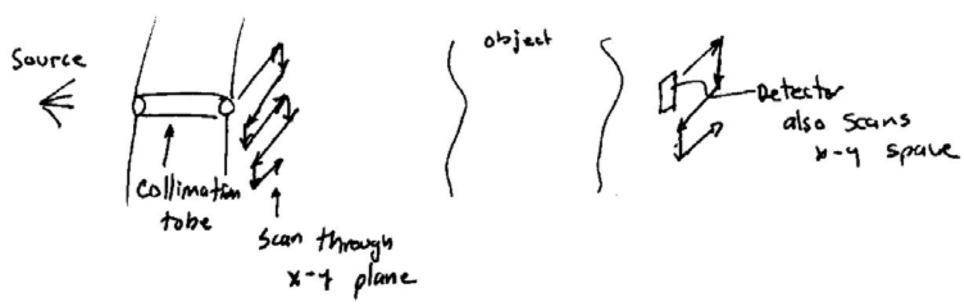
The Parallel X-ray Imaging System

Earlier, we considered a parallel ray system with an incident intensity I_0 that passes through a 3D object having a distribution of attenuation coefficients $\mu(x,y,z)$ and projects to an image $I_d(x,y)$:



$$I_d(x,y) = I_0 \exp\left(-\int \mu(x,y,z)dz\right)$$

There are essentially no practical medical project x-ray systems where the source has parallel rays. There are some scanning systems that might be appropriate for industrial inspection operations, for example:

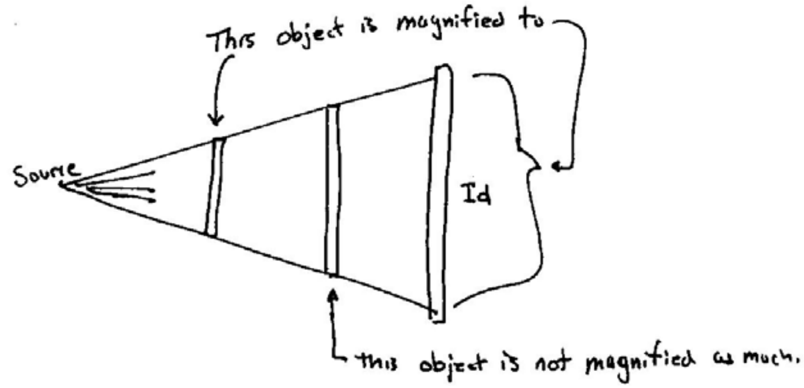


but these kinds of systems are too slow for medical applications.

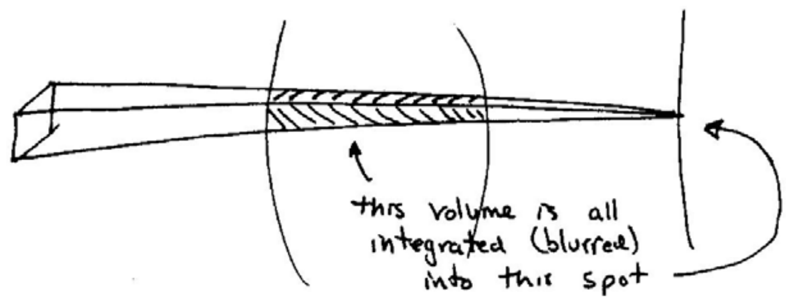
Practical X-ray Sources

There are two main issues associated with practical x-ray sources:

1. Geometric distortions due to point geometry – "depth dependent magnification."

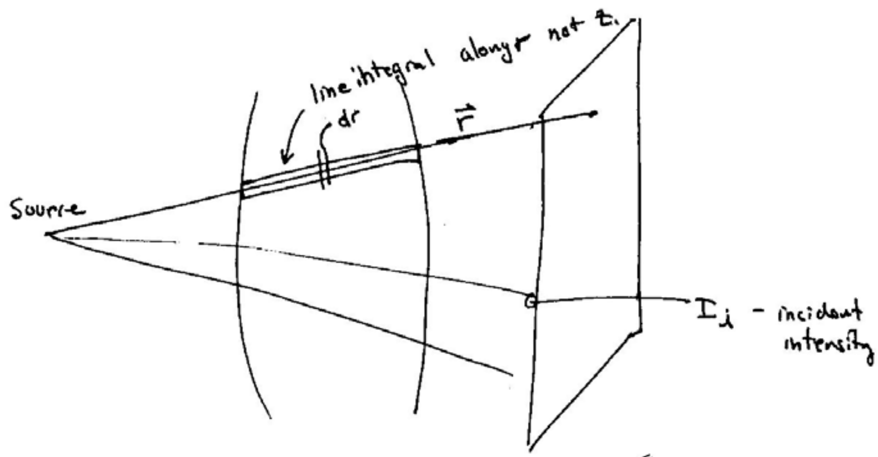


2. Resolution loss (blurring) due to finite (large) source sizes



Point Source Geometry

First, we will find expressions for the image intensity, $I_d(x_d, y_d)$, for a point source geometry:



$$I_d(x_d, y_d) = I_i(x_d, y_d)\exp\left(-\int \mu(x, y, z)dr\right)$$

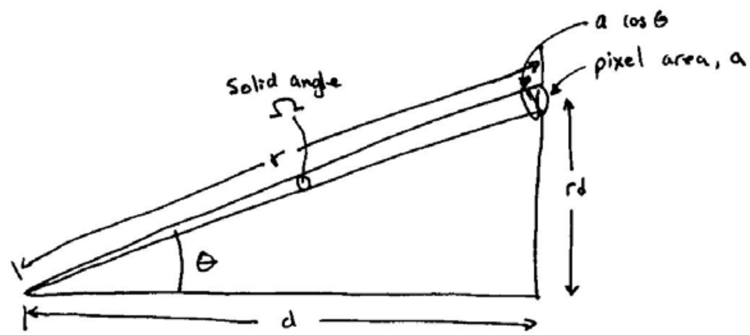
Comments:

1. (x_d, y_d) is the coordinate system in the output detector plane.
2. (x,y,z) is the coordinate system of the object.

3. Notice that $I_i(x_d,y_d)$ a spatially variant incident intensity replaces I_0.
4. Notice that the integration is along some path r with variable of integration dr.

Intensity Variations

The incident intensity is maximal at the center of the coordinate system and falls off towards the edges. This has two components – an increases in distance from the source and the rays obliquely striking the detector.



Intensity has really power/unit area. We can write an expression for the intensity I_i as:

$$I_i = \frac{\text{(photons)(mean photon E)}}{\text{(unit area)(exposure time)}} = \frac{kN}{a}\frac{\Omega}{4\pi}$$

where k is a scaling coefficient, N is the number of photon that are emitted during the observation time (we assume here that photons are emitted isotropically over a sphere), and $\Omega/4\pi$ is fraction of the surface of a sphere that is subtended by pixel area a.
[Ω is known as the *solid angle* and has units of *steradians* of which there are 4π over the surface of a sphere. This is similar to there being 2π radians over circumference of a circle.]
For a pixel of area a at some position angle θ away from the origin, the part of a sphere covered will be $a\cos\theta$. Thus:

$$\frac{\Omega}{4\pi} = \frac{a\cos\theta}{4\pi r^2} \quad \text{or} \quad \Omega = \frac{a\cos\theta}{r^2}$$

We now define the intensity at the origin to be I_0. At the origin, $\theta = 0$ and the distance from the source to the detector is $r = d$, thus $\Omega = a/d^2$ and:

$$I_0 = I_i(0,0) = \frac{kN}{4\pi d^2}$$

Note that the intensity, I_0, falls off with $1/d^2$ as the detector moves away from the source. The constant k can now be found in terms of I_0:

$$k = I_0 \frac{4\pi d^2}{N}$$

Substituting:

$$I_i = \frac{kN}{a}\frac{\Omega}{4\pi} = I_0 d^2 \frac{\cos\theta}{r^2}$$

Observing that $\cos\theta = \frac{d}{r}$, we get:

$$I_i = I_0 \cos^3\theta = I_0\left(\frac{d}{r}\right)^3$$

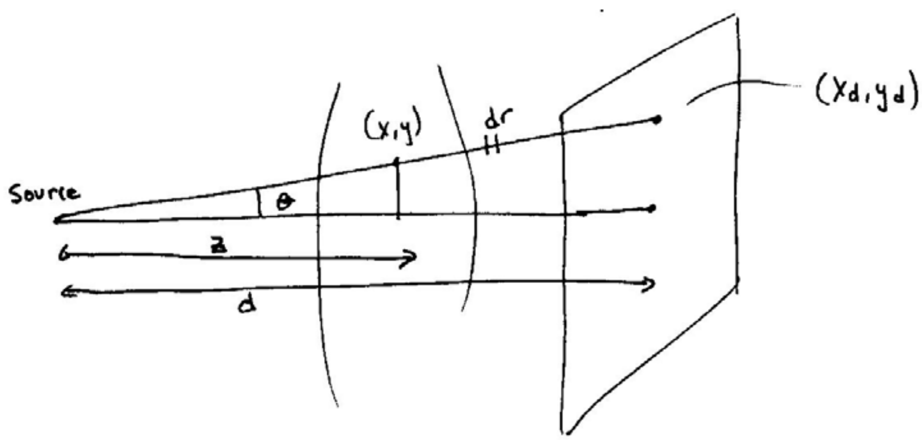
we can put this expression in the coordinate system of the detector using $r_d^2 = x_d^2 + y_d^2$ and $r^2 = d^2 + r_d^2$:

$$I_i(x_d, y_d) = I_0\left(\frac{d}{\sqrt{d^2 + r_d^2}}\right)^3 = I_0 \frac{1}{\left(1 + \left(\frac{r_d}{d}\right)^2\right)^{3/2}}$$

The $\cos^3\theta$ term (or its other representations) is called the *incident intensity obliquity term* and this has two components: the $\cos^2\theta$ term for an increase in distance from the source to the detector and the $\cos\theta$ term for rays obliquely striking the detector. The $\cos^2\theta$ term is really a $1/r^2$ term, the inverse square law for fallout of intensity. The $\cos\theta$ term can be easily visualized if you think of a flashlight beam hitting a wall obliquely – the oblique beam spreads the photons over a larger area of the wall.

Oblique Path Integration

If we look at some point in the object (x,y) at depth z, we see that it will strike the detector at a position $(x_d, y_d) = \left(x\frac{d}{z}, y\frac{d}{z}\right)$:



where $M(z) = \dfrac{d}{z}$ is the magnification factor for an object at depth z. We can now write the attenuation coefficient at location (x,y) in terms of the output coordinate system:

$$\mu(x, y, z) = \mu\left(\frac{x_d}{M(z)}, \frac{y_d}{M(z)}, z\right)$$

Also, instead of integrating along the path r, we can rewrite the expression to integrate in z:

$$\begin{aligned} dr &= \sqrt{dx^2 + dy^2 + dz^2} \\ &= dz\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} \\ &= dz\sqrt{1 + \left(\frac{x_d}{d}\right)^2 + \left(\frac{y_d}{d}\right)^2} \\ &= dz\sqrt{1 + \left(\frac{r_d}{d}\right)^2} \end{aligned}$$

This expression says that if with integrate in z instead of r, the integral will need to be increased by $\sqrt{1 + \left(\frac{r_d}{d}\right)^2}$ in order to account of the longer path length in r (than z). This term is sometimes known as the *pathlength obliquity term*.

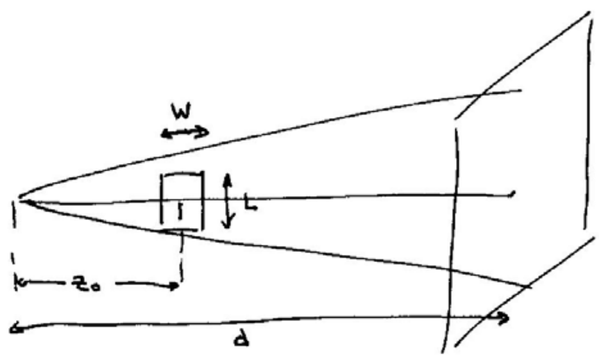
Finally, we put it all together and we get an expression for the output intensity from a point source:

$$I_d(x_d, y_d) = I_0 \frac{1}{\left(1+\left(\frac{r_d}{d}\right)^2\right)^{3/2}} \exp\left(-\sqrt{1+\left(\frac{r_d}{d}\right)^2}\int \mu\left(\frac{x_d}{M(z)}, \frac{y_d}{M(z)}, z\right)dz\right)$$

Example

For the example, we will reduce the dimensions of the problem to 2 – y and z, and thus $r_d = y_d$. Now, let's look at a rectangular object at depth z_0:

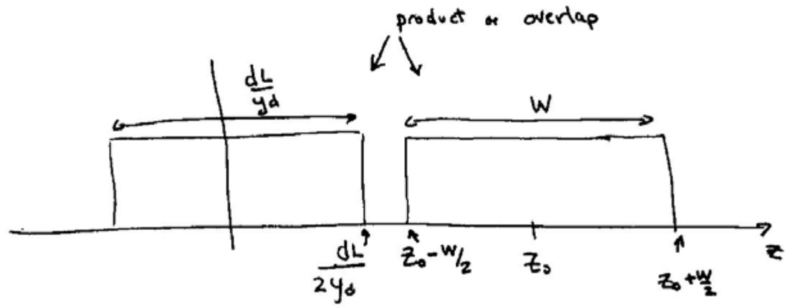
$$\mu(y,z) = \mu_0 \mathrm{rect}\left(\frac{y}{L}\right)\mathrm{rect}\left(\frac{z-z_0}{W}\right)$$



The expression for the image intensity will be:

$$I_d(y_d) = I_0 \frac{1}{\left(1+\left(\frac{y_d}{d}\right)^2\right)^{3/2}} \exp\left(-\sqrt{1+\left(\frac{y_d}{d}\right)^2}\,\mu_0 \int \mathrm{rect}\left(\frac{y_d z}{dL}\right)\mathrm{rect}\left(\frac{z-z_0}{W}\right)dz\right)$$

The use of the magnification factor allowed the function of y to be converted to a function of z for each location y_d in the detector plane. The first rect in the above expression has width dL/y_d and is centered at z=0. The second rect has width W and is centered at z=z_0. The integral is the area under the overlap of these two rect functions.

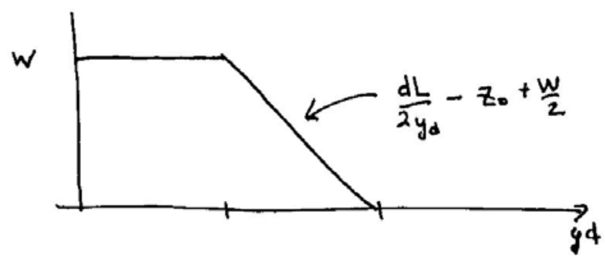


The integral is:

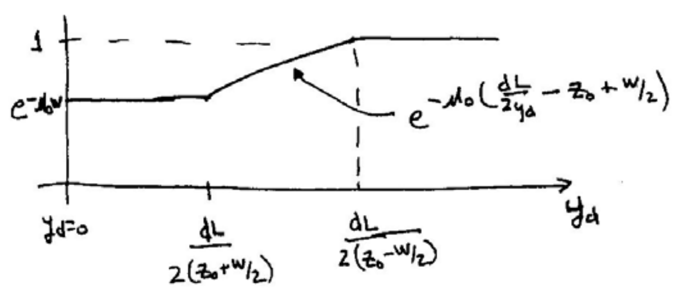
$$0 \quad \text{for} \quad \frac{dl}{2|y_d|} < z_0 - \frac{W}{2} \quad \text{or} \quad |y_d| > \frac{dL}{2(z_0 - W/2)}$$

$$W \quad \text{for} \quad \frac{dl}{2|y_d|} > z_0 + \frac{W}{2} \quad \text{or} \quad |y_d| < \frac{dL}{2(z_0 + W/2)}$$

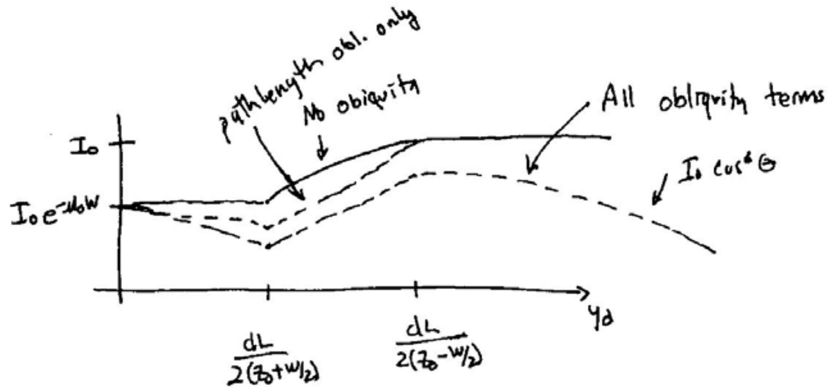
$$\frac{dL}{2y_d} - z_0 + \frac{W}{2} \qquad \text{otherwise}$$



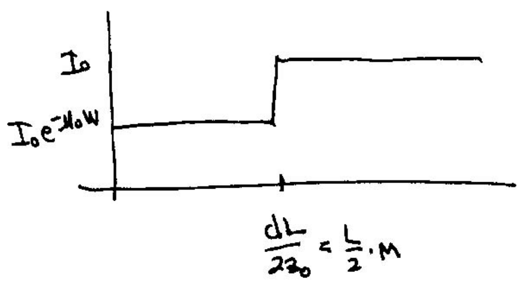
If we ignore all obliquity terms, we get the following:



Including the pathlength and incident intensity obliquity terms we get:



Under a parallel ray geometry we get the following:



As we can see, the depth dependent magnification has significantly distorted the appearance of the object in the image. We can define a fractional transition width be:

$$\frac{\dfrac{dL}{2(z_0 - W/2)} - \dfrac{dL}{2(z_0 + W/2)}}{\dfrac{dL}{4(z_0 - W/2)} + \dfrac{dL}{4(z_0 + W/2)}} = \frac{W}{z_0}$$

Thus, we can minimize the geometric distortions by placing the object as far from the source as possible (make z_0 large).

Finite (Large) Sources

To gain an understanding of this issue, we will first consider a "thin" object. Specifically, we will let the attenuation coefficient be:

$$\mu(x, y, z) = \tau(x, y)\delta(z - z_0)$$

and then:

$$I_d(x_d, y_d) = I_i \exp\left(-\sqrt{1+\left(\frac{r_d}{d}\right)^2} \int \tau\left(\frac{x_d}{M(z)}, \frac{y_d}{M(z)} \right) \delta(z - z_0) dz \right)$$

$$= I_i \exp\left(-\sqrt{1+\left(\frac{r_d}{d}\right)^2} \tau\left(\frac{x_d}{M(z_0)}, \frac{y_d}{M(z_0)} \right) \right)$$

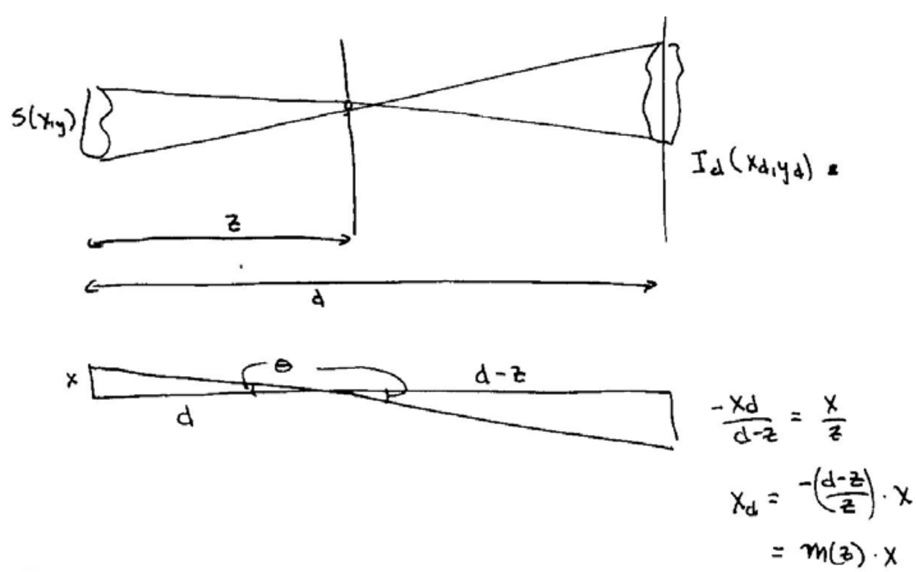
We let $M = M(z_0) = d / z_0$ the *object magnification factor*, and we will ignore the pathlength obliquity term to get:

$$I_d(x_d, y_d) = I_i \exp\left(-\tau\left(\frac{x_d}{M}, \frac{y_d}{M} \right) \right) = I_i t\left(\frac{x_d}{M}, \frac{y_d}{M} \right)$$

where $t = \exp(-\tau)$ is the transmission function. Ignoring all obliquity terms we get:

$$I_d(x_d, y_d) = I_0 t\left(\frac{x_d}{M}, \frac{y_d}{M} \right)$$

Now we consider a finite source function $s(x,y)$ and a very small pinhole transmission function:



The image will now be an image of the source with the *source magnification factor*, $m = m(z) = -\frac{d-z}{z}$:

$$I_d(x_d, y_d) = ks\left(\frac{x_d}{m}, \frac{y_d}{m} \right)$$

where k is a scaling factor that is proportional to the area of the pinhole, $1/d^2$, etc. If we want the above I_d to represent the impulse response of the system, we need to make the pinhole equal to $\delta(x,y)$ and account for all of the scaling terms [$t(x,y) = \delta(x,y)$ is not a realizable transmission function since t can never exceed 1, nevertheless, we will allow it for mathematical convenience.]

The area of the pinhole is $\iint \delta(x,y)dxdy = 1$. The capture efficiency of the pinhole is the fraction of all photons emitted from the source that pass through the pinhole. This will be equal to:

$$\eta = \frac{\text{pinhole area}}{4\pi z^2} = \frac{1}{4\pi z^2}$$

Letting the total number of photon emitted be:

$$N = \iint s(x,y)dxdy$$

and the total number of photons to get through the pinhole will be:

$$N\eta = \frac{N}{4\pi z^2}.$$

This must be the same number at the detector:

$$\iint ks\left(\frac{x_d}{m}, \frac{y_d}{m}\right)dx_d dy_d = kNm^2 = \frac{N}{4\pi z^2}$$

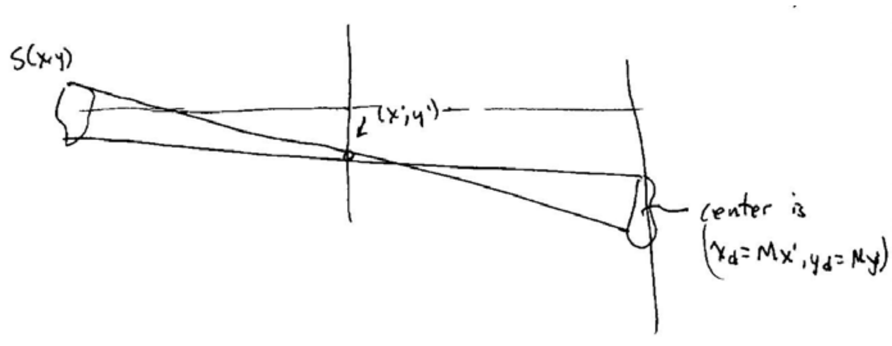
The scaling coefficient will therefore be:

$$k = \frac{1}{4\pi z^2 m^2}$$

so:

$$I_d(x_d, y_d) = \frac{1}{4\pi z^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

Now we let the pinhole be at position (x',y'), that is, $t(x,y) = \delta(x-x', y-y')$:



The image of the source is not located at (x_d=Mx', y_d=My') where M is the object magnification factor. Thus, the impulse response function is:

$$h(x_d, y_d; x', y') = I_d(x_d, y_d) = \frac{1}{4\pi z^2 m^2} s\left(\frac{x_d - Mx'}{m}, \frac{y_d - My'}{m}\right)$$

Now we can calculate the image for an arbitrary transmission function using the superposition integral:

$$I_d(x_d, y_d) = \iint t(x' y') h(x_d, y_d; x', y') dx' dy'$$

$$= \frac{1}{4\pi z^2 m^2} \iint t(x' y') s\left(\frac{x_d - Mx'}{m}, \frac{y_d - My'}{m}\right) dx' dy' \text{ and sub } Mx' = x$$

$$= \frac{1}{4\pi z^2 m^2 M^2} \iint t\left(\frac{x}{M}, \frac{y}{M}\right) s\left(\frac{x_d - x}{m}, \frac{y_d - y}{m}\right) dxdy$$

$$= \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** t\left(\frac{x_d}{M}, \frac{y_d}{M}\right)$$

<u>Thus, the final image is equal to the convolution of the magnified source and the magnified object. The object is blurred by the source function.</u>

The frequency domain equivalent is:

$$F_{2D}\{I_d(x_d, y_d)\} = \frac{1}{4\pi z^2} S(mu, mv) T(Mu, Mv)$$

Consider $z_0 = d/2$ which yields M=2 and |m|=1. The object is magnified by a factor of 2 and is blurred by the unmagnified source.

Comments:

1. The least blurring come when $|m|$ is made small. Thus, it is desirable to make the depth plane as far from the source as possible: $z_0 \rightarrow d$. Then $|m| = (d\text{-}z)/z \rightarrow 0$ and $M \rightarrow 1$. As we was above, making $z_0 \rightarrow d$ also reduces geometric distortions. The common practice for x-ray imaging, then, is to position the subject immediately next to or on top of the detector.
2. If the thickness of the body is a limiting factor, then let $d, z \rightarrow \infty$. This will make the system close to a parallel ray geometry with $|m| = \rightarrow 0$ and $M \rightarrow 1$. The main problem with this approach is $I_0 \propto 1/d^2 \rightarrow 0$ and SNR $\propto \sqrt{I_0} \rightarrow 0$.
3. We would also like the make $s(x,y)$ as small as possible to reduce blurring, but $I_0 \propto \iint s(x,y)dxdy$ and making it small might reduce the number of photons created and thus reduce SNR.
4. For a complex object, we can make $\mu(x,y,z) = \sum \tau_i(x,y)\delta(z - z_i)$ and each plane will have its own magnification factors. This is not particularly useful, but it can give you some idea of how blurring and magnification might affect different parts of a real object differently.

Overall System Response

Now we can add the detector response to the other system elements:

$$I_d(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** t\left(\frac{x_d}{M}, \frac{y_d}{M}\right) ** h(r_d)$$

The impulse response function will then be:

$$h(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** h(r_d)$$

or for a circularly symmetric source function:

$$h(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{r_d}{m}\right) ** h(r_d)$$

Object Blurring

One issue is how much does the detector response blur the object. It is important to realize that the detector blurs the magnified object. Our intuition would be to make the object as large as possible by making $M = d/z$ very large. This would dictate moving the object as close to the source as possible, which is exactly opposite as what we would like to do to minimize source blurring.

Consider also, that the magnified source also blurs the magnified object (source and object have different magnification factors). One way to look at this is to examine the response in the coordinate system of the object (x,y) rather than the detector (x_d,y_d):

$$I(x,y) = ks\left(\frac{Mx}{m}, \frac{My}{m}\right) **t(x,y)**h(Mr_d)$$

the effective magnification of the source is:

$$\left|\frac{m}{M}\right| = \frac{d-z}{d}$$

and the effective magnification of the detector response is:

$$\frac{1}{M} = \frac{z}{d}$$

These are in competition:

- to make the source blurring small, make $z \rightarrow d$
- to make the detector response small, make $z \rightarrow 0$

Comments:

1. For most film systems, the detector response is very small and the source is almost always bigger. Therefore, we would like to make $z \rightarrow d$.
2. For other kinds of systems, e.g. digital fluoroscopy systems, the detector resolution is much larger (e.g. 0.5 mm) and for these systems an intermediate z may be appropriate.