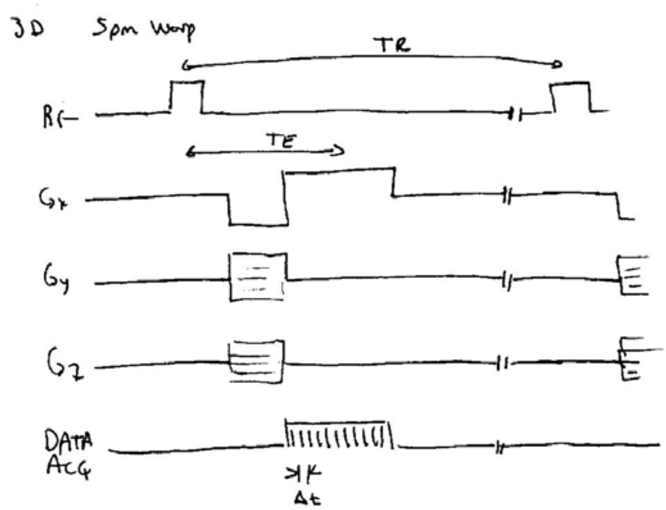
Notes on MRI, Part III

The 3rd Dimension - Z

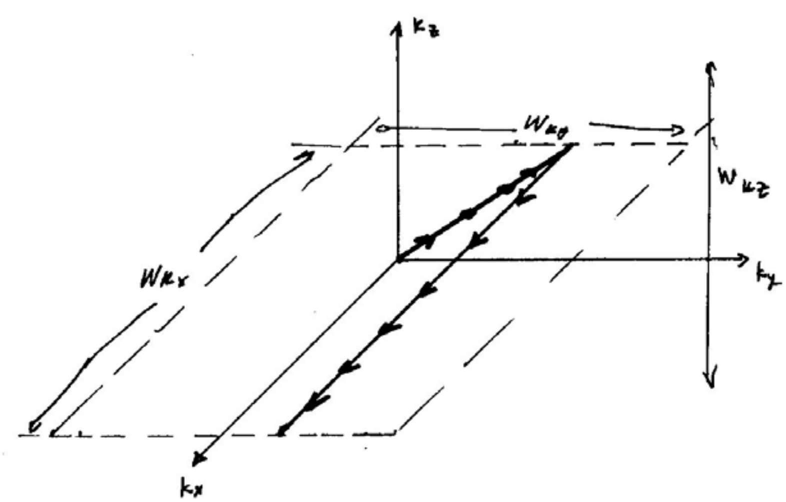
The 3D signal equation can be written as follows:

$$s(t) = \iiint m(x, y, z) \exp(-i2\pi(k_x(t)x + k_y(t)y + k_z(t)z)dxdydz$$
$$= M(u, v, w) \big|_{u=k_x(t), v=k_y(t), w=k_z(t)},$$

where $M(u,v,w)$ is the 3D FT of $m(x,y,z)$. In the spin-warp method for 2D acquisition, one line at a time is acquired in the 2D Fourier domain (or k-space). This method is easily extended to 3D by using phase encoding in two dimensions (rather than 1) and frequency encoding in the remaining dimension:



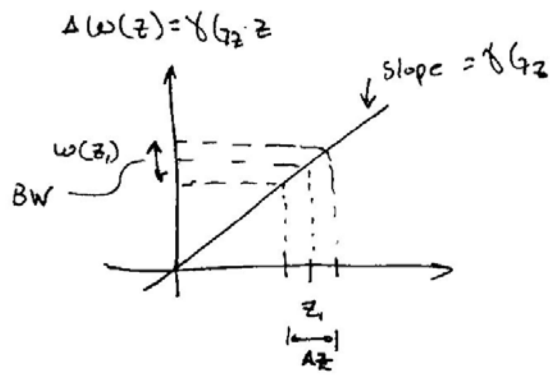
This results in the acquisition of a cubic data set one line at a time:



The sampling requirements and spatial resolution requirements are the same as they would be for the 2D spin-warp method ($FOV_z = 1/\Delta k_z$; $\Delta z = 1/W_{kz}$). If there are N_y and N_z samples in the y and z directions, respectively, then the total time to acquire the 3D volume is N_y*N_z*TR. For example, for $N_y = N_z = 128$ and TR = 33 ms, the overall image acquisition time is 9 min – rather long!

Slice Selective Excitation

The most common approach for dealing with the 3rd (z) dimension is to use slice selective excitation. This is done by applying a z-gradient so that the resonance frequency varies in the z-direction and applying a bandpass RF pulse to excite only the those spins whose resonant frequency lies within the band:



We will again solve the Bloch equations for this specific case. We will let B_1 be a time-varying magnetic field rotating at ω_0. For this analysis, we'll let the rotating frame be at $\omega_{frame} = \omega_0$.

$$\mathbf{B_1}(t) = B_1(t)(\cos\omega_0 t\mathbf{i} + \sin\omega_0 t\mathbf{j})$$

$$\mathbf{B}_{\mathbf{1},eff}(t) = B_1(t)\mathbf{i}'$$

A z-gradient is applied, so the component in the z-direction is:

$$\mathbf{B_z}(z) = (B_0 + G_z \cdot z)\mathbf{k}$$

$$\mathbf{B}_{\mathbf{z},eff}(z) = (G_z \cdot z)\mathbf{k}$$

and the net effective applied field is:

$$\mathbf{B}_{eff} = B_1(t)\mathbf{i}' + (G_z \cdot z)\mathbf{k}$$

The Bloch equation for this case reduces to the following:

$$\frac{d\mathbf{M_{rot}}}{dt} = \mathbf{M_{rot}} \times \gamma \mathbf{B_{eff}} = \begin{bmatrix} 0 & \gamma G_z z & 0 \\ -\gamma G_z z & 0 & \gamma B_1(t) \\ 0 & -\gamma B_1(t) & 0 \end{bmatrix} \mathbf{M_{rot}}$$

What we would like to know is how the magnetization, $\mathbf{M_{rot}}$, varies as a function of z position following the application of the specified B_1 field. This is, in general, a very difficult equation to solve because it is non-linear.

Small Tip Angle Approximation

One particularly useful approach to the solution to the above Bloch equation is to use the "small tip angle approximation." Here, we assume the B_1 produces a small net rotation angle, say,

$$\int \gamma B_1(t) dt < \frac{\pi}{6} \quad (30°)$$

In this case, we can assume the z component of the magnetization, m_z, is approximately equal to m_0 during the RF pulse. Essentially, we are saying that:

$$\cos\left(\int_0^t \gamma B_1(\tau) d\tau \right) \approx 1$$

Under this assumption, $\frac{dm_z}{dt} = 0$, $m_z(t) = m_0$, and thus $m_z(t) = m_0$. The above Bloch equation can then be rewritten as:

$$\frac{d\mathbf{M_{rot}}}{dt} = \begin{bmatrix} 0 & \gamma G_z z & 0 \\ -\gamma G_z z & 0 & \gamma B_1(t) \\ 0 & \underline{0} & 0 \end{bmatrix} \begin{bmatrix} m_{x,rot} \\ m_{y,rot} \\ \underline{m_0} \end{bmatrix}$$

We now would like to solve for $m_{xy,rot}(z,t) = m_{x,rot}(z,t) + i\, m_{y,rot}(z,t)$. We can then write a differential equation using for the transverse component:

$$\frac{dm_{xy,rot}}{dt} = \frac{dm_{x,rot}}{dt} + i \frac{dm_{y,rot}}{dt}$$

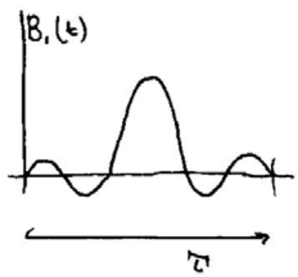
$$= \gamma G_z z m_{y,rot} - i\gamma G_z z m_{x,rot} + i\gamma B_1(t) m_0$$

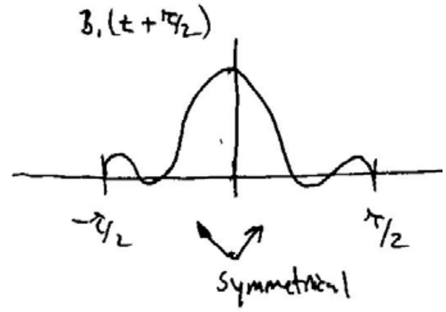
$$= -i\gamma G_z z m_{xy,rot} + i\gamma B_1(t) m_0$$

Observe that $i\gamma G_z z$ is a constant with respect to time and thus we have a first order differential equation with a driving function $i\gamma B_1(t)m_0$. For initial condition, $m_{xy,rot}(z,t) = 0$, the solution to this differential equation can be shown be:

$$m_{xy,rot}(z,t) = i m_0 e^{-i\gamma G_z z t}\int_0^t \exp(i\gamma G_z z\xi)\gamma B_1(\xi)d\xi$$

Again, we want the solution to this differential equation at the time of the end of the RF pulse, which we define as τ, $m_{xy,rot}(z,\tau)$:





We now make a variable substitution, $s = \xi - \tau/2$. We also assume that the RF pulse that is symmetrical (even) around $\tau/2$ and that it is zero outside of the interval $[0,\tau]$. The magnetization can now be described as:

$$m_{xy,rot}(z,\tau) = i m_0 e^{-i\gamma G_z z\tau}\int_{-\tau/2}^{\tau/2} \exp(i\gamma G_z z(s+\tau/2))\gamma B_1(s+\tau/2)ds$$

$$= i\gamma m_0 e^{-i\gamma G_z z\tau/2}\int_{-\tau/2}^{\tau/2} \exp(i\gamma G_z z s)B_1(s+\tau/2)ds$$

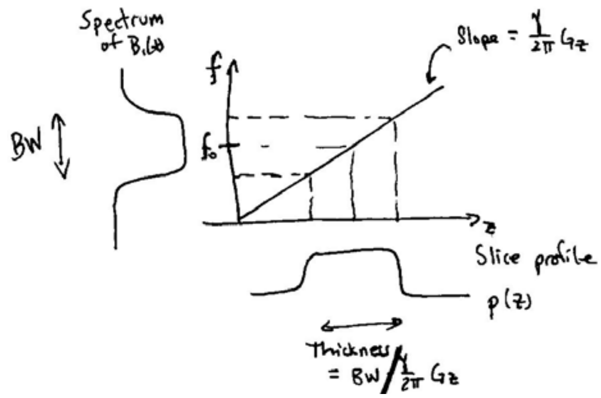
$$= i\gamma m_0 e^{-i\gamma G_z z\tau/2}\int_{-\infty}^{\infty} B_1(s+\tau/2)\exp\left(i2\pi\left(\frac{\gamma}{2\pi}G_z z\right)s\right)ds$$

$$= i\gamma m_0 e^{-i\gamma G_z z\tau/2}F^{-1}\left\{B_1(s+\tau/2)\right\}\Big|_{x=\frac{\gamma}{2\pi}G_z z}$$

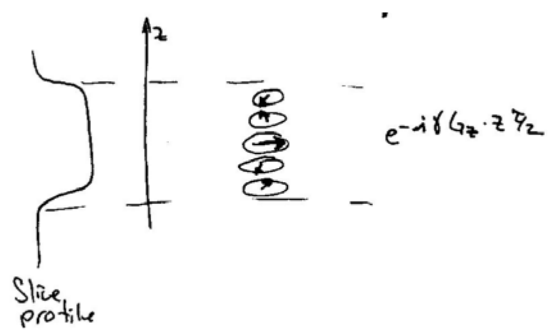
For symmetrical RF pulses, the forward and inverse FT are the same. Thus, under the small tip angle approximation, the slice profile (variation of $m_{xy,rot}$ with z) is related to the spectrum of the RF pulse:

$$\left|m_{xy,rot}(z,\tau)\right| \propto \left|F\left\{B_1(t)\right\}\right|_{f=\frac{\gamma}{2\pi}G_z z}$$

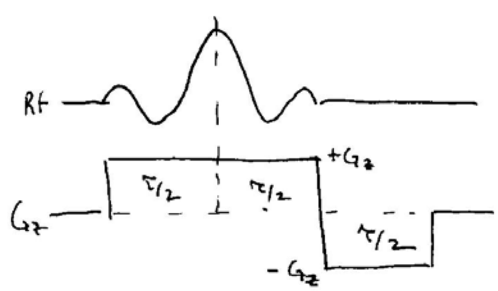
and $f = \frac{\gamma}{2\pi} G_z z$ is the conversion between spectrum and the z location:



We're almost there, but we still have some undesired phase variation in the z-direction, $\exp(-i\gamma G_z z\tau/2)$, the can lead to undesired phase destruction when integrated by the RF coil.



How is this fixed? We simply apply a negative G_z for a period $\tau/2$. This is often called a slice rephasing pulse.



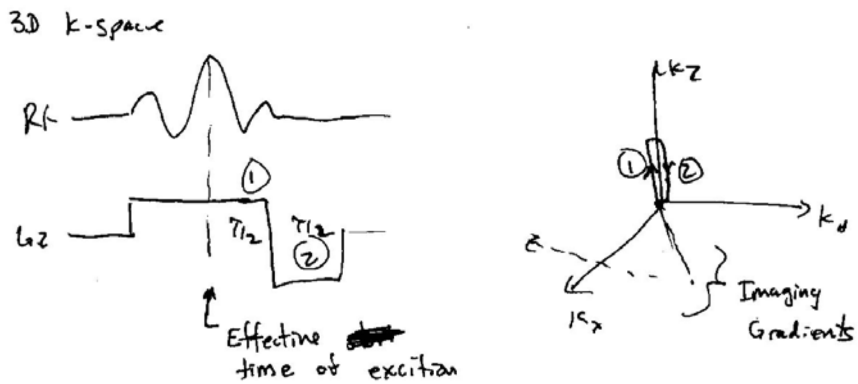
This will result in phase accumulation of:

$$\exp\left(-i\gamma \int_{\tau}^{3\tau/2} \Delta B(z,t)dt\right) = \exp\left(i\gamma G_z z\tau/2\right)$$

and thus:

$$m_{xy,rot}(z, 3\tau/2) = i\gamma m_0 F\{B_1(s+\tau/2)\}\Big|_{x=\frac{\gamma}{2\pi}G_z z}$$

There is a k-space picture to this. For this, we assume that the RF pulse occurs instantaneously a the center of the pulse (at $\tau/2$) and we begin accumulation area in k-z after that point. By applying a negative gradient for the same duration as the last half of the pulse, the areas cancel and the k-space location in the z direction is returned to the origin.



Notice for an RF pulse applied along the real (**i'**) axis, the magnetization will end up along the imaginary (**j'**)axis. Also observe the flip angle at the center of the slice is:

$$\alpha = \int_{-\tau/2}^{\tau/2} \gamma B_1(s+\tau/2)ds = \gamma F\{B_1(s+\tau/2)\}\,\big|_{z=0}$$

Previously, we discussed the that transverse magnetization after an α pulse was $m_0 \sin\alpha$, but for small α, $\sin\alpha \sim \alpha$. So here, the transverse magnetization is also $m_0\alpha$.

Example – the sinc RF pulse

Consider an RF pulse roughly in the form:

$$B_1(t) = A\,\mathrm{sinc}\left(\frac{t-\tau/2}{T}\right)$$

which has a spectrum:

$$F\{B_1(s+\tau/2)\} = AT\,\mathrm{rect}\left(\frac{f}{BW}\right), \text{ where } BW = 1/T$$

The magnetization will be:

$$m_{xy,rot}(z, 3\tau/2) = i\gamma m_0 AT\,\mathrm{rect}\left(\frac{zG_z\gamma}{2\pi BW}\right) = i\alpha m_0\,\mathrm{rect}\left(\frac{z}{\Delta z}\right)$$

where the slice thickness is $\Delta z = \dfrac{2\pi BW}{\gamma G_z}$ and flip angle is $\alpha = \gamma AT$.

Putting Slice Selection with the Signal Equation

Let's define our slice profile function:

$$p(z) = \gamma F\{B_1(t + \tau/2)\} \big|_{f=\frac{\gamma}{2\pi}G_z z}$$

then

$$m_{xy,rot}(z,3\tau/2) = im_0 p(z)$$

Now we go back to the case where we have a 3D distribution of magnetization by substituting $im_0 = m(x,y,z)$ and putting it into the signal equation (again the RF coil integrates across the object):

$$s(t) = \iiint m(x,y,z)\,p(z)\exp\left(-i2\pi\left(xk_x(t) + yk_y(t)\right)\right)dxdydz$$

Here we are performing 2D imaging while integrating across the slice profile.

Larger Flip Angles

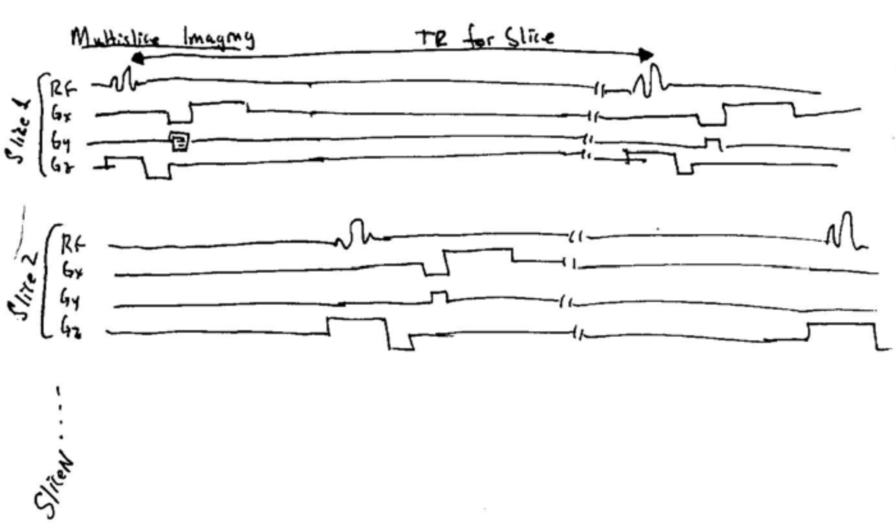
The above analysis was for small flip angles, e.g. $\alpha < \pi/6$, but it turns out the pulses created using the small tip angle approximation, also perform well for large flip angles (e.g. 90 degrees). Here, a better approximation for the transverse component of the magnetization is:

$$m_{xy,rot}(z,3\tau/2) = im_0 \sin\left(p(z)\right)$$

although due to the non-linearity of the Bloch equations, an exact solution would require numeric simulation of the RF pulse.

Mutlislice Imaging

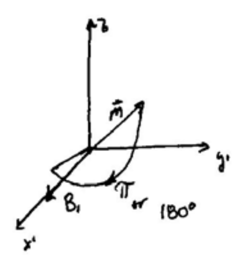
The most common way to image 3D volumes in MRI uses interleaved slice selective excitation. Here, slice 1 is excited and part of the k-space data are acquired, then slice 2 is excited and acquired, then slice 3 and so on. After all have been acquired, we come back to slice 1 to acquire additional parts of the k-space data, etc. When one slice is excited, the others are not perturbed and thus each slice has it's own T1 recovery time (TR). Slice selection allows the efficient use of longer TR's by simultaneously acquiring many slices:



For a slice-selective, 2D spin-warp acquisition the overall acquisition time will be N_y*TR. For example, if we are interested in acquiring a T1-weighted image with 20 slices and a 500 ms TR and 128 phase encoding lines in k-space, the total acquisition time for these 20 slices is N_y*TR = ~ 1 minute.

Spin Echo Pulses

Earlier we described 180 degree RF pulses for purposes of inverting the m_z magnetization. Let's consider the effect of a 180 degree B1 pulse applied to the x' axis in the rotating frame on a magnetization vector, $\mathbf{m} = [m_x, m_y, m_z]$:



For t_{180}- and t_{180}+ being the time just before and after the 180 degree pulse, the magnetization will be:

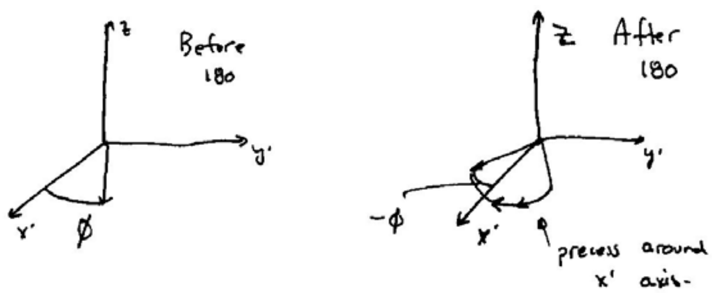
$$m_{x,rot}(t_{180}+) = m_{x,rot}(t_{180}-)$$
$$m_{y,rot}(t_{180}+) = -m_{y,rot}(t_{180}-)$$
$$m_{z,rot}(t_{180}+) = -m_{z,rot}(t_{180}-)$$

Now, let's look at a magnetization vector that is lying in the transverse plane ($m_z = 0$). Suppose the vector was originally positioned on the x' axis and phase, $\phi(\mathbf{r}, t)$ has accumulated due to ΔB terms. The phase after the 180 degree pulse will be:

$$\phi(\mathbf{r}, t_{180}+) = -\phi(\mathbf{r}, t_{180}-)$$

or equivalently:

$$m_{xy,rot}(t_{180}+) = m_{xy,rot}(t_{180}-)^*$$



When imaging, the phase term results from gradients and can be written as:

$$\phi(\mathbf{r}, t) = 2\pi(\mathbf{k}(t) \cdot \mathbf{r})$$

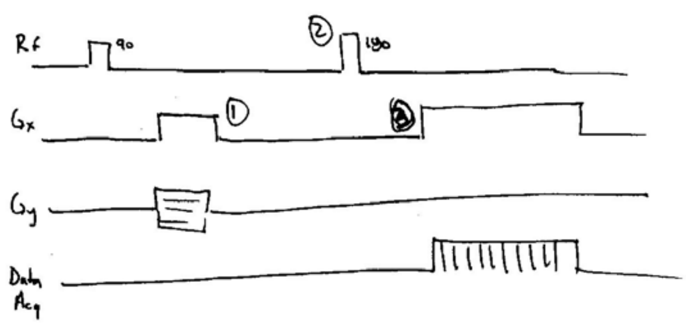
and thus:

$$\mathbf{k}(t_{180}+) = -\mathbf{k}(t_{180}-)$$

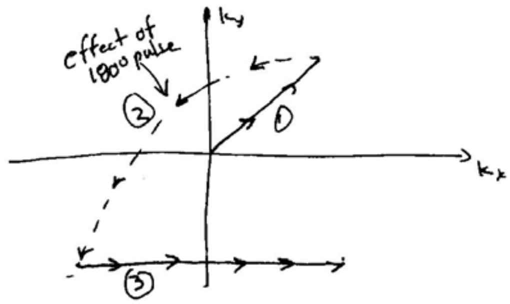
What this says is that a 180 degree pulse will invert the location in k-space. Spin-echo 180 degree pulses are at times known as "phase reversal" or "time reversal" pulses.

Spin-echo Spin-warp Pulse Sequence

Consider this pulse sequence:



Here the 180 degree pulse inverts the position in k-space as shown here:



Why do spin-echo pulses?

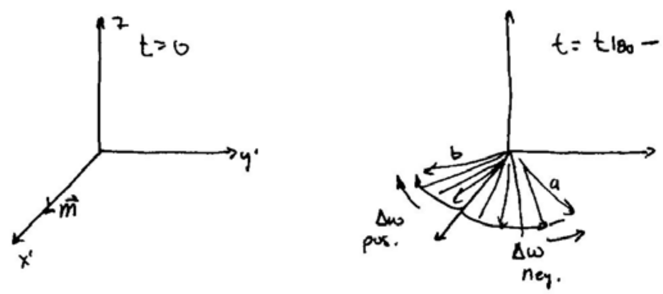
Magnetic field inhomogeneity can results in intra-voxel signal dephasing. Consider a magnetic field inhomogeneity function $\Delta B(\mathbf{r})$. The effective magnetic field (rotating frame) will be:

$$B_{z,\text{eff}} = \mathbf{G}(t)\cdot\mathbf{r} + \Delta B(\mathbf{r})$$

and the corresponding phase function is:

$$\phi(\mathbf{r}, t) = 2\pi(\mathbf{k}(t)\cdot\mathbf{r}) + \Delta\omega(\mathbf{r})t$$

when integrating $\Delta\omega(\mathbf{r})t$ across a voxel some signal may be lost. The spin-echo pulse brings this phase back together again. Consider the following example:

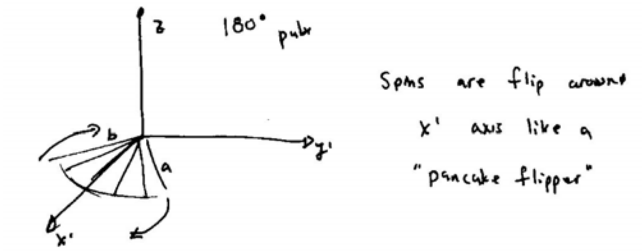


Ignoring the k-space term, the phase accumulation at the time of the 180 will be:

$$\phi(\mathbf{r}, t_{180}-) = \Delta\omega(\mathbf{r})t_{180}$$

and just after the 180 is will be:

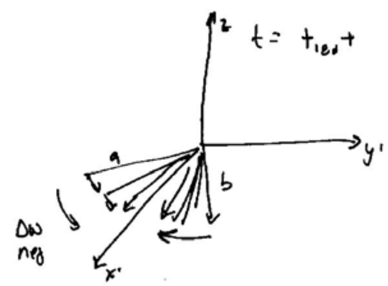
$$\phi(\mathbf{r}, t_{180}+) = -\Delta\omega(\mathbf{r})t_{180}$$

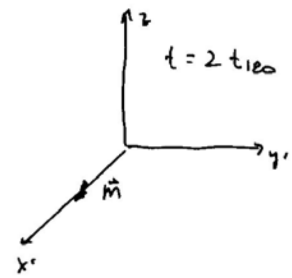


At this point the phase continues to accumulate and if we look at time 2 t_{180}, we will have a total phase accumulation of:

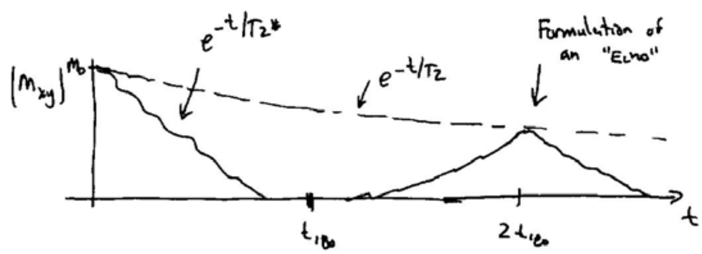
$$\phi(\mathbf{r}, 2\, t_{180}) = \phi(\mathbf{r}, t_{180}+) + \omega(\mathbf{r})t_{180} = -\Delta\omega(\mathbf{r})t_{180} + \Delta\omega(\mathbf{r})t_{180} = 0$$

That is, all the phase accumulation due to magnetic field inhomogeneity cancels to zero at 2 times the time of 180 degree pulse, or 2 t_{180}.





The size of the signal in the transverse plane ($|m_{xy}|$) will look like this:



As shown here, the signal comes back together again in an "echo" at 2 t_{180}. The more rapid decay of the signal due to T2 decay plus inhomogeneity effects is given another decay term – T2*. When all dephasing is cancelled by the spin-echo, however, the T2 decay still remains.

Noise in MRI

Sources of noise in MRI

- Thermal noise from body – thermal vibration of ions, electrons, etc. [Dominant source of noise in most MRI systems]
- Quantization noise in the A/D devices

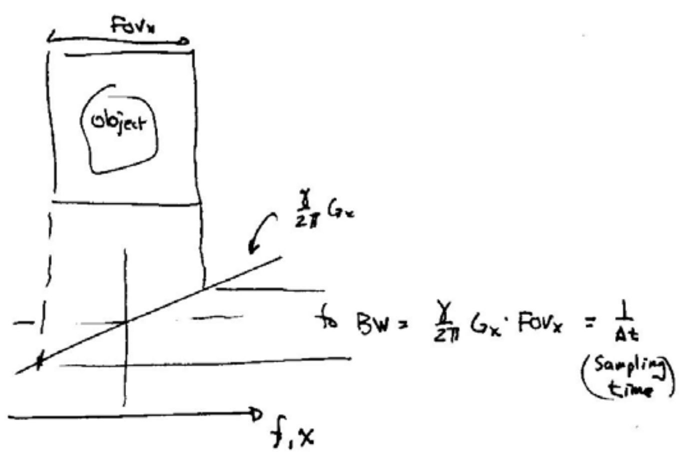
- Preamp/electronic noise
- Thermal noise in RF coil

Some comments on thermal noise:

- Not related to the NMR
  - Present with or without B_0, RF, Gradients
- Uniform spectral density (near ω_0) – white
- Comes from the whole body – amount of noise depends on the amount of the body to which the receive coil is sensitive

Noise in Spin-warp imaging – consider the frequency encoding gradient, Gx, and assuming that the field of view is FOVx, then the bandwidth of the receiver will be set to:

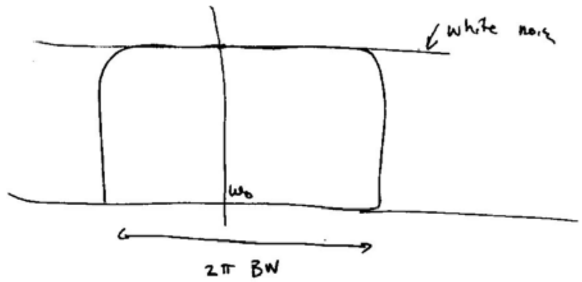
$$BW = \frac{\gamma}{2\pi} G_x FOV_x = \frac{1}{\Delta t}.$$



Noise characteristics:

- Noise is zero-mean and additive $s(t) = M(k_x(t), k_y(t)) + n$
- Samples are independent (due to the whiteness of the spectrum)
- Gaussian distributed (it results from large numbers of vibrating particles)
- Bi-variate - independent noise in real + imaginary (the channels of the complex demodulator are orthogonal) ($n = n_i + i\, n_q$)

- Noise variance for each sample is proportional to the presampling bandwidth $\sigma_n^2 \propto BW$:



- A 1D image is reconstructed by an N point DFT:

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} X_m \exp\left(i2\pi \frac{nm}{N}\right)$$

where X_m are samples with independent noise having variance σ_n^2. The noise in image pixels x_n will also be zero-mean, additive, independent, bi-variate Gaussian noise, but with variance σ_n^2 / N. The derivation of the variance is:

$$\begin{aligned}\operatorname{var}(x_n) = E\left[x_n x_n^*\right] &= E\left[\frac{1}{N}\sum_{m=0}^{N-1} X_m e^{i2\pi\frac{nm}{N}} \frac{1}{N}\sum_{l=0}^{N-1} X_l e^{-i2\pi\frac{nl}{N}}\right] \\ &= \frac{1}{N^2}\left(E\left[\sum_{m=0}^{N-1} X_m^2\right] + E\left[\sum_{m\neq l} X_m X_l e^{i2\pi\frac{n(m-l)}{N}}\right]\right) \\ &= \frac{1}{N^2}\sum_{m=0}^{N-1} E\left[X_m^2\right] = \frac{1}{N^2} N\sigma_n^2 = \frac{\sigma_n^2}{N}\end{aligned}$$

- A 2D image is reconstructed by a N_x by N_y inverse 2D DFT. Again the resultant noise in the image will be zero-mean, additive, independent, bi-variate Gaussian noise, but with variance $\sigma_n^2 /(N_x N_y)$.

Signal to Noise Ratio

- The noise/pixel in a 2D image will be then be:

$$\frac{\sigma_n^2}{N_x N_y} \propto \frac{BW}{N_x N_y} = \frac{1}{N_x N_y \Delta t} = \frac{1}{T_{A/D}}$$

where $T_{A/D}$ is the total time the A/D is sampling.

- The signal ($x_{i,j}$) represents the total amount of magnetization in a particular voxel (recall that $\sum\sum x_{i,j} = X_{0,0} = M(0,0)$. Thus the signal is proportional to $m_0 V$ where $V = \Delta x \Delta y \Delta z$ is the "voxel" volume and Δz is the slice thickness.
- The signal to noise ratio is then:

$$SNR \propto \frac{signal}{\sigma_n} = m_0 V \sqrt{T_{A/D}}$$

(m_0 is proportional to ρ - the concentration the nucleus of interest, B_0, and γ.)

Examples

Case 1: Suppose we find that we have an image that is too noisy, so we average together neighboring pixels to achieve $\Delta y' = \Delta y*2$ (all other dimensions remain the same and Δt hasn't changed either). Since by averaging in image domain, we effectively are discarding samples in k-space, $T_{A/D}' = T_{A/D}/2$ and:

$$SNR' = 2\Delta x \Delta y \sqrt{\frac{T_{A/D}}{2}} = \sqrt{2} SNR_{orig}$$

That is, we've improved the SNR by sqrt(2).

Case 2: Suppose we knew in advance that the SNR of an image was too noisy, and we compensated by acquiring a lower resolution $\Delta y' = \Delta y*2$ (all other dimensions remain the same) but we've compensated so as to preserve the original acquisition time $T_{A/D}' = T_{A/D}$. Thus:

$$SNR' = 2\Delta x \Delta y \sqrt{T_{A/D}} = 2 SNR_{orig}$$

From these two examples, we see that it is preferable to anticipate the SNR that is necessary for a given image an set the acquisition accordingly. We don't achieve as good of an SNR by smoothing the image after it is acquired than if we had acquired at the appropriate resolution originally.

Case 3: Suppose we average each k-space sample 2 times (and thus increase our image acquisition time by a factor of 2). There the resolution is the same and $T_{A/D}' = 2\, T_{A/D}$, and:

$$SNR' = \Delta x \Delta y \sqrt{2 T_{A/D}} = \sqrt{2} SNR_{orig}$$

Averaging increases the SNR by sqrt(N) where N is the number of averages.

Case 4: Suppose we increased the field strength by a factor of 2, $B_0' = 2B_0$. Then

$$SNR' = 2SNR_{orig}$$

Keeping resolution constant, we can use this additional SNR to reduce the number of averages (and thus overall imaging time) by a factor of 4! (Keep in mind that in medical imaging time=money.)