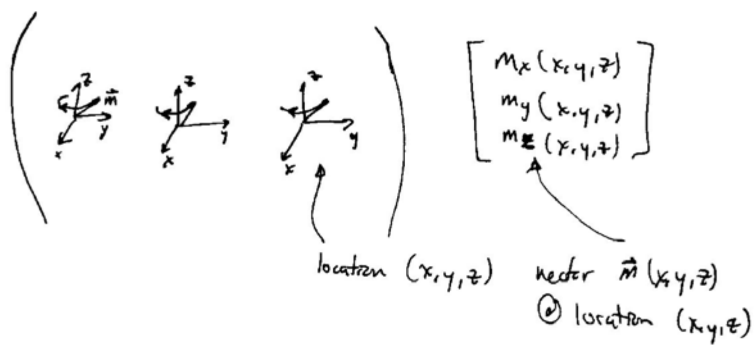
Notes on MRI, Part II

Spatial and Temporal Variations

We will now generalize our solution to the Bloch equations to functions in the object domain, for example:

$$m_{xy}(\mathbf{r},t) = m_x(x,y,z,t) + i m_y(x,y,z,t)$$

Please note the distinction between the subscript *x*, which denotes the direction of a magnetization vector, and the argument *x*, which denotes the spatial location of that magnetization vector.



We also will allow the applied magnetic field to be a function of both space and time, but as before, we will first consider the case where the applied field is only in the *z* direction:

$$\mathbf{B}(\mathbf{r},t) = \left(B_0 + \Delta B(\mathbf{r},t)\right)\mathbf{k}$$

The Bloch equation again simplifies to separate solutions for the m_z and m_{xy} components. The m_{xy} equations is written here:

$$\frac{dm_{xy}(\mathbf{r},t)}{dt} = -(i\gamma\left(B_0 + \Delta B(\mathbf{r},t)\right)\omega_0 + \frac{1}{T_2(\mathbf{r})})m_{xy}(\mathbf{r},t) \quad \text{and} \quad m_{xy}(\mathbf{r},0) = m_0(\mathbf{r})$$

which has a solution:

$$m_{xy}(\mathbf{r},t) = m_0(\mathbf{r}) e^{-i\omega_0 t} e^{-t/T_2(\mathbf{r})} \exp\left(-i\gamma \int_0^t \Delta B(\mathbf{r},t')dt'\right)$$

It is also useful to note the spatially variant version of the signal in a frame rotating at ω_0:

$$m_{xy,rot}(\mathbf{r},t) = m_0(\mathbf{r}) e^{-t/T_2(\mathbf{r})} \exp\left(-i\gamma \int_0^t \Delta B(\mathbf{r},t')dt'\right)$$

Keep in mind that time, t, begins with each RF pulse the bring magnetization from the longitudinal axis into the transverse plane where it is observable.

Let's look at the example of a constant, linear variation in the applied field (known as a "gradient"). Specifically, let the variation be the x direction, $\Delta B(\mathbf{r},t) = G_x \cdot x$, then the solution to the Bloch equation is:

$$m_{xy}(\mathbf{r},t) = m_0(\mathbf{r})e^{-i\omega_0 t}e^{-t/T_2(\mathbf{r})}\exp(-i\gamma G_x x t)$$

$$= m_0(\mathbf{r})e^{-i\omega_0 t}e^{-t/T_2(\mathbf{r})}\exp(-i\Delta\omega(x)t)$$

where the spins will precess at a frequency related to x location ($\Delta\omega(x) = \gamma G_x x$).

For a gradient in an arbitrary direction, G, the frequency-position relationship is $\Delta\omega(\mathbf{r}) = \gamma\mathbf{G}\cdot\mathbf{r}$, and the solution to the Bloch equation is:

$$\begin{aligned} m_{xy}(\mathbf{r},t) &= m_0(\mathbf{r})e^{-i\omega_0 t}e^{-t/T_2(\mathbf{r})}\exp(-i\gamma\mathbf{G}\cdot\mathbf{r}t) \\ &= m_0(\mathbf{r})e^{-i\omega_0 t}e^{-t/T_2(\mathbf{r})}\exp(-i\Delta\omega(\mathbf{r})t) \end{aligned}$$

For a time-varying gradient, we have $\Delta\omega(\mathbf{r},t) = \gamma\mathbf{G}(t)\cdot\mathbf{r}$, and the solution to the Bloch equation is:

$$\begin{aligned} m_{xy}(\mathbf{r},t) &= m_0(\mathbf{r})e^{-i\omega_0 t}e^{-t/T_2(\mathbf{r})}\exp\left(-i\gamma\int_0^t \mathbf{G}(t')\cdot\mathbf{r}\,dt'\right) \\ &= m_0(\mathbf{r})e^{-i\omega_0 t}e^{-t/T_2(\mathbf{r})}\exp(-i\phi(\mathbf{r},t)) \end{aligned}$$

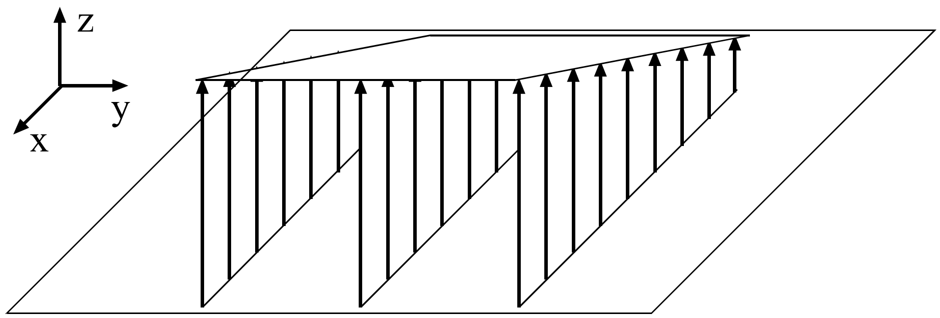
where $\phi(\mathbf{r},t)$ is a spatially and temporally varying phase variation.

Gradients

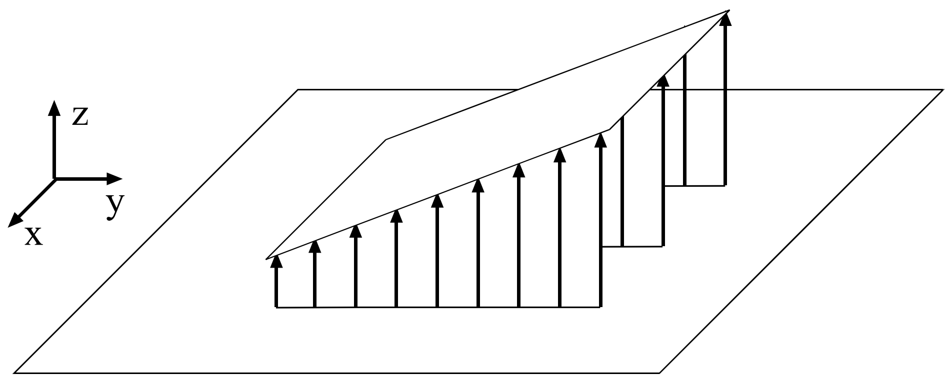
Gradient fields are the principle tool for localization in MRI.

It is important to remember the gradient fields vary along some spatial direction, but that field lines are aligned to the main magnetic field. For example:

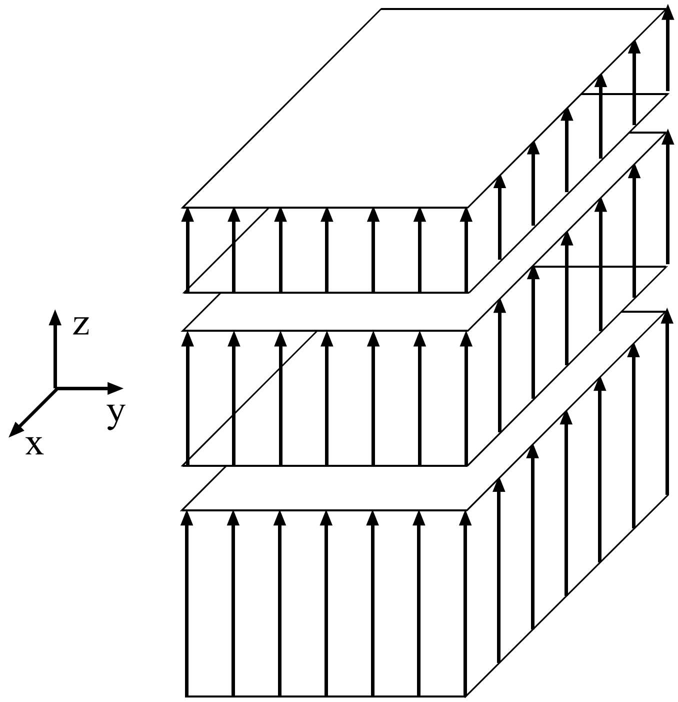
The X-Gradient



The Y-Gradient



The Z-Gradient

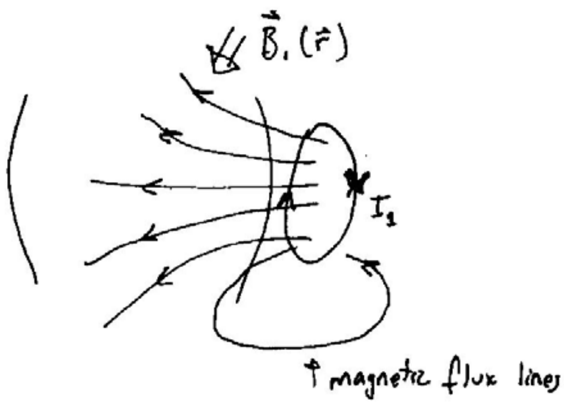


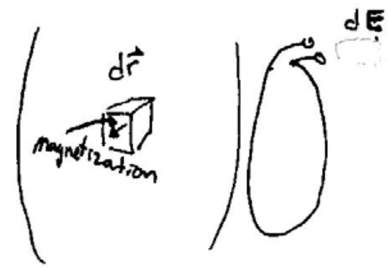
Signal Reception in MRI

The signal that we detect in MRI is a voltage induced in an RF coil by changes in magnetic flux from the precessing magnetization in the object. One expression for the voltage induced in a coil is:

$$E = -\frac{d\Phi}{dt}$$

where Φ is the flux in the coil. A common configuration is to use the same RF coil to transmit B_1 fields to the object and to receive signal from the magnetization. Assume, for a given coil configuration and current I_1, the RF field generated is $\mathbf{B_1}$. By the principle of reciprocity, the coil's receive sensitivity can be defined as $\mathbf{C_1} = \mathbf{B_1}/I_1$.





The incremental voltage produced by magnetization in an element $d\mathbf{r}$ is:

$$dE = -\left[\mathbf{C_1}(\mathbf{r}) \cdot \frac{\partial}{\partial t}\mathbf{m}(\mathbf{r},t)\right] d\mathbf{r}$$

The received signal is then:

$$s_r(t) = E = \int_V dE = -\int_V \left[\mathbf{C_1}(\mathbf{r}) \cdot \frac{\partial}{\partial t}\mathbf{m}(\mathbf{r},t)\right] d\mathbf{r}$$

We will now make a number of simplifying assumptions:

1. Note that $\dfrac{\partial m_z}{\partial t}$ varies at a rate similar to $1/T_1$, about 1 Hz, but $\dfrac{\partial m_{xy}}{\partial t}$ varies at a rate similar to ω_0, on the order of 10^7 Hz. Thus the voltage induced by the x-y components is about 7 orders of magnitude bigger than that induced by variations in the z component. Thus, we will consider only the x-y components of the sensitivity and the magnetization (e.g. $m_{xy}(\mathbf{r},t)$) in the above expression.
2. We will assume that coil geometry and placement will make it sensitive only the y-component of the magnetization and furthermore, that the sensitivity will be uniform across the volume of interest. Thus: $\mathbf{C_1}(\mathbf{r}) = \begin{bmatrix} 0 \\ C_{1y} \\ 0 \end{bmatrix}$.

Observe that $m_y(\mathbf{r},t) = \mathrm{Im}\{m_{xy}(\mathbf{r},t)\} = m_0(\mathbf{r}) e^{-t/T_2(\mathbf{r})} \sin\left(-\omega_0 t - \gamma \int_0^t \Delta B(\mathbf{r},t')dt'\right)$

Under these assumptions:

$$s_r(t) = -\int_V C_{1y} \frac{\partial}{\partial t}\left\{m_0(\mathbf{r}) e^{-t/T_2(\mathbf{r})} \sin\left(-\omega_0 t - \gamma \int_0^t \Delta B(\mathbf{r},t')dt'\right)\right\} d\mathbf{r}$$

where the derivative of the magnetization is:

$$\frac{\partial}{\partial t} m_y(\mathbf{r},t) = \frac{\partial}{\partial t}\left\{ m_0(\mathbf{r}) e^{-t/T_2(\mathbf{r})} \sin\left(-\omega_0 t - \gamma \int_0^t \Delta B(\mathbf{r},t')dt' \right)\right\}$$

$$= \left(-\omega_0 - \frac{1}{T_2(\mathbf{r})} - \gamma \Delta B(\mathbf{r},t) \right) m_0(\mathbf{r}) e^{-t/T_2(\mathbf{r})} \cos\left(\omega_0 t + \gamma \int_0^t \Delta B(\mathbf{r},t')dt' \right)$$

Further simplifications:

3. The terms at the front of the second line of this expression can be simplified by observing that $\omega_0 >> 1/T_2$ and $\omega_0 >> \gamma \Delta B = \gamma \mathbf{G} \cdot \mathbf{r}$ and thus the leading terms can be replaced by just $-\omega_0$.
4. We will neglect T2 decay, $e^{-t/T_2(\mathbf{r})}$ (for now).
5. We will absorb the coil sensitivity, ω_0 into a constant $C = C_{1y}\omega_0$, and for convenience, we will set this constant to 1: $C = 1$.

Now, the coil sensitivity times the derivative of the magnetization is:

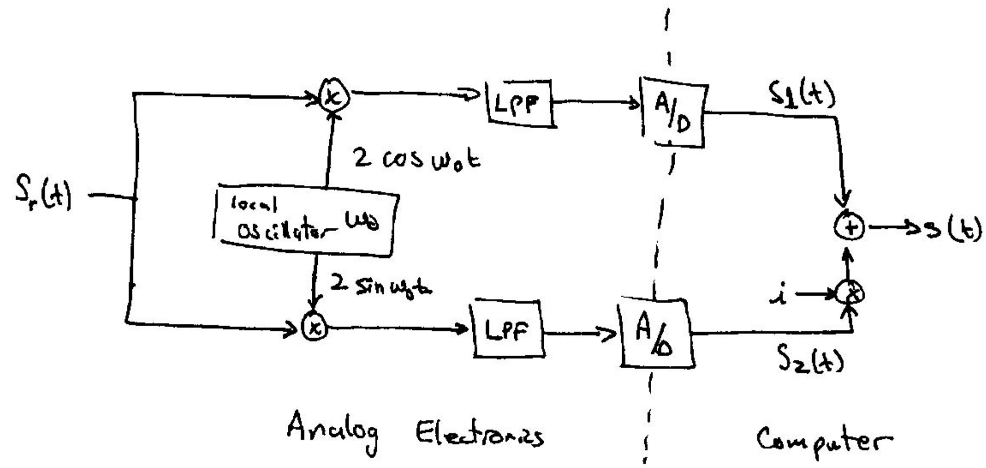
$$C_{1y} \frac{\partial}{\partial t} m_y(\mathbf{r},t) = -m_0(\mathbf{r}) \cos\left(\omega_0 t + \gamma \int_0^t \Delta B(\mathbf{r},t')dt' \right)$$

and the received signal is:

$$s_r(t) = \int_V m_0(\mathbf{r}) \cos\left(\omega_0 t + \gamma \int_0^t \Delta B(\mathbf{r},t')dt' \right) d\mathbf{r}$$

Complex Demodulation

The received signal, $s_r(t)$, is a real-valued voltage. We transform this to a baseband signal using a complex demodulator, as shown here:



We first look at the upper and lower channels of the complex demodulator for a single component of the received signal at location **r**. The upper channel of the demodulator yields:

$$s_1(t) = LPF\{2m_0(\mathbf{r})\cos(\omega_0 t + \phi(\mathbf{r},t))\cos(\omega_0 t)\}$$

$$= LPF\{m_0(\mathbf{r})[\cos(\phi(\mathbf{r},t)) + \cos(2\omega_0 t + \phi(\mathbf{r},t))]\}$$

$$= m_0(\mathbf{r})\cos(\phi(\mathbf{r},t))$$

and the lower channel yields:

$$s_2(t) = LPF\{2m_0(\mathbf{r})\cos(\omega_0 t + \phi(\mathbf{r},t))\sin(\omega_0 t)\}$$

$$= LPF\{m_0(\mathbf{r})[-\sin(\phi(\mathbf{r},t)) + \sin(2\omega_0 t + \phi(\mathbf{r},t))]\}$$

$$= -m_0(\mathbf{r})\sin(\phi(\mathbf{r},t))$$

We can then construct the combined signal, $s(t)$:

$$s(t) = s_1(t) + is_2(t)$$

$$= m_0(\mathbf{r})\exp(-i\phi(\mathbf{r},t))$$

The Signal Equation. It is relatively easy to show that the above demodulation process is linear (scaling and superposition), so in general we will get:

$$s(t) = \int_V m_0(\mathbf{r}) \exp(-i\phi(\mathbf{r},t))d\mathbf{r}$$

$$= \int_V m_0(\mathbf{r}) \exp\left(-i\gamma\int_0^t \Delta B(\mathbf{r},t')dt'\right)d\mathbf{r}$$

$$= \int_V m_{xy,rot}(\mathbf{r},t)d\mathbf{r}$$

Thus, the baseband signal, $s(t)$, can be represented by the integral over the transverse component of the transverse magnetization in the rotating frame.

Important points!

1. Through complex demodulation, we have access to the signal in the rotating frame where the frame frequency is determined by the local oscillator of the demodulator.
2. The RF coil integrates this signal from the entire object (or for the part of the object to which the coil is sensitive.

k-Space

We first will look at the case of planar imaging, that is, we will let $m_{xy,rot}(\mathbf{r},t) = m_{xy,rot}(x,y,t)$, the spatially variant transverse magnetization. The received signal is then:

$$s(t) = C\iint m_{xy,rot}(x,y,t)dxdy$$

Again, please bear in mind that we are mixing coordinate systems here. For example, in $m_{xy,rot}(x,y,t)$, the x,y in the argument refers to physical (x,y) locations in space, whereas the xy in the subscript refers to a mini-coordinate frame to describe direction of the magnetization vector at each point in space.

Let's consider a spatially and temporally varying applied magnetic fields introduced by time-varying gradient fields:

$$B(\mathbf{r},t) = B(x,y,t) = B_0 + G_x(t)\cdot x + G_y(t)\cdot y$$

The instantaneous frequency at each point is space is then:

$$\gamma B(x,y,t) = \gamma\left(B_0 + G_x(t)\cdot x + G_y(t)\cdot y\right)$$

which in the rotating frame is:

$$\Delta\omega(x,y,t) = \gamma\left(G_x(t)\cdot x + G_y(t)\cdot y\right)$$

and the spatially variant phase distribution is:

$$\phi(\mathbf{r},t) = \phi(x,y,t) = \int_0^t \gamma\left(G_x(\tau)\cdot x + G_y(\tau)\cdot y\right)d\tau$$

where time, t, begins with each RF pulse the bring magnetization from the longitudinal axis into the transverse plane where it is observable.

The Signal Equation, revisited. For convenience, we will let $C = 1$ and we will define $m(x,y) = m_0(x,y) = m_0(\mathbf{r})$. We now get a revised version of the signal equation:

$$\begin{aligned}
s(t) &= \iint m_{xy,rot}(x,y,t)\,dxdy \\
&= \iint m(x,y)\exp(-i\phi(x,y,t))\,dxdy \\
&= \iint m(x,y)\exp\left(-i\int_0^t \gamma\left(G_x(\tau)\cdot x + G_y(\tau)\cdot y\right)d\tau\right)dxdy \\
&= \iint m(x,y)\exp\left(-i\gamma\left(\int_0^t G_x(\tau)d\tau\cdot x + \int_0^t G_y(\tau)d\tau\cdot y\right)\right)dxdy
\end{aligned}$$

Finally, we define two quantities:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

And substituting into the above signal equation:

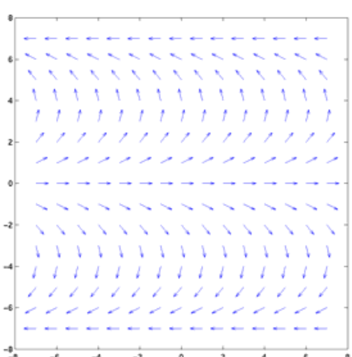
$$\begin{aligned} s(t) &= \iint m(x,y)\exp\left(-i2\pi\left(xk_x(t)+yk_y(t)\right)\right)dxdy \\ &= F_{2D}\{m(x,y)\}|_{u=k_x(t),v=k_y(t)} = M\left(k_x(t),k_y(t)\right) \end{aligned}$$

That is, the signal is equal to the Fourier transform of the initial magnetization evaluated at locations defined by k_x and k_y above. Reminder:

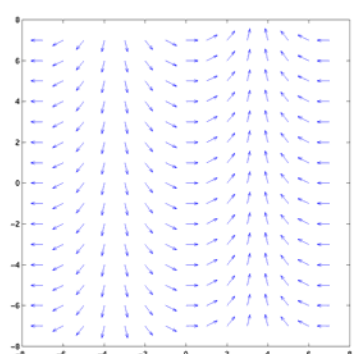
$$G(u,v) = F_{2D}\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i2\pi(xu+vy)}dxdy$$

The signal equation says that samples of the received signal are equal to samples of the 2D Fourier transform of the object. This make sense if we think about what exactly the expression for the 2D FT means – the FT at any point (u,v) is the integral over the object modified by a spatially variant (linear) rotation in the complex plane.

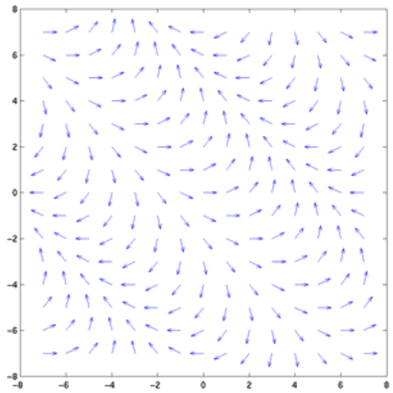
exp(-$i2\pi y$)



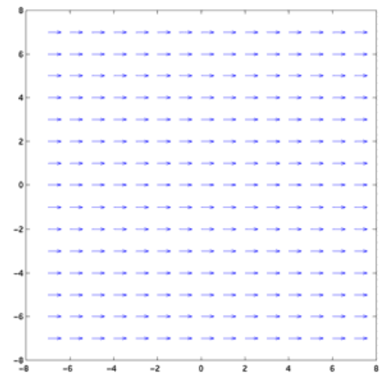
exp(-$i2\pi x$)



$\exp(-i2\pi(x+y))$



$\exp(-i0)$



In MRI, the integration is performed by the integration of voltages in the RF coil. The phase variation is performed by the gradients – by shifting the field (and thus frequency) in a spatially linear fashion for a period of time, the magnetization will rotate to a new orientation (in the complex plane). Thus MRI has exactly the same mechanisms as the FT operation.

K-space always begins at the origin (0,0). Why? After the excitation pulse, all spins across the object are pointing in the same direction (e.g. $\exp(-i0)$) and the integral of this is the DC value of the FT.

If we want to determine the object, we must fully sample its Fourier transform. A sequence of samples can be viewed as samples along a pathway determined by "running integral" under the gradient waveforms as defined by k_x and k_y above. The final object $m(x,y)$ can be reconstructed simply by taking the inverse 2D FT of the sampled Fourier data (k-space data):

$$m(x,y) = F_{2D}^{-1}\left\{M\left(k_x(t), k_y(t)\right)\right\}$$

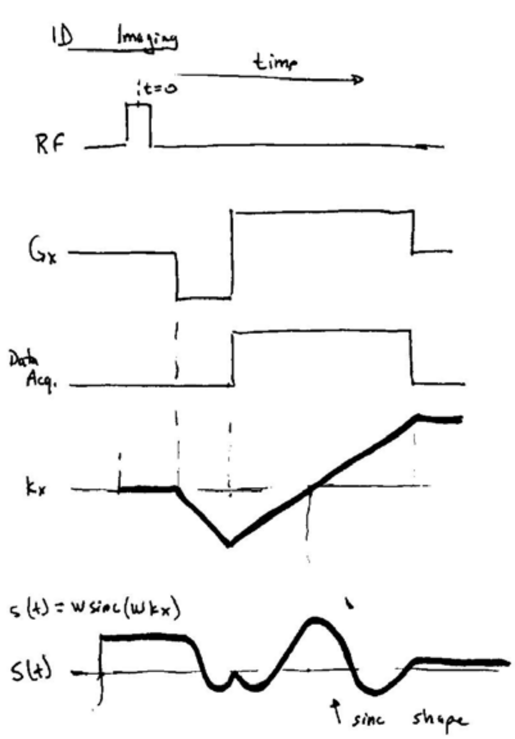
Does it make sense that our samples in time are actually samples of spatial frequency data? Remember the 1D case – we sampled time, FT'ed to get a spectrum. Since there was a 1-1 correspondence between frequency and spatial position, the FT of the frequency data produces time-domain data thus there is 1-1 correspondence between time and spatial-frequency.

Comments:

- Fourier space is called "k-space" in the MRI literature
- $G_x(t)$ and $G_y(t)$ control the k-space "trajectories" or paths on which sample locations fall.
- To create an image, we must sample $M(u,v)$ densely enough to prevent aliasing and of a large enough extent to have sufficient spatial resolution.

1D Imaging

We first examine the case of a 1D object $m(x) = \text{rect}(x/W)$. The received signal will be the Fourier transform $M(u) = W\,\text{sinc}(Wu)$ evaluated at particular k_x locations as dictated by the integration of the G_x gradient waveform. Presented here is "pulse sequence" for a 1D imaging experiment along with the k-space values and the received signal:

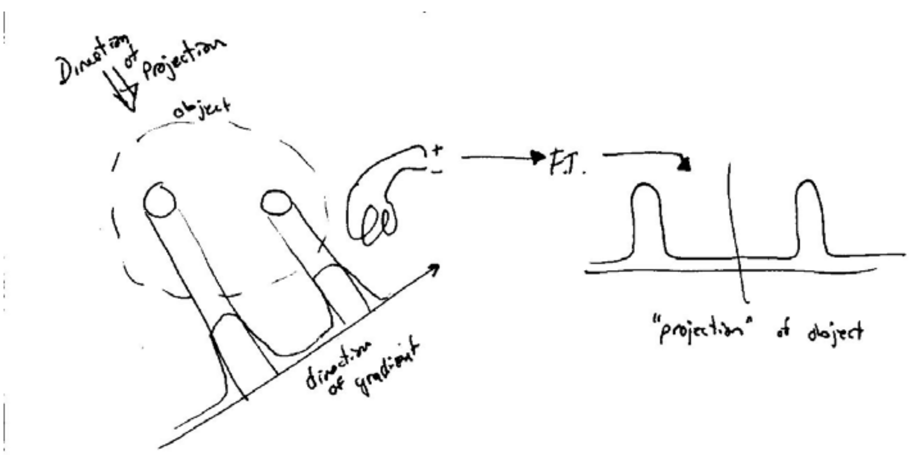


Recall the in the first MRI lecture we talked about taking the FT of the received signal to get a 1D view of the object. In this case, the received signal is a sinc function and thus the 1D FT of the sinc function is a rect function, which is in fact the object. During the entire time that data is acquired (see the Data Acq. line in the pulse sequence) the gradient is constant – during this situation there is 1-1 correspondence between frequency and spatial position. This is know as "frequency encoding" since spatial location is encoded as frequency.

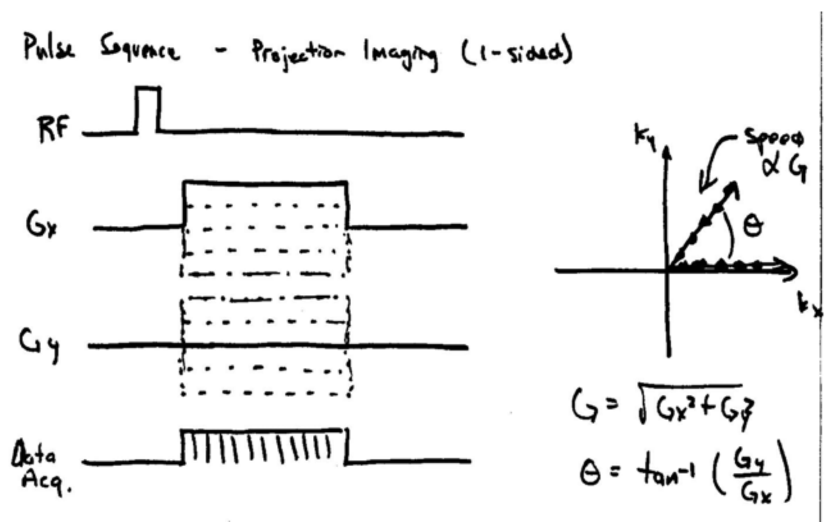
Notice also that we use a negative gradient before the positive gradient. Without the negative gradient, we can only acquire the positive spatial frequencies – or only ½ of the FT of the object.

2D Imaging using Projections

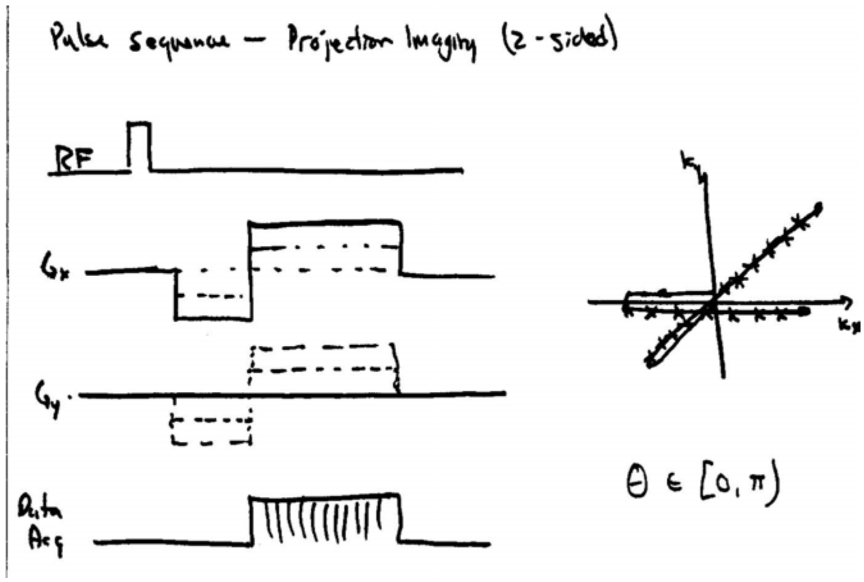
The first 2D imaging method implemented in MRI (by Paul Lauterbur while at SUNY Stony Brook), used a series of 1D acquisition with the gradients in different directions. Please not that by applying 1D gradients in x and y simultaneously, we get a single 1D gradient at an angle $\theta = \tan^{-1}(G_y/G_x)$. Thus we can get 1D views of the object (or projections) from many different angles.



We'll discuss (in the section on computed tomography) the methods for reconstructing images from 1D projections. For now, suffice it to say that if we acquire enough projections, we can fully determine the underlying object. The pulse sequence used by Lauterbur is given here:

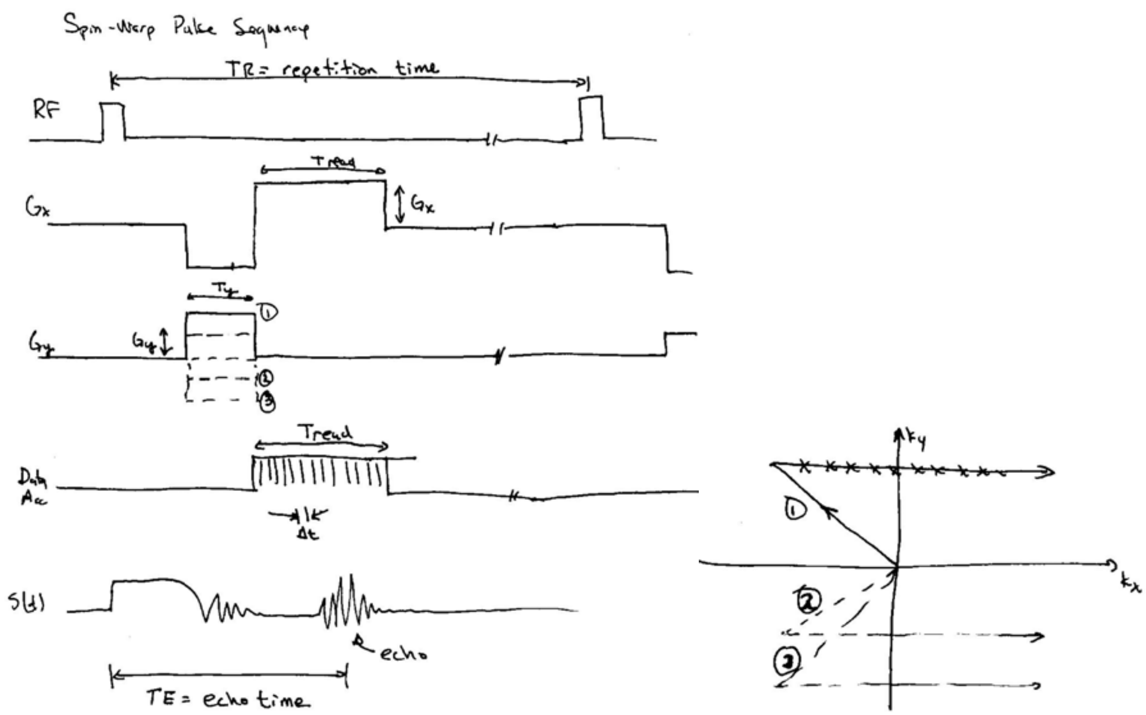


By sweeping through angles [0,2π) we acquire the full k-space (Fourier) data for the object. There is also a variant on projection imaging in which the positive gradient is preceded by a negative gradient – this will allow both positive and negative frequencies to be acquired along a particular line in k-space. One advantage to this approach is that one only needs to sweep through angles [0,π) in order to fully acquire the k-space data.



2D Spin-Warp Imaging

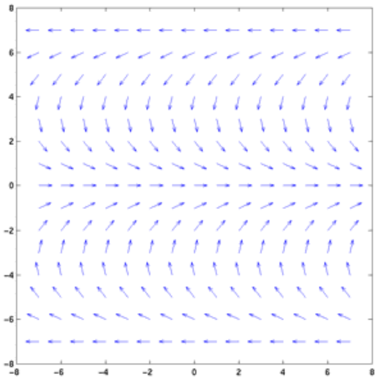
The most common acquisition used in MRI today is known as the "spin-warp" acquisition. The pulse sequence is given here along with the corresponding k-space trajectory:



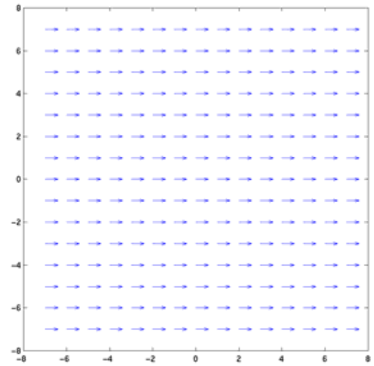
This is a repeated pulse sequence with a different y-gradient value for each RF excitation (TR interval). Let's look at the x-gradient – as in the case of 1D imaging, above, this gradient encodes the *x* spatial position into frequency. This is often called the "frequency encoding" gradient or *x*, in this case, is known as the "frequency direction."

The y-gradient is on briefly before each acquisition but is not on during data acquisition. Thus, whatever encoding performed by the y-gradient is done. In this case, the y-gradient sets up a spatially dependent phase distribution that remains fixed during the frequency encoding process. In other words, the y-gradient encodes spatial position into the phase of the magnetization (direction of the **m** vector), which is known as "phase encoding." *y*, in this case, is known as the "phase direction." Below are depictions of the phase distribution set up for the –1, 0, +1 and +2 phase encoding steps. These correspond to –1, 0, +1 and +2 cycles of phase across the field of view, respectively.

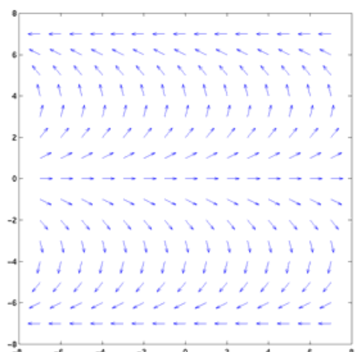
exp(-$i\gamma$(-1)$\Delta G_y T_y$)



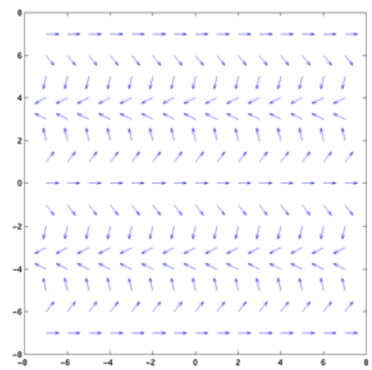
exp(-$i\gamma$(0)$\Delta G_y T_y$)



exp(-$i\gamma$(1)$\Delta G_y T_y$)



exp(-$i\gamma$(2)$\Delta G_y T_y$)



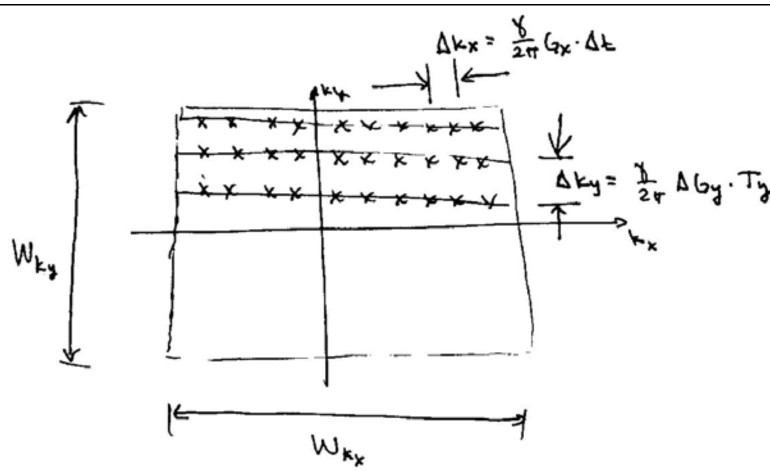
In terms of parameters described in the above pulse sequence, we can define several parameters of interest in the acquired space. The sample spacing and width of the k-space are:

$$\Delta k_x = \frac{\gamma}{2\pi} G_x \Delta t$$

$$\Delta k_y = \frac{\gamma}{2\pi} \Delta G_y T_y$$

$$W_{kx} = N_x \Delta k_x = \frac{\gamma}{2\pi} G_x T_{read}$$

$$W_{ky} = N_y \Delta k_y = \frac{\gamma}{2\pi} 2 G_{y,\max} T_y$$



Sampling in k-space (spatial frequency domain). Previously we discussed sampling of the object and its effect on the spectrum. Here we have the reverse – sampling in Fourier domain and its effect on the reconstructed object. Again, we will perform our sampling by multiplying a function times the 2D comb function. With sample spacing of Δk_x and Δk_y, in the k_x and k_y directions, the sampled Fourier data is:

$$\widetilde{M}(u,v) = M(u,v)\mathrm{comb}\left(\frac{u}{\Delta k_x},\frac{v}{\Delta k_y}\right)$$

$$= \Delta k_x \Delta k_y \sum_{n,m=-\infty}^{\infty} \delta(u - n\Delta k_x, v - m\Delta k_y) M(n\Delta k_x, m\Delta k_y)$$

The image (space) domain equivalent is:

$$\widetilde{m}(x,y) = m(x,y) ** \Delta k_x \Delta k_y \mathrm{comb}(\Delta k_x u, \Delta k_y v)$$

$$= m(u,v) ** \sum_{n,m=-\infty}^{\infty} \delta\left(u - \frac{n}{\Delta k_x}, v - \frac{m}{\Delta k_y}\right)$$

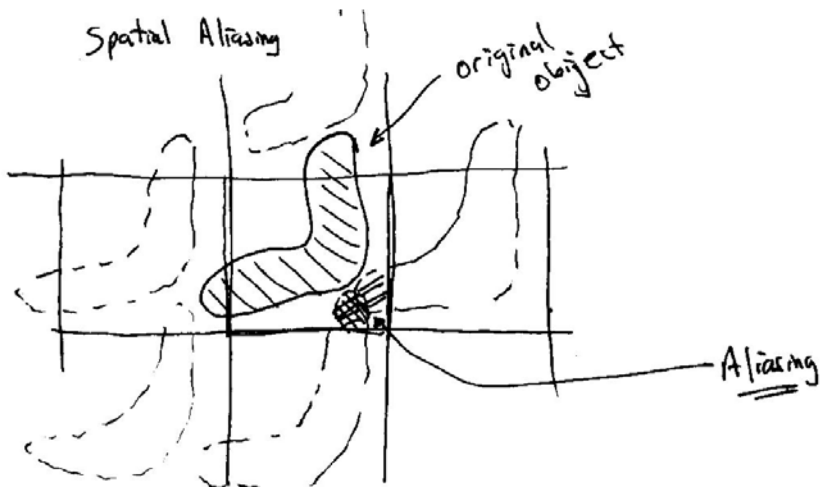
$$= \sum_{n,m=-\infty}^{\infty} m\left(u - \frac{n}{\Delta k_x}, v - \frac{m}{\Delta k_y}\right)$$

Thus, sampling in the Fourier domain leads to replication in the image domain. Spacing of the replicated image (object) is $(1/\Delta k_x, 1/\Delta k_y)$. The replicated images will not overlap the original image if the highest spatial position in x is $x_{\max} \le \frac{1}{2\Delta kx}$ and the highest spatial position in y is $y_{\max} \le \frac{1}{2\Delta ky}$. If this is not satisfied, then there will be spatial overlap in the images (or aliasing).

The field of view of an acquisition is typically defined as one over the k-space sample spacing:

$$\mathrm{FOV}_x = 1/\Delta k_x \quad \text{and} \quad \mathrm{FOV}_y = 1/\Delta k_y$$

and aliasing will not occur if $x_{max} < ½\ \mathrm{FOV}_x$ and $y_{max} < ½\ \mathrm{FOV}_y$.



Point Spread Function.

Observe that the practical k-space is not of infinite extent, but rather is limited to W_{kx} and W_{ky}. The sampled k-space can be written as:

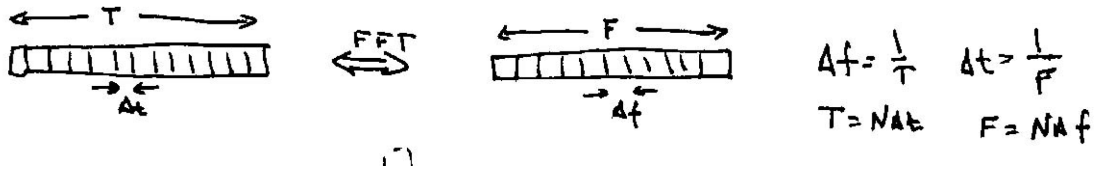
$$\widetilde{M}(u,v) = M(u,v)\mathrm{rect}\left(\frac{u}{W_{kx}},\frac{v}{W_{ky}}\right)\mathrm{comb}\left(\frac{u}{\Delta k_x},\frac{u}{\Delta k_y}\right)$$

which results in an image of the following form:

$$\widetilde{m}(x,y) = m(x,y) ** W_{kx}W_{ky}\mathrm{sinc}(W_{kx}x)\mathrm{sinc}(W_{ky}y) ** \Delta k_x \Delta k_y \mathrm{comb}(\Delta k_x u, \Delta k_y v)$$

The sinc functions are the point spread function and the comb function generates replicated versions of the object (does the aliasing). Observe that the sinc functions have approximate widths (in x and) of $\Delta x = 1/W_{kx}$ and $\Delta y = 1/W_{ky}$. This defines, in essence, the spatial resolution of an MRI acquisition – in order to get better (finer) spatial resolution, we need to acquire a larger area in k-space.

Resolution of the FFT. Most forms of the FFT work this way – for an N point input function the FFT will produce an N point output. Each output point corresponds to an integer number, n, of cycles in $\exp(-i2\pi nx)$ across the object and go from n = [-N/2:N/2-1]. Observe the DFT of -N/2 and N/2 are the same and thus this represents the entire, unaliased frequency domain of the object. Thus F/2 (see below) = ½ of $1/\Delta t$ or $F = 1/\Delta t$. A similar argument can be made in reverse to get $T = 1/\Delta f$.



Resolution and Object and Sample Spacing in MRI. For data acquired on a 2D rectilinear grid in k-space and reconstructed with a 2D FFT the spatial resolution and Field of View relationships are:

$$\mathrm{FOV}_x = 1/\Delta k_x \quad \text{and} \quad \mathrm{FOV}_y = 1/\Delta k_y$$

$$\Delta x = 1/W_{kx} \quad \text{and} \quad \Delta y = 1/W_{ky}$$