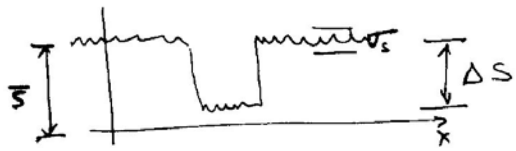
X-Ray Notes, Part III

Noise in Detectors

Consider an output to an x-ray system that looks like this:



We define a number of quantities (slightly different definitions that used by Macovski):

Contrast: $C = \Delta S / \bar{S}$

Signal to Noise Ratio: $SNR = \bar{S} / \sigma_s$

Contrast to Noise Ratio: $CNR = \Delta S / \sigma_s = C \cdot SNR$

Previously, we described the SNR for a system having pixels distributed according to a Poisson R.V. If the mean value of the photon counts for a pixel is $\mu = N$, then the signal to noise ratio of for that pixel will be:

$$SNR = \frac{\bar{S}}{\sigma_S} = \frac{N}{\sqrt{N}} = \sqrt{N}$$

The probability density function for the Poisson R.V. is:

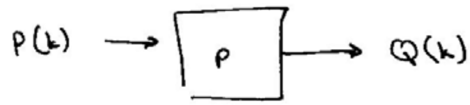
$$p(k) = \frac{e^{-\mu} (\mu)^k}{k!}$$

and

$$E[k] = N$$

$$E[k^2] = N^2 + N$$

We will now describe the probability distribution of detected photons. Suppose the incident x-ray photons arriving at the detector are Poisson(N) and that the detector has efficiency η, as describe previously. We can view the detector as a binary random system in which the photon is detected with probability $p = \eta$:



where

$$
\begin{aligned}
Q(k) &= \Pr\{k \text{ photons are detected}\} \\
&= \sum_{n=0}^{\infty} \Pr\{k \text{ photons are detected}/n+k \text{ photons are incident}\} \cdot \Pr\{n+k \text{ photons are incident}\} \\
&= \sum_{n=0}^{\infty} \text{Binomial}(n+k)_k \cdot \text{Poisson}(N)_{n+k} \\
&= \sum_{n=0}^{\infty} \frac{(n+k)!}{k!n!} p^k (1-p)^n \cdot \frac{e^{-N} N^{n+k}}{(n+k)!} \\
&= \frac{e^{-N} p^k N^k}{k!} \sum_{n=0}^{\infty} \frac{N^n (1-p)^n}{n!} \\
&= \frac{e^{-N} p^k N^k}{k!} e^{N(1-p)} \\
&= \frac{e^{-pN} (pN)^k}{k!} \\
&= \text{Poisson}(pN)
\end{aligned}
$$

Thus, the detected photons are also Poisson distributed, but will have probability ηN and the SNR of the detected photons is now:

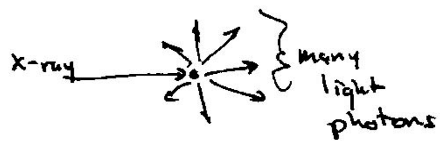
$$SNR_{\text{det}} = \frac{\eta N}{\sqrt{\eta N}} = \sqrt{\eta N}$$

Comments:

1. It is also easy to show that the number of photons that are not detected is also a Poisson process with parameter probability $(1-\eta)N$.
2. The sum of Poisson processes is also Poisson.
3. Finally, if the incident photons are Poisson, then the number of photons that reach the detector will also be Poisson. Attenuation processes that independently affect photons work exactly as above.

Cascaded Poisson Processes

Consider an x-ray photon that interacts with the detector and generates a shower of photons:



We can model the number of light photons, *Y*, as the number of random variables:

$$Y = \sum_{i=1}^{M} X_i$$

where X_i is the number of light photons for x-ray photon number *i* and *M* is the number of x-ray photons (Poisson(ηN)). For now, g_1 is also viewed as the gain of light conversion process). We now determined the characteristics of *Y*:

$$\bar{Y} = E[Y] = E_M\left[E_{Y/M}[Y]\right]$$

$$= E_M\left[E_{Y/M}\left[\sum_{i=1}^{M} X_i\right]\right]$$

$$= E_M\left[Mg_1\right]$$

$$= \eta N g_1$$

and

$$E[Y^2] = E_M\left[E_{Y/M}[Y]\right]$$

$$= E_M\left[E_{Y/M}\left[\sum_{i=1}^{M} X_i^2 + \sum_{i=1}^{M} \sum_{j \neq i} X_i X_j\right]\right]$$

$$= E_M\left[M(g_1^2 + g_1) + M(M-1)g_1^2\right]$$

$$= E_M\left[Mg_1 + M^2 g_1^2\right]$$

$$= \eta N g_1 + E[M^2]g_1^2$$

and finally:

$$\sigma_Y^2 = E[Y^2] - (\eta N g_1)^2$$

$$= \eta N g_1 + E[M^2] g_1^2 - (\eta N g_1)^2$$

$$= \eta N g_1 + g_1^2 \left(E[M^2] - (E[M])^2 \right)$$

$$= E[M]\sigma_X^2 + E[X]^2 \sigma_M^2$$

This has two components: the first represents the variance in the number of photons for a given sum length and the second represents the variation that comes from changes in the length of the sum. This can also be written as:

$$\sigma_Y^2 = \eta N g_1 + g_1^2 \left(E[M^2] - (E[M])^2 \right)$$

$$= \eta N g_1 + \eta N g_1^2$$

$$= \eta N (g_1 + g_1^2)$$

we can now write a new expression for the SNR:

$$SNR = \frac{\eta N g_1}{\sqrt{\eta N g_1 + \eta N g_1^2}} = \sqrt{\eta N} \frac{1}{\sqrt{1 + \frac{1}{g_1}}}$$

This last part of this expression can be thought of as the SNR degradation term. If the gain is very large, then this process will result in essentially no loss of SNR.

For additional cascaded Poisson processes, e.g.:

$$W = \sum_{i=1}^{Y} Z_i$$

where Z_i as Poisson(g_2) and then:

$$E[W] = E[Y]E[Z]$$

$$\sigma_W^2 = E[Y]\sigma_Z^2 + E[Z]^2 \sigma_Y^2$$

it follows that:

$$E[W] = \eta N g_1 g_2$$

$$\sigma_W^2 = \eta N g_1 g_2 + \eta N g_1 g_2^{\,2} + \eta N g_1^{\,2} g_2^{\,2}$$

and the SNR can be written as:

$$SNR = \sqrt{\eta N} \frac{1}{\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2}}}$$

For additional cascaded processes, this situation continues:

$$SNR = \sqrt{\eta N} \frac{1}{\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2} + \frac{1}{g_1 g_2 g_3} + \ldots}}$$

If any of the cascaded gains are small (or even close to one) then the SNR will be reduced. Thus, in the design of detector systems, it is important that the product of all of the gain is kept large. For example, we want $g_1 >> 1$, $g_1 g_2 >> 1$, $g_1 g_2 g_3 >> 1$, …

Example

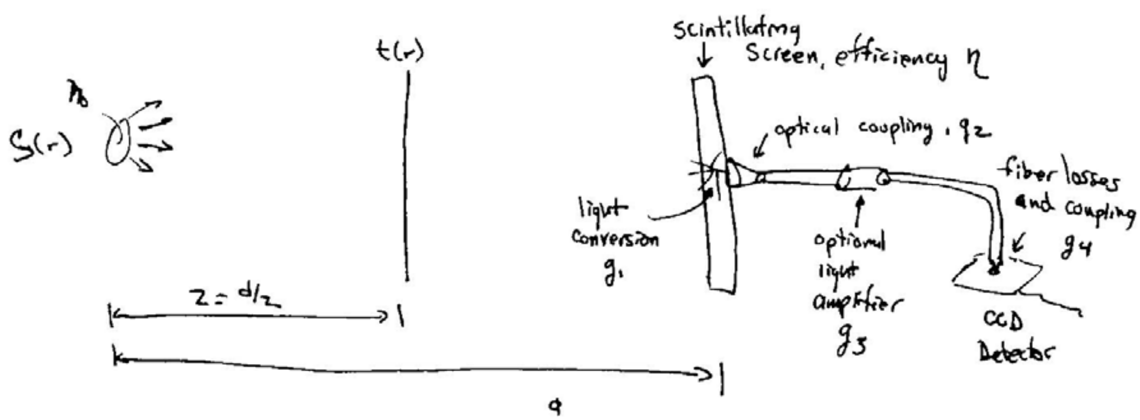
Consider a scintillating screen that produces $g_1 = 500$ light photons/x-ray photon detected and that it takes roughly 200 light photons to convert a silver halide particle to an observable grain in the developed film – that is, $g_2 = 1/200$. The SNR reduction factor will then be:

$$\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2}} = \sqrt{1 + \frac{1}{500} + \frac{1}{2.5}} \approx \sqrt{1 + \frac{1}{2.5}} = \sqrt{1.4} \approx 1.2$$

The $1/g_1$ term, which is large, doesn't contribute to any loss of SNR. The $1/g_1 g_2$ term is the source of SNR reduction.

Overall System Response and SNR Example

Let's consider the following system:



1. The source function be: $s(r) = n_0 \exp(-\pi(r^2 / a^2))$ where n_0 has units of photons/cm^2.
2. The transmission function is: $t(r) = \frac{1}{2} + \frac{1}{2}\exp(-\pi(r^2 / b^2))$ where $\frac{1}{2}\exp(-\pi(r^2 / b^2))$ represents the lesions that we are trying to detect. The contrast in the transmission function is $C = 1$.
3. We'll estimate the recorder response function as the region of light captured by the aperture of the fiber optic coupling: $h(r) = \exp(-\pi(r^2 / c^2))$.
4. Let a = 10 mm, b = 10 mm and c = 1 mm.
5. $z=d/2$: the object magnification factor is $M = 2$ and
   the source magnification factor is $m = -1$.
6. The gains are g_1 = 500 light photons/interaction, g_2 = 0.1 are captured by the optical coupling, the optional g_3 = 100 photons/photon in the light amplifier, and g_4 = 0.05 for the fiber losses, optical coupling and inefficiencies of the CCD.

The function in the transducer is:

$$
\begin{aligned}
I_d(r_d) &= \frac{1}{4\pi d^2 m^2} s\left(\frac{r_d}{m}\right) ** t\left(\frac{r_d}{M}\right) ** h(r_d) \\
&= \frac{n_0}{4\pi d^2 m^2}\frac{1}{2}\exp\left(-\pi\left(\frac{r_d}{am}\right)^2\right) ** \left[1 + \exp\left(-\pi\left(\frac{r_d}{bM}\right)^2\right)\right] ** \exp\left(-\pi\left(\frac{r_d}{c}\right)^2\right)
\end{aligned}
$$

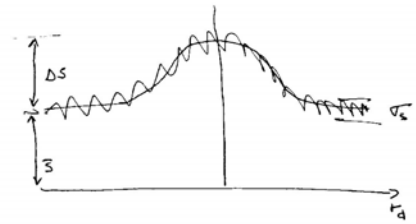
This convolution is most easily solved in the Fourier domain:

$$
\begin{aligned}
I_d(\rho) &= \frac{n_0 a^2 c^2}{4\pi d^2}\frac{1}{2}\exp\left(-\pi(am\rho)^2\right)\left[\delta(u,v) + b^2M^2\exp\left(-\pi(bM\rho)^2\right)\right]\exp\left(-\pi(c\rho)^2\right) \\
&= \frac{n_0 a^2 c^2}{8\pi d^2}\left[\delta(u,v) + b^2M^2\exp\left(-\pi\left(a^2m^2 + b^2M^2 + c^2\right)\rho^2\right)\right]
\end{aligned}
$$

and back to the image domain:

$$
I_d(r_d) = \frac{n_0 a^2 c^2}{8\pi d^2}\left[1 + \frac{b^2M^2}{\left(a^2m^2 + b^2M^2 + c^2\right)}\exp\left(-\pi\frac{r_d^{\,2}}{\left(a^2m^2 + b^2M^2 + c^2\right)}\right)\right]
$$

From this expression, it is clear the lesion contrast after the system response will be:



$$C = \frac{b^2 M^2}{\left(a^2 m^2 + b^2 M^2 + c^2\right)} = 0.798$$

which is reduced by about 20% from the input function. The number of x-ray photons that fall within a pixel (the area of the optical coupling) is:

$$N = \frac{n_0 a^2 c^2}{8\pi d^2}$$

The final SNR with out g_3 is:

$$CNR = \frac{b^2 M^2}{\left(a^2 m^2 + b^2 M^2 + c^2\right)} \sqrt{\eta \frac{n_0 a^2 c^2}{8\pi d^2}} \frac{1}{\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2} + \frac{1}{g_1 g_2 g_4}}}$$

where the SNR reduction factor is:

$$\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2} + \frac{1}{g_1 g_2 g_4}} = \sqrt{1 + \frac{1}{500} + \frac{1}{50} + \frac{1}{2.5}} \approx 1.2$$

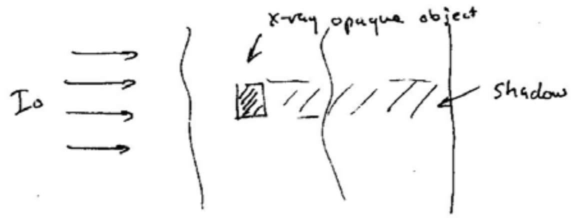
and with the optional light amplifier, the SNR reduction factor is:

$$\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2} + \frac{1}{g_1 g_2 g_3} + \frac{1}{g_1 g_2 g_3 g_4}} = \sqrt{1 + \frac{1}{500} + \frac{1}{50} + \frac{1}{5000} + \frac{1}{250}} \approx 1$$

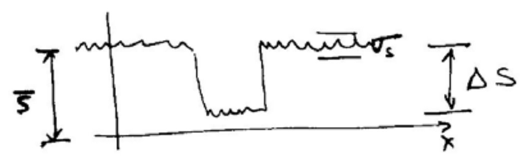
which demonstrates a common practice of inserting gain system (image intensifiers, for example) before the low gain components (including the human eye).

Compton Scattered X-rays

Consider the following object with an x-ray opaque core:

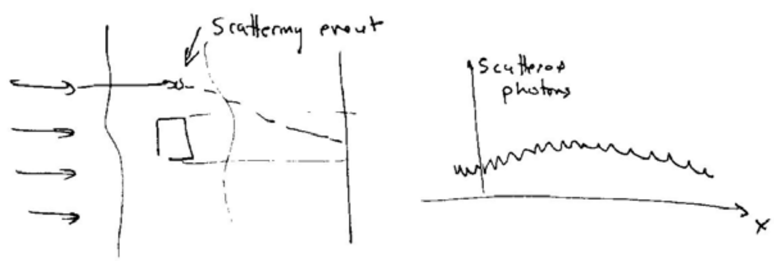


the output image might look like this:



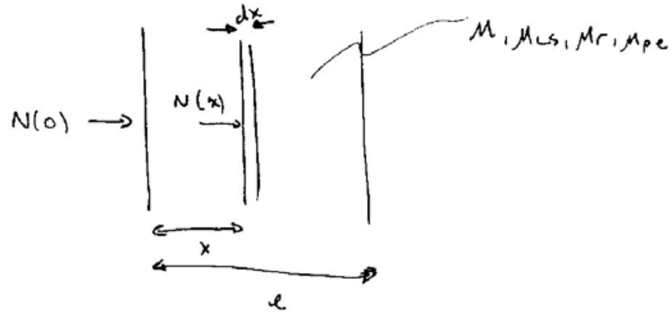
on which we can define a contrast $C = \Delta S / \overline{S}$ and a contrast to noise ratio $CNR = \Delta S / \sigma_s$. Now, consider that the scattered photons – here some fraction of the scattered photons will scatter forward and will generate additional photons in the final image.

The distribution of the scattered photons will look something like the object convolved with the forward scattering distribution. The final image will be the sum of the transmitted photons and the scattered photons.



By increasing $\overline{S}$ and σ_s the scattered photons will reduce both the contrast and the contrast to noise ratio.

How many photons are scattered? (Derived from Macovski, Problem 3.4) Let's look at an object of length *l* having an attenuation coefficient $\mu = \mu_{rt} + \mu_{pe} + \mu_{cs}$. Let $N(0)$ be the number photons incident upon the object and that the number of photons that have not interacted at depth *x* is *N*(*x*).



The number of scattered photons in an interval *dx* will be:

$$N_{cs}(x) = \mu_{cs} N(x) dx$$

and the total number of scattered photons will be:

$$N_{cs} = \int_0^l N_{cs}(x) dx$$

$$= \int_0^l \mu_{cs} N(x) dx$$

$$= \int_0^l \mu_{cs} N(0) \exp(-\mu x) dx$$

$$= \frac{\mu_{cs}}{\mu} N(0)(1 - \exp(-\mu l))$$

Note that $N(0)(1\text{-}\exp(-\mu l))$ is the total number of photons that interact with the object.

Additive Noise in X-Ray Imaging

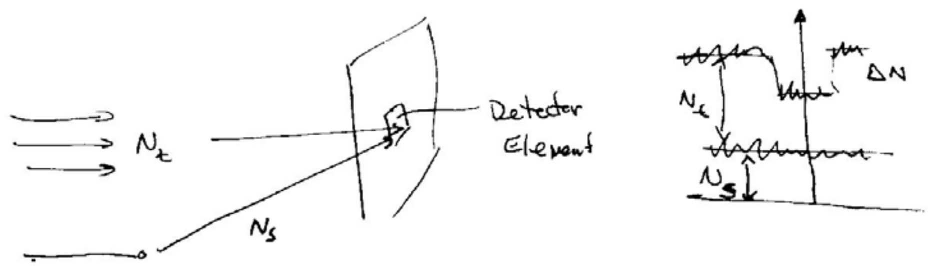
There are two kinds of additive noise that we will consider. The first is zero-mean additive noise, for example, electronic noise in digitized images. In this case, the mean value doesn't change, but the variance does. In most cases, the additive noise will be independent of the Poisson variation in the received photons and thus, the variances will add:

$$\sigma^2 = \sigma_s^2 + \sigma_a^2$$

where σ_a^2 is the variance of the additive noise and σ_s^2 is the Poisson variance. Thus, the SNR is:

$$SNR = \frac{\bar{S}}{\sigma} = \frac{\eta N}{\sqrt{\eta N + \sigma_a^2}} = \sqrt{\eta N}\,\frac{1}{\sqrt{1 + \sigma_a^2 / \eta N}}$$

The other kind of additive noise that we will consider is scatter. Since scatter is also Poisson distributed, it isn't zero mean, and as we've discussed before, it does affect the contrast.



Consider the case where we have $N_t = N$ transmitted photons and N_s scattered photons and a signal difference of ΔN. The original contrast was:

$$C = \frac{\Delta N}{N}$$

and our reduced contrast is:

$$C_r = \frac{\Delta N}{N + N_s}$$

We also know that the variance of the background signal will be the sum of the variances of the to constituent Poisson processes:

$$\sigma = \sqrt{\eta N + \eta N_s}$$

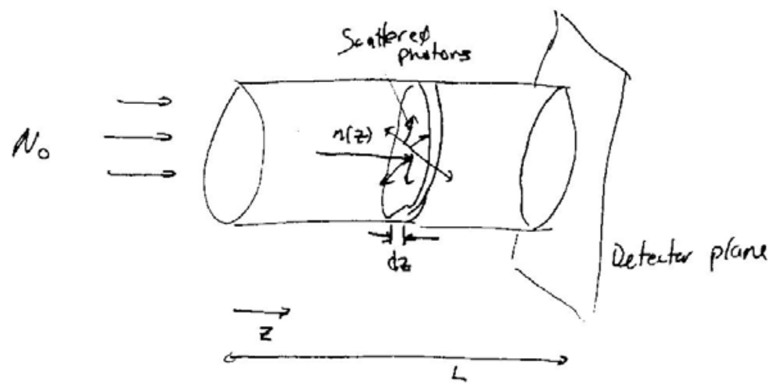
and thus, the reduced contrast to noise ratio will be:

$$CNR_r = \frac{\eta \Delta N}{\sqrt{\eta N + \eta N_s}} = C\,\frac{\eta N}{\sqrt{\eta N + \eta N_s}} = C\sqrt{\eta N}\,\frac{1}{\sqrt{1 + N_s/N}}$$

The scatter CNR reduction factor is $\sqrt{1 + N_s/N}$ or $\sqrt{1+\Psi}$, where $\Psi = \frac{N_s}{N}$ the ratio of scatter photons to transmitted photons.

SNR Reduction

To get an idea of how many photons are scattered and strike the detector, we can look at an example with an isotropic object with attenuation coefficient μ and an scattering component μ_s:



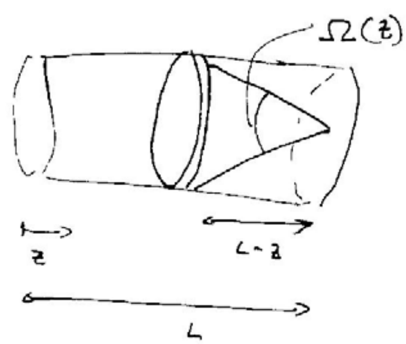
First, the number of scattered photons generated in each incremental thickness is:

$$n_s(z) = n(z)\mu_s dz = N_0 e^{-\mu z}\mu_s dz$$

We make a variety of assumptions:

1. Ignore obliquities
2. Assume a parallel ray geometry for incident intensity
3. Assume μ is energy independent
4. Neglect multiple scatters
5. Assume isotropic scattering

Thus, for the number of scattered photons, some fraction, $F(z)$, will be captured:



$$F(z) = \frac{\Omega(z)}{4\pi} e^{-\mu(L-z)}$$

The number of scattered photons at the detector will then be:

$$N_s = \int_0^L F(z) n_s(z) dz = \int_0^L \frac{\Omega(z)}{4\pi} e^{-\mu(L-z)} N_0 e^{-\mu z} \mu_s dz = e^{-\mu L} N_0 \mu_s \int_0^L \frac{\Omega(z)}{4\pi} dz = N_t \mu_s G$$

where G is a geometric, object dependent factor (which has units of length). For thin and wide object, $G \rightarrow 0.5L$ (this results from $\Omega(z) = 2\pi$) and for long, slim objects $G \rightarrow 0$. Therefore:

$$\Psi = \frac{N_s}{N} = \mu_s G$$

If we take typical values for attenuation coefficients for water at 100 keV, $\mu \approx \mu_s \approx 0.16\,\mathrm{cm}^{-1}$, and L = 20 cm and we will let $G = 0.4L$, then:

$$\Psi = 1.28$$

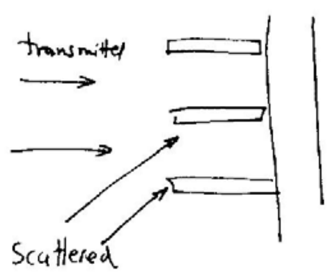
resulting in a reduction of SNR of:

$$\sqrt{1+\Psi} = 1.5$$

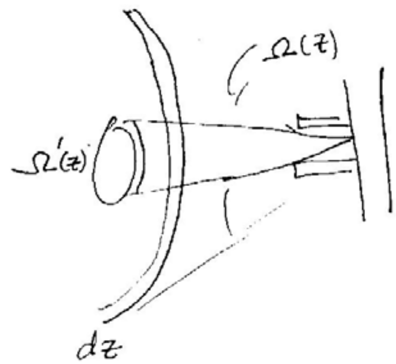
a 50% reduction in SNR.

Scatter Reduction Grids

The most common way of reducing scatter is through the use of a scatter reduction grid:



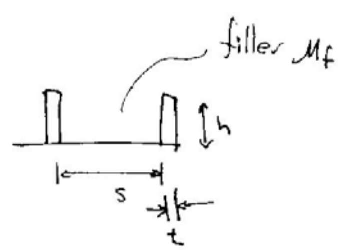
where the gird is made out of some high μ material (like Pb, W) that will block any photons that strike it. The grid works principally by cutting down on the acceptance angle for scattered photons, Ω:



We can define a scatter reduction factor:

$$R_s = \frac{\int \Omega'(z)dz}{\int \Omega(z)dz}$$

where Ω'(z) is the acceptance angle of the scatter reduction grid. In addition to reducing the scatter, this also results in a reduction of transmitted photons. We can define an efficiency of the grid by considering transmitted photons blocked by grid and attenuated by the filler material:



in equations, this will be:

$$\eta_t = \frac{s-t}{s}\exp(-\mu_f h)$$

The CNR will now be:

$$CNR = C\frac{\sqrt{\eta\eta_t N}}{\sqrt{\eta\eta_t N + \eta R_s N_s}} = C\sqrt{\eta\eta_t N}\frac{1}{\sqrt{1+R_s\Psi/\eta_t}}$$

where $\sqrt{1+R_s\Psi/\eta_t}$ is the new SNR reduction factor.

The Relationship of a Poisson Process to the Exponential R.V.

Let T be an exponential R.V. that describes the time between events in a Poisson process. The derivation follows. Recall that the probability that an event occurs in interval Δt will be $p = \lambda\Delta t$. Also, note that the probability that no event occurs in interval Δt will be $q = (1 - \lambda\Delta t)$. Now, suppose the we want to know what is the probability that no event occurred between 0 and t. This is the same as saying that we have $N = t/\Delta t$ inverval in which no event can occur. If these intervals are independent (that is saying that the photons don't interact with each other or tend to come in groups or whatever), the probability that no event occurred between 0 and t will be q^N:

$$\Pr\{\text{no event occurs in } (0,\text{t})\} = (1-\lambda\Delta t)^{t/\Delta t}$$

we again determine this function as $\Delta t \rightarrow 0$:

$$\Pr\{\text{no event occurs in } (0,\text{t})\} = \lim_{\Delta t \to 0}\left[(1-\lambda\Delta t)^{-1/(\lambda\Delta t)}\right]^{-\lambda t} = e^{-\lambda t}$$

The probability density function of T, $f(t)$, describes the probability that an event occurs at time t and probability distribution function of T (integral of $f(t)$) describes the probability that an event occurs by time t will be equal to:

$$F(t) = 1 - \Pr\{\text{no event occurs in } (0,\text{t})\} = \begin{cases} 1 - e^{-\lambda t}, \text{ for } t \geq 0 \\ 0, \text{ for } t < 0 \end{cases}$$

and the probability density function is the derivative of this function:

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, \text{ for } t \geq 0 \\ 0, \text{ for } t < 0 \end{cases}$$

The exponential R.V. is a continuous R.V. of the times between events and is described as:

$$T \sim \text{Exponential}(\lambda)$$

which has a mean an variance of:

$$\overline{T} = \frac{1}{\lambda}$$

$$\sigma_T^2 = \frac{1}{\lambda^2}$$

Memoryless Property

The exponential R.V. is "memoryless," meaning that distribution (and density) of event times in the future is not affected by past events, that is, at any point in time, the time until the next event is an exponential R.V. with parameter λ. This is the same as saying that just because we haven't seen an event in a long time, we're not more likely to have an event soon. (Just like the "gambler's fallacy.") Specifically,

$$\Pr\{T > t + t_0 \mid T > t_0\} = \Pr\{T > t\}$$

which says "given that an event hasn't occurred by time t_0, the probability that an event will not occur by time ($t_0 + t$) will be same as the probability that no event occurs in (0,t)." Proof:

$$\Pr\{T > t + t_0 \mid T > t_0\} = \frac{1 - F(t + t_0)}{1 - F(t_0)}$$

$$= \frac{\exp(-\lambda(t + t_0))}{\exp(-\lambda t_0)} = \exp(-\lambda t) = 1 - F(t)$$

$$= \Pr\{T > t\}$$