X-Ray Notes, Part II

Noise in X-ray Systems (part 1)

In an x-ray system, images typically are created from intensity values that are related to the number of photons that strike a detector element in a finite period of time. The photons are generated by electrons randomly striking a source and thus the photons at the detector are also random in nature. We typically describe this kind of random process as one having a rate parameter, λ (units: events/time), and an observation time, T. Let X be the random variable (R.V.) that describes the number of events (photons striking the detector element) in time T.

X will be a Poisson distributed random variable with parameter λT. E.g.

$$X \sim \text{Poisson}(\lambda T)$$

Derivation of Poisson Distribution

Below, we will derive the Poisson distribution from a set of independent Bernoulli R.V.'s. Let Δt be some small time interval and $N = T/\Delta t$ be the number of independent trials. The probability of an event (photon) in interval Δt will be $\lambda\Delta t$. Each Bernoulli trail will then be an R.V.:

$$Y_i \sim \text{Bernoulli}(\lambda\Delta t)$$

$$Y_i = \begin{cases} \text{event A, with probability } p = \lambda\Delta t \\ \text{event B, with probability } q = 1 - p \end{cases}$$

We also assume that Δt is chosen to be small enough so that the probability that there are two events is very small (later we will let Δt go to zero, so this is a non-issue).

Now we consider the sum of the N events, which yields a binomial R.V.

$$X = \sum_{i=1}^{N} Y_i$$

$$X \sim \text{Binomial}(N, \lambda T)$$

The probability density function is $f(x)$ = Probability$\{X = x\}$ (the probability that there were x events in time T). For a binomial R.V., this is derived from the following:

$$\left.\begin{matrix} \mathrm{A} \\ \vdots \\ \mathrm{A} \end{matrix}\right\} \quad x \text{ of event type A}$$

$$\left.\begin{matrix} \mathrm{B} \\ \vdots \\ \mathrm{B} \end{matrix}\right\} \quad N - x \text{ of event type B}$$

which will occur with probability $p^x q^{N-x}$ and there are $\binom{N}{x} = \dfrac{N!}{x!(N-x)!}$ different ways to get x of event type A. This yields the following p.d.f.:

$$f(x) = \frac{N!}{x!(N-x)!} p^x q^{N-x}$$

Please also observe that

$$\sum_{x=0}^{N} f(x) = 1$$

The mean of X is:

$$\overline{X} = E[X] = \sum_{x=0}^{N} x \frac{N!}{x!(N-x)!} p^x q^{N-x}$$

$$= Np \sum_{x=1}^{N} \frac{(N-1)!}{(x-1)!(N-x)!} p^{x-1} q^{N-x}, \text{ and letting } N' = N, \text{ and } y = x\text{-}1$$

$$= Np \sum_{y=0}^{N'} \frac{N'!}{(y)!(N'-y)!} p^y q^{N'-y} = Np \sum_{y=0}^{N'} f_{N'}(y) = Np$$

In a similar fashion we can show that

$$E\left[X^2\right] = Np + N^2 p^2 - Np^2, \quad \text{and}$$

$$\sigma_X^2 = Np(1-p) = Npq$$

Finally, we will let $\Delta t \rightarrow 0$, $N = T/\Delta t \rightarrow \infty$, $p = \lambda \Delta t \rightarrow 0$, and $q \rightarrow 1$. In the following, keep in mind that $q = 1 - p$, $Np = \lambda T$, $N = \lambda T/p$. The Poisson probability distribution is therefore:

$$\lim_{\Delta t \to 0} f(x) = \lim_{\Delta t \to 0} \frac{N!}{x!(N-x)!} p^x q^{N-x}$$

$$= \lim_{\Delta t \to 0} \left[\frac{N(N-1)\cdots(N-x+1)}{N^x} \right] \left[\frac{N^x p^x}{x!} \right] \left[\frac{1}{q^x} \right] \left[q^N \right]$$

$$= \left[\lim_{N \to \infty} \frac{N(N-1)\cdots(N-x+1)}{N^x} \right] \left[\frac{(\lambda T)^x}{x!} \right] \left[\lim_{q \to 1} \frac{1}{q^x} \right] \left[\lim_{p \to 0} (1-p)^{-1/p} \right]^{-\lambda T}$$

$$= [1] \left[\frac{(\lambda T)^x}{x!} \right] [1] [e]^{-\lambda T}$$

$$= \frac{e^{-\lambda T} (\lambda T)^x}{x!}$$

[The exponential limit comes from $e^{-\varepsilon} \approx 1-\varepsilon \to e \approx (1-\varepsilon)^{-1/\varepsilon}$.]

The mean and variance are:

$$\overline{X} = \lim_{\Delta t \to 0} Np = \lim_{\Delta t \to 0} \frac{T}{\Delta t} \lambda \Delta t = \lambda T$$

$$\sigma_X^2 = \lim_{\Delta t \to 0} Npq = \lim_{\Delta t \to 0} \frac{T}{\Delta t} \lambda \Delta t (1 - \lambda \Delta t) = \lambda T$$

Here X is a Poisson R.V. with parameter λT:

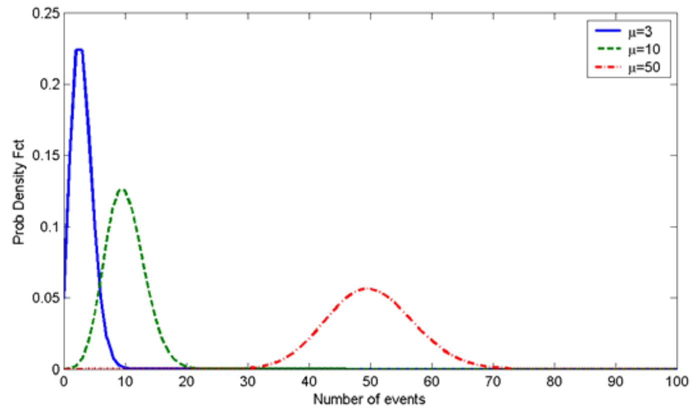
$$X \sim \text{Poisson}(\lambda T).$$

SNR of a Poisson Measurement

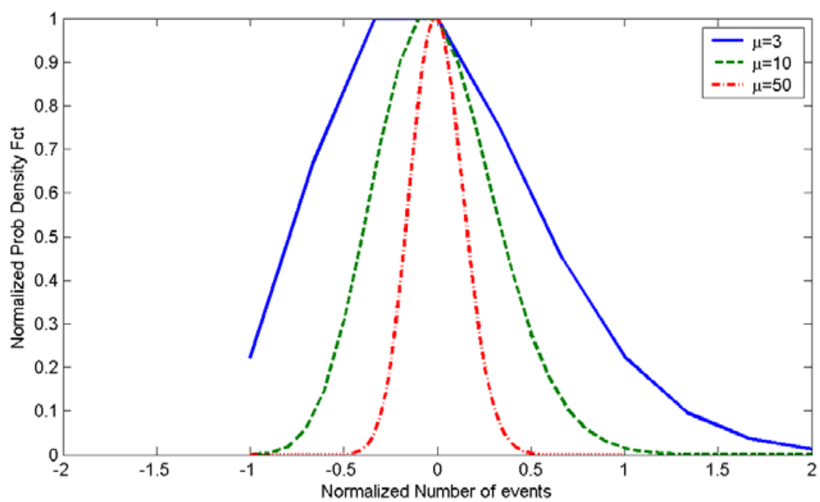
In general, the pixel values in an x-ray image are distributed according to a Poisson R.V. If the mean value of the photon counts for a pixel is μ, then the signal to noise ratio of for that pixel will be:

$$SNR = \frac{\overline{X}}{\sigma_X} = \frac{\mu}{\sqrt{\mu}} = \sqrt{\mu}$$

The SNR increases as the square root of the number of photons. *Thus, the SNR increases as the square root of the dose to the patient.* Finally, by averaging together two neighboring pixels, we can roughly double the photon counts and improve the SNR by $\sqrt{2}$.



The above figure shows Poisson distributions as the mean increases from 3 to 50. We can see that the distribution becomes more symmetric and Gaussian.

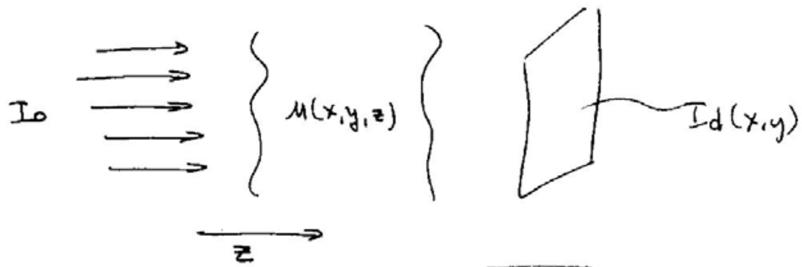


The above figure takes Poisson distributions and normalizes them by their mean, that is, we subtract the mean and divide the x-axis by the mean. This plot show demonstrates that the width of the distribution as a fraction of the mean. As the mean gets larger, the distribution gets proportionately narrower – the std. dev. vs. mean ratio is smaller (SNR is higher).

Source Issues

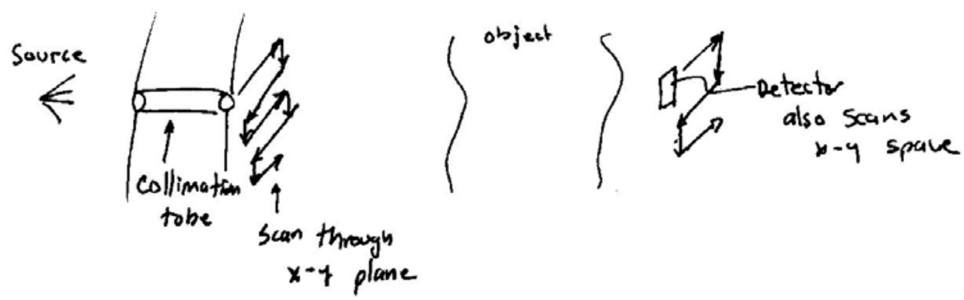
The Parallel X-ray Imaging System

Earlier, we considered a parallel ray system with an incident intensity I_0 that passes through a 3D object having a distribution of attenuation coefficients $\mu(x,y,z)$ and projects to an image $I_d(x,y)$:



$$I_d(x,y) = I_0 \exp\left(-\int \mu(x,y,z)dz\right)$$

There are essentially no practical medical project x-ray systems where the source has parallel rays. There are some scanning systems that might be appropriate for industrial inspection operations, for example:

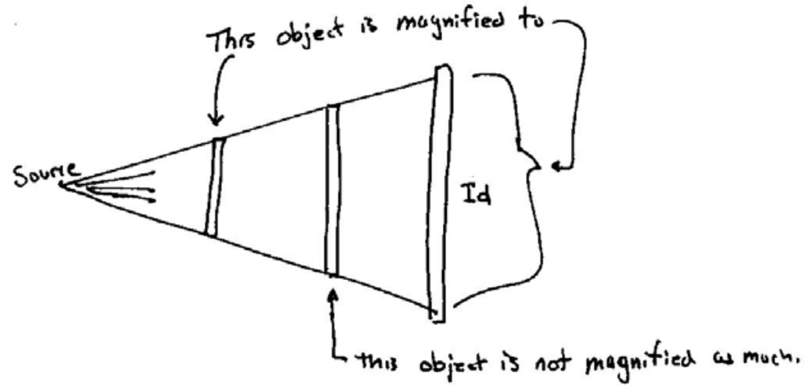


but these kinds of systems are too slow for medical applications.

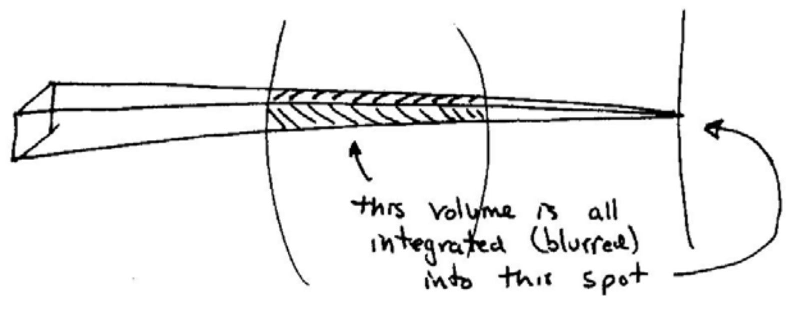
Practical X-ray Sources

There are two main issues associated with practical x-ray sources:

1. Geometric distortions due to point geometry – “depth dependent magnification.”

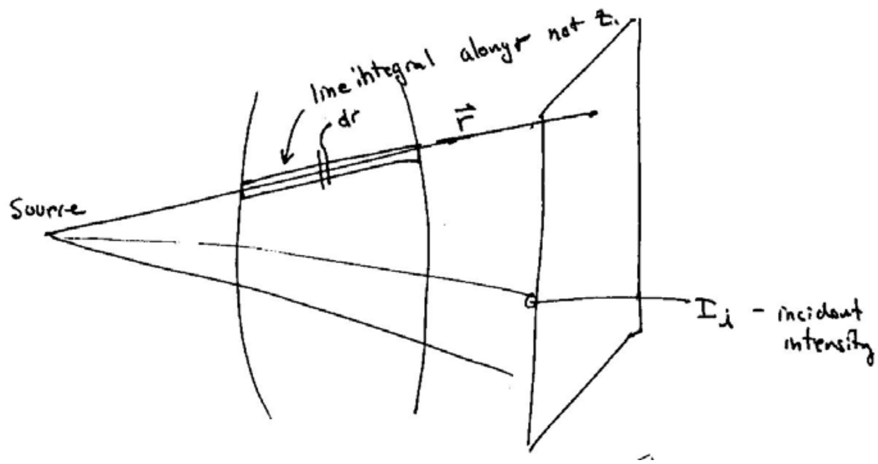


2. Resolution loss (blurring) due to finite (large) source sizes



Point Source Geometry

First, we will find expressions for the image intensity, $I_d(x_d,y_d)$, for a point source geometry:



$$I_d(x_d, y_d) = I_i(x_d, y_d)\exp\left(-\int \mu(x, y, z)dr\right)$$

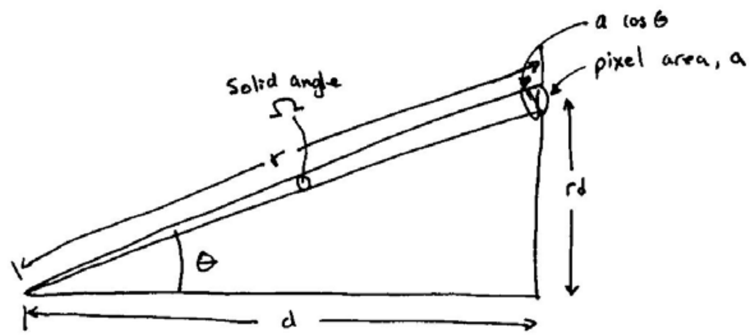
Comments:

1. (x_d,y_d) is the coordinate system in the output detector plane.
2. (x,y,z) is the coordinate system of the object.

3. Notice that $I_i(x_d,y_d)$ a spatially variant incident intensity replaces I_0.
4. Notice that the integration is along some path r with variable of integration dr.

Intensity Variations

The incident intensity is maximal at the center of the coordinate system and falls off towards the edges. This has two components – an increases in distance from the source and the rays obliquely striking the detector.



Intensity has really power/unit area. We can write an expression for the intensity I_i as:

$$I_i = \frac{\text{(photons)(mean photon E)}}{\text{(unit area)(exposure time)}} = \frac{kN}{a}\frac{\Omega}{4\pi}$$

where k is a scaling coefficient, N is the number of photon that are emitted during the observation time (we assume here that photons are emitted isotropically over a sphere), and $\Omega/4\pi$ is fraction of the surface of a sphere that is subtended by pixel area a.
[Ω is known as the *solid angle* and has units of *steradians* of which there are 4π over the surface of a sphere. This is similar to there being 2π radians over circumference of a circle.]
For a pixel of area a at some position angle θ away from the origin, the part of a sphere covered will be $a\cos\theta$. Thus:

$$\frac{\Omega}{4\pi} = \frac{a\cos\theta}{4\pi r^2} \quad \text{or} \quad \Omega = \frac{a\cos\theta}{r^2}$$

We now define the intensity at the origin to be I_0. At the origin, $\theta = 0$ and the distance from the source to the detector is $r = d$, thus $\Omega = a/d^2$ and:

$$I_0 = I_i(0,0) = \frac{kN}{4\pi d^2}$$

Note that the intensity, I_0, falls off with $1/d^2$ as the detector moves away from the source. The constant k can now be found in terms of I_0:

$$k = I_0 \frac{4\pi d^2}{N}$$

Substituting:

$$I_i = \frac{kN}{a}\frac{\Omega}{4\pi} = I_0 d^2 \frac{\cos\theta}{r^2}$$

Observing that $\cos\theta = \frac{d}{r}$, we get:

$$I_i = I_0 \cos^3\theta = I_0\left(\frac{d}{r}\right)^3$$

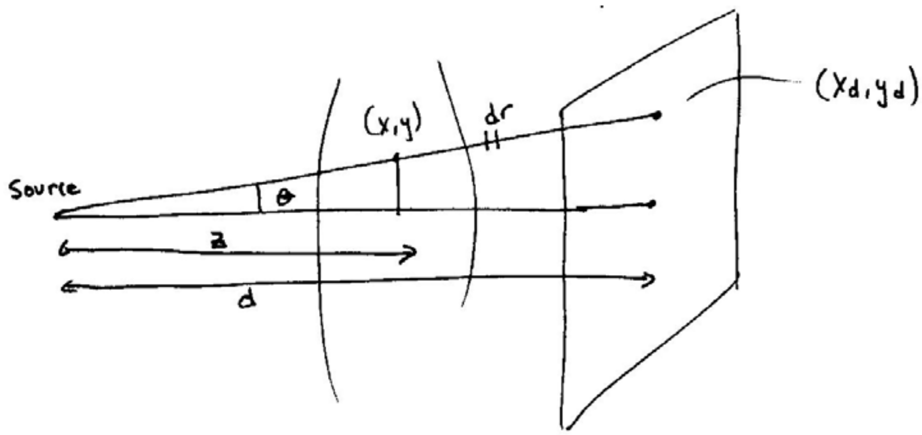
we can put this expression in the coordinate system of the detector using $r_d^2 = x_d^2 + y_d^2$ and $r^2 = d^2 + r_d^2$:

$$I_i(x_d, y_d) = I_0\left(\frac{d}{\sqrt{d^2 + r_d^2}}\right)^3 = I_0 \frac{1}{\left(1+\left(\frac{r_d}{d}\right)^2\right)^{3/2}}$$

The $\cos^3\theta$ term (or its other representations) is called the *incident intensity obliquity term* and this has two components: the $\cos^2\theta$ term for an increase in distance from the source to the detector and the $\cos\theta$ term for rays obliquely striking the detector. The $\cos^2\theta$ term is really a $1/r^2$ term, the inverse square law for fallout of intensity. The $\cos\theta$ term can be easily visualized if you think of a flashlight beam hitting a wall obliquely – the oblique beam spreads the photons over a larger area of the wall.

Oblique Path Integration

If we look at some point in the object (x,y) at depth z, we see that it will strike the detector at a position $(x_d, y_d) = \left(x\frac{d}{z}, y\frac{d}{z}\right)$:



where $M(z)=\dfrac{d}{z}$ is the magnification factor for an object at depth z. We can now write the attenuation coefficient at location (x,y) in terms of the output coordinate system:

$$\mu(x,y,z)=\mu\left(\frac{x_d}{M(z)},\frac{y_d}{M(z)},z\right)$$

Also, instead of integrating along the path r, we can rewrite the expression to integrate in z:

$$\begin{aligned} dr &= \sqrt{dx^2+dy^2+dz^2} \\ &= dz\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2} \\ &= dz\sqrt{1+\left(\frac{x_d}{d}\right)^2+\left(\frac{y_d}{d}\right)^2} \\ &= dz\sqrt{1+\left(\frac{r_d}{d}\right)^2} \end{aligned}$$

This expression says that if with integrate in z instead of r, the integral will need to be increased by $\sqrt{1+\left(\frac{r_d}{d}\right)^2}$ in order to account of the longer path length in r (than z). This term is sometimes known as the *pathlength obliquity term*.

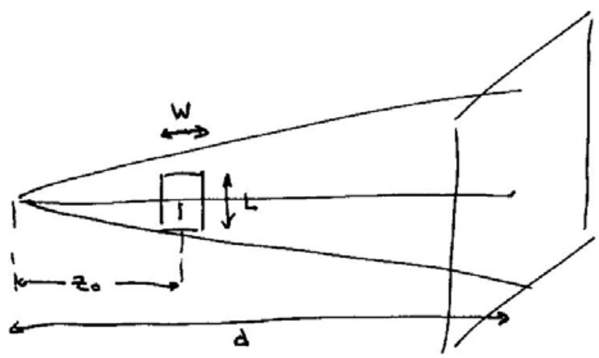
Finally, we put it all together and we get an expression for the output intensity from a point source:

$$I_d(x_d, y_d) = I_0 \frac{1}{\left(1+\left(\frac{r_d}{d}\right)^2\right)^{3/2}} \exp\left(-\sqrt{1+\left(\frac{r_d}{d}\right)^2} \int \mu\left(\frac{x_d}{M(z)}, \frac{y_d}{M(z)}, z\right) dz\right)$$

Example

For the example, we will reduce the dimensions of the problem to 2 – y and z, and thus $r_d = y_d$. Now, let's look at a rectangular object at depth z_0:

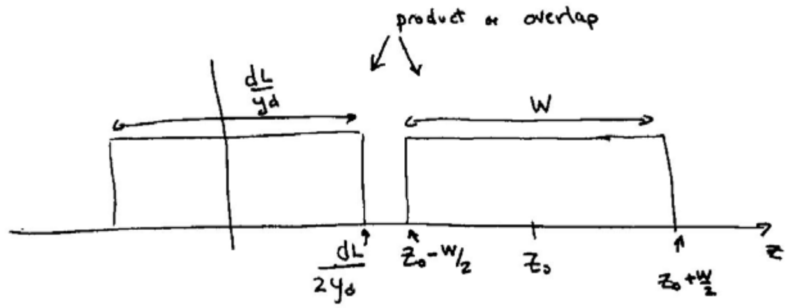
$$\mu(y,z) = \mu_0 \mathrm{rect}\left(\frac{y}{L}\right)\mathrm{rect}\left(\frac{z-z_0}{W}\right)$$



The expression for the image intensity will be:

$$I_d(y_d) = I_0 \frac{1}{\left(1+\left(\frac{y_d}{d}\right)^2\right)^{3/2}} \exp\left(-\sqrt{1+\left(\frac{y_d}{d}\right)^2} \mu_0 \int \mathrm{rect}\left(\frac{y_d z}{dL}\right)\mathrm{rect}\left(\frac{z-z_0}{W}\right) dz\right)$$

The use of the magnification factor allowed the function of y to be converted to a function of z for each location y_d in the detector plane. The first rect in the above expression has width dL/y_d and is centered at z=0. The second rect has width W and is centered at z=z_0. The integral is the area under the overlap of these two rect functions.

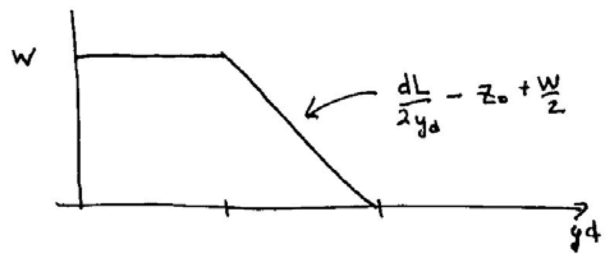


The integral is:

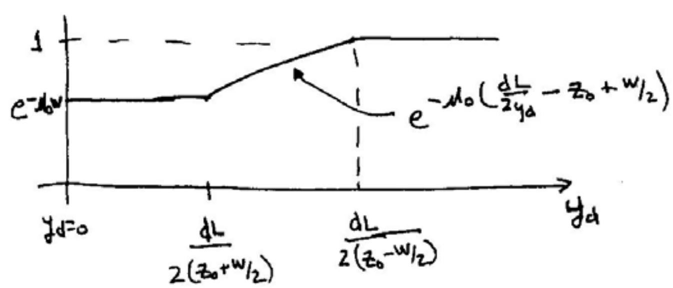
0 for $\frac{dl}{2|y_d|} < z_0 - \frac{W}{2}$ or $|y_d| > \frac{dL}{2(z_0 - W/2)}$

W for $\frac{dl}{2|y_d|} > z_0 + \frac{W}{2}$ or $|y_d| < \frac{dL}{2(z_0 + W/2)}$

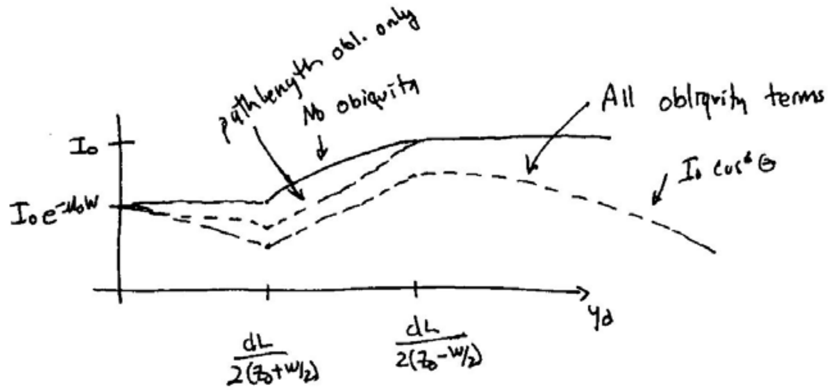
$\frac{dL}{2y_d} - z_0 + \frac{W}{2}$ otherwise



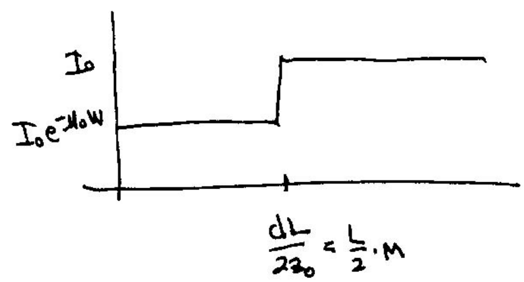
If we ignore all obliquity terms, we get the following:



Including the pathlength and incident intensity obliquity terms we get:



Under a parallel ray geometry we get the following:



As we can see, the depth dependent magnification has significantly distorted the appearance of the object in the image. We can define a fractional transition width be:

$$\frac{\dfrac{dL}{2(z_0-W/2)}-\dfrac{dL}{2(z_0+W/2)}}{\dfrac{dL}{4(z_0-W/2)}+\dfrac{dL}{4(z_0+W/2)}}=\frac{W}{z_0}$$

Thus, we can minimize the geometric distortions by placing the object as far from the source as possible (make z_0 large).

Finite (Large) Sources

To gain an understanding of this issue, we will first consider a "thin" object. Specifically, we will let the attenuation coefficient be:

$$\mu(x,y,z)=\tau(x,y)\delta(z-z_0)$$

and then:

$$I_d(x_d, y_d) = I_i \exp\left(-\sqrt{1+\left(\frac{r_d}{d}\right)^2} \int \tau\left(\frac{x_d}{M(z)}, \frac{y_d}{M(z)}\right) \delta(z - z_0) dz \right)$$

$$= I_i \exp\left(-\sqrt{1+\left(\frac{r_d}{d}\right)^2} \tau\left(\frac{x_d}{M(z_0)}, \frac{y_d}{M(z_0)}\right) \right)$$

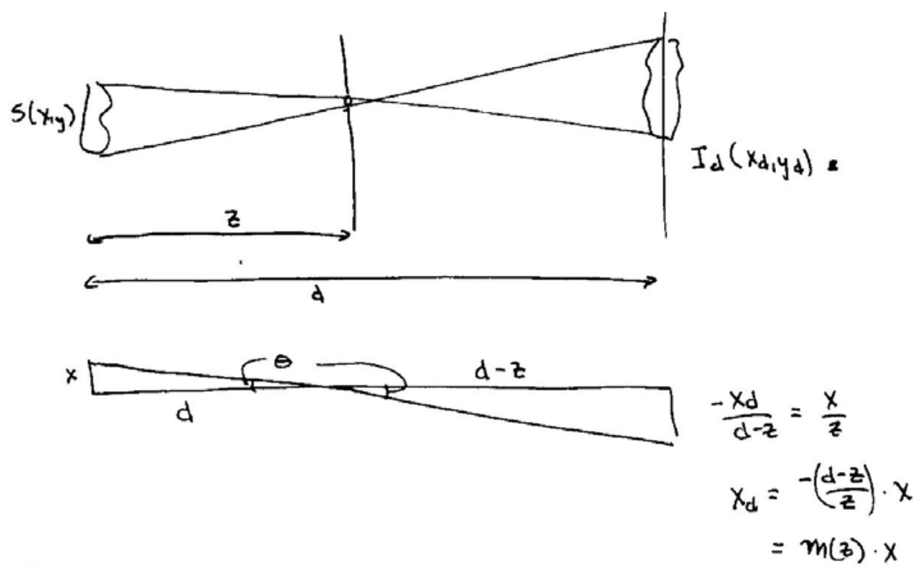
We let $M = M(z_0) = d / z_0$ the *object magnification factor*, and we will ignore the pathlength obliquity term to get:

$$I_d(x_d, y_d) = I_i \exp\left(-\tau\left(\frac{x_d}{M}, \frac{y_d}{M}\right)\right) = I_i t\left(\frac{x_d}{M}, \frac{y_d}{M}\right)$$

where $t = \exp(-\tau)$ is the transmission function. Ignoring all obliquity terms we get:

$$I_d(x_d, y_d) = I_0 t\left(\frac{x_d}{M}, \frac{y_d}{M}\right)$$

Now we consider a finite source function $s(x,y)$ and a very small pinhole transmission function:



The image will now be an image of the source with the *source magnification factor*, $m = m(z) = -\dfrac{d-z}{z}$:

$$I_d(x_d, y_d) = ks\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

where k is a scaling factor that is proportional to the area of the pinhole, $1/d^2$, etc. If we want the above I_d to represent the impulse response of the system, we need to make the pinhole equal to $\delta(x,y)$ and account for all of the scaling terms [$t(x,y) = \delta(x,y)$ is not a realizable transmission function since t can never exceed 1, nevertheless, we will allow it for mathematical convenience.]

The area of the pinhole is $\iint \delta(x, y)dxdy = 1$. The capture efficiency of the pinhole is the fraction of all photons emitted from the source that pass through the pinhole. This will be equal to:

$$\eta = \frac{\text{pinhole area}}{4\pi z^2} = \frac{1}{4\pi z^2}$$

Letting the total number of photon emitted be:

$$N = \iint s(x, y)dxdy$$

and the total number of photons to get through the pinhole will be:

$$N\eta = \frac{N}{4\pi z^2}.$$

This must be the same number at the detector:

$$\iint ks\left(\frac{x_d}{m}, \frac{y_d}{m}\right)dx_d dy_d = kNm^2 = \frac{N}{4\pi z^2}$$

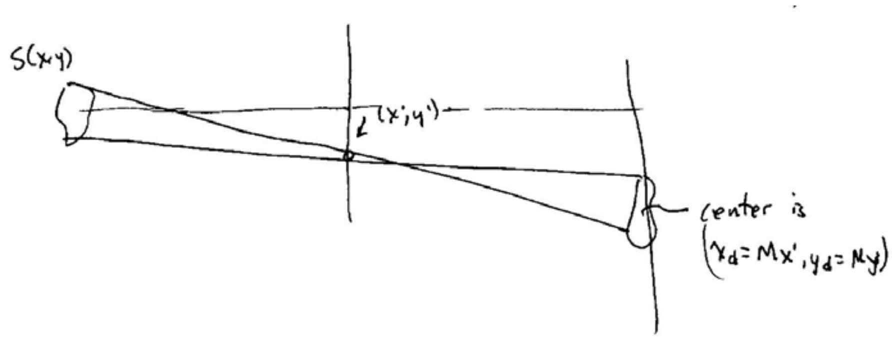
The scaling coefficient will therefore be:

$$k = \frac{1}{4\pi z^2 m^2}$$

so:

$$I_d(x_d, y_d) = \frac{1}{4\pi z^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

Now we let the pinhole be at position (x’,y’), that is, $t(x,y) = \delta(x\text{-}x', y\text{-}y')$:



The image of the source is not located at (x_d=Mx', y_d=My') where M is the object magnification factor. Thus, the impulse response function is:

$$h(x_d, y_d; x', y') = I_d(x_d, y_d) = \frac{1}{4\pi z^2 m^2} s\left(\frac{x_d - Mx'}{m}, \frac{y_d - My'}{m}\right)$$

Now we can calculate the image for an arbitrary transmission function using the superposition integral:

$$I_d(x_d, y_d) = \iint t(x' y') h(x_d, y_d; x', y') dx' dy'$$

$$= \frac{1}{4\pi z^2 m^2} \iint t(x' y') s\left(\frac{x_d - Mx'}{m}, \frac{y_d - My'}{m}\right) dx' dy' \text{ and sub } Mx' = x$$

$$= \frac{1}{4\pi z^2 m^2 M^2} \iint t\left(\frac{x}{M}, \frac{y}{M}\right) s\left(\frac{x_d - x}{m}, \frac{y_d - y}{m}\right) dx dy$$

$$= \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** t\left(\frac{x_d}{M}, \frac{y_d}{M}\right)$$

Thus, the final image is equal to the convolution of the magnified source and the magnified object. The object is blurred by the source function.

The frequency domain equivalent is:

$$F_{2D}\{I_d(x_d, y_d)\} = \frac{1}{4\pi z^2} S(mu, mv) T(Mu, Mv)$$

Consider $z_0 = d/2$ which yields M=2 and |m|=1. The object is magnified by a factor of 2 and is blurred by the unmagnified source.

Comments:

1. The least blurring come when $|m|$ is made small. Thus, it is desirable to make the depth plane as far from the source as possible: $z_0 \rightarrow d$. Then $|m| = (d\text{-}z)/z \rightarrow 0$ and $M \rightarrow 1$. As we was above, making $z_0 \rightarrow d$ also reduces geometric distortions. The common practice for x-ray imaging, then, is to position the subject immediately next to or on top of the detector.
2. If the thickness of the body is a limiting factor, then let $d, z \rightarrow \infty$. This will make the system close to a parallel ray geometry with $|m| = \rightarrow 0$ and $M \rightarrow 1$. The main problem with this approach is $I_0 \propto 1/d^2 \rightarrow 0$ and SNR $\propto \sqrt{I_0} \rightarrow 0$.
3. We would also like the make $s(x,y)$ as small as possible to reduce blurring, but $I_0 \propto \iint s(x,y)dxdy$ and making it small might reduce the number of photons created and thus reduce SNR.
4. For a complex object, we can make $\mu(x,y,z) = \sum \tau_i(x,y)\delta(z - z_i)$ and each plane will have its own magnification factors. This is not particularly useful, but it can give you some idea of how blurring and magnification might affect different parts of a real object differently.

Overall System Response

Now we can add the detector response to the other system elements:

$$I_d(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** t\left(\frac{x_d}{M}, \frac{y_d}{M}\right) ** h(r_d)$$

The impulse response function will then be:

$$h(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** h(r_d)$$

or for a circularly symmetric source function:

$$h(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{r_d}{m}\right) ** h(r_d)$$

Object Blurring

One issue is how much does the detector response blur the object. It is important to realize that the detector blurs the magnified object. Our intuition would be to make the object as large as possible by making $M = d/z$ very large. This would dictate moving the object as close to the source as possible, which is exactly opposite as what we would like to do to minimize source blurring.

Consider also, that the magnified source also blurs the magnified object (source and object have different magnification factors). One way to look at this is to examine the response in the coordinate system of the object (x,y) rather than the detector (x_d,y_d):

$$I(x,y) = ks\left(\frac{Mx}{m}, \frac{My}{m}\right) ** t(x,y) ** h(Mr_d)$$

the effective magnification of the source is:

$$\left|\frac{m}{M}\right| = \frac{d-z}{d}$$

and the effective magnification of the detector response is:

$$\frac{1}{M} = \frac{z}{d}$$

These are in competition:

- to make the source blurring small, make $z \rightarrow d$
- to make the detector response small, make $z \rightarrow 0$

Comments:

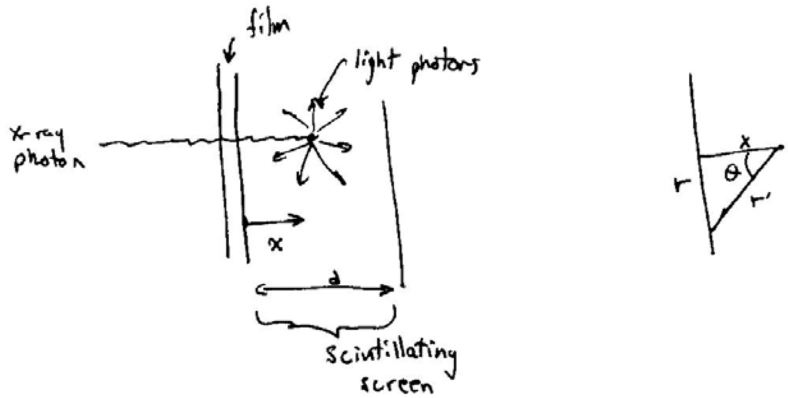
1. For most film systems, the detector response is very small and the source is almost always bigger. Therefore, we would like to make $z \rightarrow d$.
2. For other kinds of systems, e.g. digital fluoroscopy systems, the detector resolution is much larger (e.g. 0.5 mm) and for these systems an intermediate z may be appropriate.

Detector Issues

Earlier, we discussed the effect of source size and location on spatial resolution and magnification distortions in x-ray imaging. Now we will discuss detector issues. In selecting detector characteristics, we will have a resolution/SNR trade-off – this come primarily from the fact the thicker detectors have better SNR, but a larger impulse response.

Conversion of x-Rays to Film

Photographic films are generally not very sensitive to x-rays, so x-rays must first be converted to visible light by a scintillating screen:



We will now develop expressions to represent the impulse response of the detector. Suppose we have a x-ray photon enter the scintillating screen and it interacts at some depth (which we'll call *x*) and generates a shower of light photons isotropically from a point of which some eventually strike the detector. The geometry is essentially the same as a point x-ray source striking the detector. Notably:

$$h(r) = h(0)\cos^3\theta$$

$$= h(0)\left(\frac{x}{\sqrt{x^2 + r^2}}\right)^3$$

but $h(0) \propto x^{-2}$ by the inverse square law, thus:

$$h(r) = k\frac{x}{\left(x^2 + r^2\right)^{3/2}}$$

The corresponding frequency domain equivalent is:

$$H(\rho) = 2\pi k \exp(-2\pi x\rho)$$

Without loss of generality, we will select *k* to normalize this expression to have a peak frequency response of 1.

$$H(\rho) = \exp(-2\pi x\rho)$$

Notice that right next to the film ($x \rightarrow 0$):

$$H(\rho) \to 1$$
$$h(r) \to \delta(x, y)$$

Finally, we can calculate an average frequency response by taking:

$$\overline{H}(\rho) = \int_0^d H(\rho, x)\, p(x)\, dx$$

where $p(x)$ is the probability density function for an interaction occurring at depth *x*. To determine this, we first recognize that the scintillating screen has its own linear attenuation coefficient μ. The number of photons that pass through at any depth x is:

$$N(x) = N(0)\exp(-\mu x)$$

and the number absorbed will be:

$$N_{abs}(x) = N(0)\left(1 - \exp(-\mu x)\right)$$

The total fraction absorbed in the detector is:

$$\eta = 1 - \exp(-\mu d)$$

where η is "detector efficiency", which increases with *d*. We can define the cumulative distribution function as:

$$P(x) = \frac{1 - \exp(-\mu x)}{\eta}$$

and thus, the probability density function is:

$$p(x) = \frac{dP}{dx} = \frac{\mu}{\eta}\exp(-\mu x)$$

The average frequency response is then:

$$\overline{H}(\rho) = \frac{\mu}{\eta}\int_0^d \exp(-2\pi x\rho)\exp(-\mu x)\, dx$$

$$= \frac{\mu}{\eta(2\pi\rho + \mu)}\left(1 - \exp(-d(2\pi\rho + \mu))\right)$$

For large ρ, this expression looks like:

$$\bar{H}(\rho) \to \frac{\mu}{2\pi\eta\rho}$$

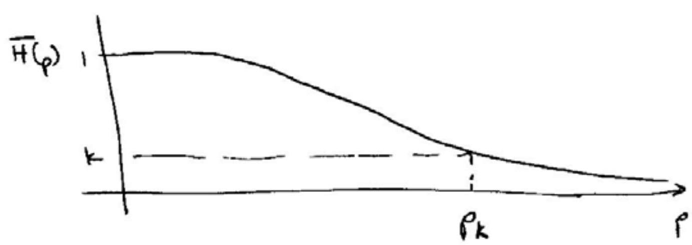
The high spatial frequencies play a large role in dictating the shape of the impulse response close to the peak (e.g. $\bar{h}(r)$ near r = 0) and the low spatial frequencies will dictate the appearance of the tails of $\bar{h}(r)$. Thus, near r = 0, the average impulse response will take on the shape:

$$\bar{h}(r) \approx \frac{\mu}{2\pi\eta r}$$

(recall the inverse Fourier-Bessel transform of $1/\rho$ is $1/r$.)

The average impulse response, then, is very peaked (infinite in amplitude). One consequence of this is the common measures of resolution or blurring (e.g. like FWHM – Full Width at Half Maximum) have no meaning.

One way to evaluate the performance of the detector system is to define a cutoff frequency, ρ_k, as the frequency at which the response falls to $k\bar{H}(0)$. For smaller values of k, this is:



$$\rho_k = \frac{\mu}{2\pi\eta k}$$

This, in essence, give the maximum spatial frequency that can be detected where k represents the level of detectability. For example, k = 0.1 is a common value and having a higher cutoff frequency, ρ_k, is desired to improve spatial resolution. We can now begin to see the SNR resolution trade-off. As d increase, the detector efficiency η, increases which leads to more x-ray photons being detected and thus the SNR improved. This, however, causes ρ_k to be smaller resulting in lower spatial resolution.

Recall that the SNR is proportion to the square root of the number x-ray photons and in order to see them, they must be detected – so the SNR is proportional to the root of the number x-ray photons that are detected. SNR is therefore proportional to $\sqrt{\eta}$.

Example

Let's look at a detector with the μ = 1.5 mm^{-1} and d = 0.25 mm, and we will use k = 0.1.

$$\eta = 0.31$$
$$\rho_k = 8mm^{-1}$$

and the limiting spatial resolution is approximately:

$$\frac{1}{\rho_k} = 125\mu m$$

Now if we double the thickness to d = 0.5 mm:

$$\eta = 0.53$$
$$\rho_k = 4.5mm^{-1}$$

and the limiting spatial resolution is approximately:

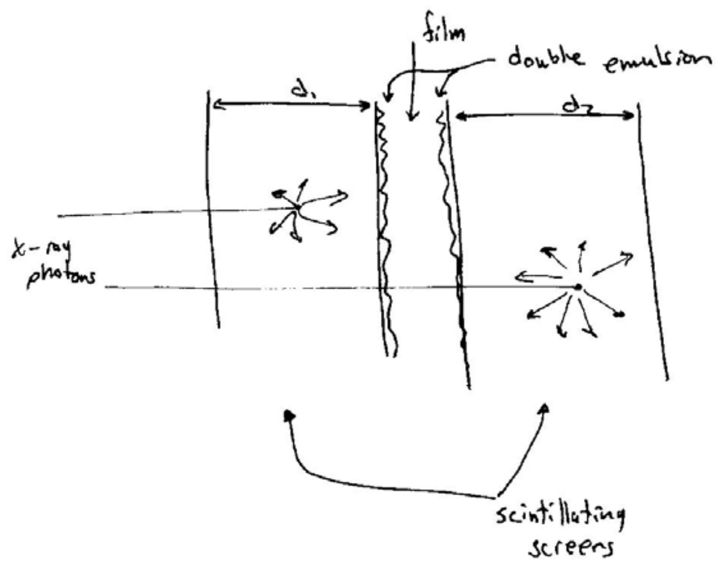
$$\frac{1}{\rho_k} = 220\mu m$$

Comments:

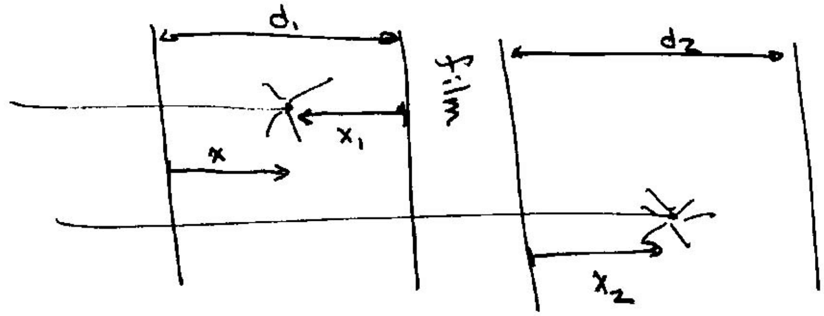
1) In general, increasing m improves both η and ρ_k.
2) What happens if we put the film on the back of the scintillator? Is the response better or worse?

Two Screen Detectors with Double Emulsion Films

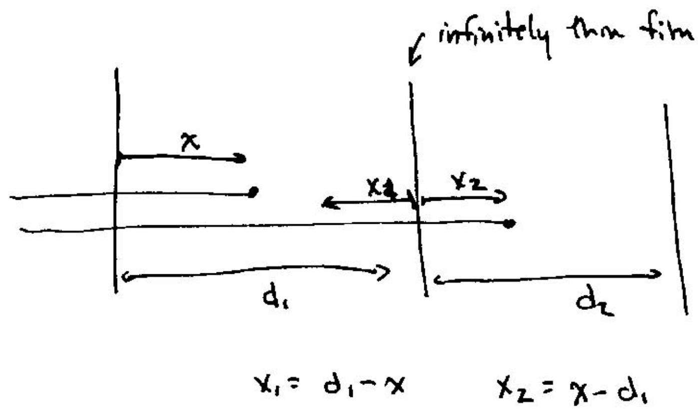
To ease the tradeoff between resolution and SNR, we can use a double emulsion film with a two screen scintillator:



We assign a coordinate system here to ease our analysis:



Since no interaction occur in the film, we can neglect its thickness:



For interactions occurring in the first screen ($0 \le x \le d_1$):

$$h(r) = k \frac{x_1}{\left(x_1^2 + r^2\right)^{3/2}}$$

which yields a frequency response of:

$$H(\rho, x) = \exp(-2\pi\rho x_1)$$
$$= \exp(-2\pi\rho(d_1 - x))$$

For interactions occurring in the second screen ($d_1 \le x \le d_2$):

$$H(\rho, x) = \exp(-2\pi\rho x_2)$$
$$= \exp(-2\pi\rho(x - d_1))$$

Finally:

$$H(\rho, x) = \begin{cases} \exp(-2\pi\rho(d_1 - x)), \text{for } 0 \le x < d_1 \\ \exp(-2\pi\rho(x - d_1)), \text{for } d_1 \le x < d_2 \end{cases}$$

where $d = d_1 + d_2$. The detector efficiency is again:

$$\eta = 1 - \exp(-\mu d).$$

We can now find the average frequency response in a similar manner as before:

$$\overline{H}(\rho) = \frac{\mu}{\eta}\left[\frac{\exp(-\mu d_1) - \exp(-2\pi d_1 \rho)}{2\pi\rho - \mu} + \frac{\exp(-\mu d_2) - \exp(-2\pi d_2 \rho + \mu d)}{2\pi\rho - \mu}\right]$$

For large ρ, this expression looks like:

$$\overline{H}(\rho) \to \frac{\mu}{2\pi\eta\rho}\left[\exp(-\mu d_1) + \exp(-\mu d_2)\right]$$

If we take $d_1 = d_2 = d/2$, then:

$$\overline{H}(\rho) \to \frac{\mu}{2\pi\eta\rho} 2\left[\exp(-\mu d)\right]^{1/2} = \frac{\mu}{2\pi\eta\rho}\left[2(1-\eta)^{1/2}\right]$$

and the cutoff frequency will be:

$$\rho_k = \frac{\mu}{2\pi\eta k}\left[2(1-\eta)^{1/2}\right]$$

where $\left[2(1-\eta)^{1/2}\right]$ is the improvement factor over the single emulsion film system.

Example

Let's look at the previous example with a detector with the $\mu = 1.5$ mm^{-1}, $d = 0.25$ mm, $d_1 = d_2 = d/2$, then and we will use $k = 0.1$.

$$\eta = 0.31$$
$$\left[2(1-\eta)^{1/2}\right] = 1.7$$
$$\rho_k = 13 mm^{-1}$$

and the limiting spatial resolution is approximately:

$$\frac{1}{\rho_k} = 76\mu m$$

Alternatively, we can hold ρ_k constant by setting $d \approx 0.4$ mm:

$$\eta = 0.45$$

$$\rho_k \approx 8mm^{-1}$$

$$\frac{1}{\rho_k} = 125\mu m$$