Notes on the Fourier Transform

Definition. The Fourier Transform (FT) relates a function to its frequency domain equivalent. The FT of a function *g(x)* is defined by the Fourier integral:

$$G(s) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{-i2\pi xs} dx$$

for $x, s \in \Re$. There are a variety of existence criteria and the FT doesn't exist for all functions. For example, the function *g(x) = cos(1/x)* has an infinite number of oscillations as $x \to 0$ and the FT integral can't be evaluated. If the FT does exist, then there is an inverse FT relationship:

$$g(x) = F^{-1}\{G(s)\} = \int_{-\infty}^{\infty} G(s) e^{i2\pi xs} ds$$

Uniqueness: Given the existence of the inverse FT, it follows that if the FT exists, it must be unique. That is, for a function forms a unique pair with its FT:

$$g(x) \leftrightarrow G(s)$$

Caveat. An exception to the uniqueness property is a class of functions called "massless" or "null" functions. An example is the continuous function $f(x) = \{ \begin{smallmatrix} 1, x=0 \\ 0, x \neq 0 \end{smallmatrix}$. This function and others like it have the same Fourier transform as *f(x)* = 0: *F(s)* = 0. Thus, the uniqueness exists only for a function plus or minus arbitrary null functions. In practice, these functions are not realizable (energyless) and thus, for the purposes of this class we will assume that the FT is unique.

Alternate FT Definition. In the above derivation, the *s* is the frequency parameter. There is another common FT definition that uses a radian frequency parameter ω:

$$G(\omega) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{-ix\omega} dx = G(s)\big|_{s=\omega/2\pi}$$

with an inverse FT of:

$$g(x) = F^{-1}\{G(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{ix\omega} d\omega$$

Units. If *x* has units of Q, then *s* will have units of "cycles/Q" or Q^{-1}. Please note that under our definition of the FT, this is not an angular frequency with units of radians/Q, but just plain Q^{-1}. Please also keep in mind that *x* is the index of variation – for example, we can have *g*(*x*) represent a velocity that varies as a function of spatial location *x*. The function *g*(*x*) has units cm/s, but *x* has units cm and *G*(*s*) has units of cm/s, but *s* has units of cm^{-1}.

Examples:

x	*s*
Time s (seconds)	Temporal Frequency s^{-1}, Hz, cycles/s
Distance cm	Spatial Frequency cm^{-1}, cycles/cm

Symmetry Definitions. We first decompose some function *g(x)* in to even and odd components, *e(x)* and *o(x)*, respectively, as follows:

$$e(x) = \tfrac{1}{2}[g(x) + g(-x)]$$

$$o(x) = \tfrac{1}{2}[g(x) - g(-x)]$$

thus,

$$g(x) = e(x) + o(x)$$

and

$$e(x) = e(-x) \text{ and } o(x) = -o(x)$$

A function, *g(x)*, is Hermitian Symmetric (Conjugate Symmetric) if:

$$\mathrm{Re}\{g(x)\} = e(x) \text{ and } \mathrm{Im}\{g(x)\} = o(x)$$

thus,

$$g(x) = e(x) + io(x) = g^*(-x)$$

Symmetry Properties of the FT. There are several related properties:

1. If *g(x)* is real, then *G(s)* is Hermitian symmetric (e.g. *G(s) = G*(-s)*).
2. If *g(x)* is real and even, *G(s)* is real and even.
3. If *g(x)* is real and odd, *G(s)* is imaginary and odd.
4. If *g(x)* is real, *G(s)* can be defined strictly by non-negative frequencies ($s \geq 0$).
5. If *g(x)* is imaginary, then *G(s)* is Anti-Hermitian symmetric (e.g. *G(s) = -G*(-s)*).

Proof of 1.

$$G(s) = \int g(x) e^{-i2\pi sx} dx$$

$$= \int [e(x) + o(x)][\cos 2\pi sx - i \sin 2\pi sx] dx \qquad \text{(cos is even, sin is odd)}$$

$$= \int e(x)\cos 2\pi sx dx + \int o(x) \cos 2\pi sx dx - i\int e(x) \sin 2\pi sx dx - i\int o(x) \sin 2\pi sx dx$$

$$= \int e'(x) dx + \int o'(x) dx - i\int o''(x) dx - i\int e''(x) dx$$

$$= E(s) + 0 - i \cdot 0 - iO(s) \quad \text{(cos is even in } s\text{, sin is odd is } s\text{, } \int_{-\infty}^{\infty} odd(x) = 0)$$

$$= E(s) - iO(s)$$

$$= E(-s) + iO(-s)$$

$$= G^*(-s) \qquad \text{Q.E.D.}$$

Comment. One interesting consequence of the symmetry properties is that if g(x) is real, the only one-half of the Fourier transform is necessary to specify the function – this follows from property 1. above. More specifically, g(x) is strictly determined by G(s) for all non-negative frequencies (s).

Comment on negative frequencies. Consider a real-valued signal – imagine a voltage on a wire or the sound pressure against your eardrum – the Fourier transform of these is completely specified by the non-negative frequencies (e.g. $G(-s) = G^*(s)$). We can argue that we have the concept of a frequency (oscillations/second), but it doesn't really make physical sense to talk about positive or negative frequencies. In this case, we could argue that the having positive and negative frequencies is merely a mathematical convenience. Are there cases where negative frequencies have meaning? Consider the bit in a drill – it can turn clockwise or counter clockwise and different rotational rates. Here positive and negative frequencies have physical meaning (the direction of rotation). As we shall see, there are cases in medical imaging where this distinction is important, for example, the magnetic moment in MRI is a case where the sign indicates the direction of precession.

Convolution Definition. The convolution operator is defined as:

$$g(x)*h(x) = \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$$

The convolution operator commutes:

$$g(x)*h(x) = \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi = \int_{-\infty}^{\infty} g(x-\xi)h(\xi)d\xi = h(x)*g(x)$$

The delta function, $\delta(x)$. The Dirac delta or impluse function is a mathematical construct that is infinitely high in amplitude, infinitely short in duration and has unity area:

$$\delta(x) = \{ \begin{matrix} \infty, x=0 \\ 0, x \neq 0 \end{matrix} \text{ and } \int \delta(x)dx = 1$$

Most properties of $\delta(x)$ can exist only in a limiting case (e.g. as a sequence of functions $g_n(x) \to \delta(x)$) or under an integral. Some important properties of $\delta(x)$:

$$\int \delta(x)g(x)dx = g(0), \text{ with } g(x) \text{ continuous at } x = 0$$

$$\int \delta(x-a)g(x)dx = g(a), \text{ with } g(x) \text{ continuous at } x = a$$

$$\int \delta(ax)g(x)dx = \frac{1}{|a|}g(0), \text{ with } g(x) \text{ continuous at } x = 0$$

$$F\{\delta(x)\} = 1$$

Delta function properties. First two are technically only defined under the integral, but we'll still talk about them.

Similarity (stretching)	$\delta(ax) = \frac{1}{\lvert a \rvert}\delta(x)$
Product/Sifting	$g(x)\delta(x-a) = g(a)\delta(x-a)$
Sifting	$\int g(x)\delta(x-a)dx = g(a)$

Convolution	$g(x)*\delta(x)=\delta(x)*g(x)=g(x)$ $g(x)*\delta(x-a)=\delta(x-a)*g(x)=g(x-a)$

Fourier Transform Theorems. There are many Fourier transform properties and theorems. This is a partial list. Assume that $F\{g(x)\}=G(s)$, $F\{h(x)\}=H(s)$ and that a and b are constants:

Linearity	$F\{ag(x)+bh(x)\}=aG(s)+bH(s)$
Similarity (stretching)	$F\{g(ax)\}=\frac{1}{\lvert a\rvert}G(\frac{s}{a})$
Shift	$F\{g(x-a)\}=G(s)e^{-i2\pi as}$
Convolution	$F\{g(x)*h(x)\}=G(s)H(s)$
Product	$F\{g(x)h(x)\}=G(s)*H(s)$
Complex Modulation	$F\{g(x)e^{i2\pi s_0 x}\}=G(s-s_0)$
Modulation	$F\{g(x)\cos(2\pi s_0 x)\}=\frac{1}{2}\left[G(s-s_0)+G(s+s_0)\right]$ $F\{g(x)\sin(2\pi s_0 x)\}=\frac{1}{2i}\left[G(s-s_0)-G(s+s_0)\right]$
Rayleigh's Power	$\int\lvert g(x)\rvert^2 dx=\int\lvert G(s)\rvert^2 ds$
Cross Power	$\int g(x)h^*(x)dx=\int G(s)H^*(s)ds$
Axis Reversal	$F\{g(-x)\}=G(-s)$
Complex Conjugation	$F\{g^*(x)\}=G^*(-s)$
Autocorrelation	$F\{g(x)*g^*(-x)\}=G(s)G^*(s)=\lvert G(s)\rvert^2$
Reverse Relationships	$F\{G(x)\}=g(-s)$
Differentiation	$F\left\{\frac{d}{dx}g(x)\right\}=i2\pi sG(s)$
Moments	$F\{xg(x)\}=\frac{i}{2\pi}\frac{d}{ds}G(s)$
DC Value	$\int g(x)dx=G(0)$

Some common FT pairs:

$g(x)$	$G(s)$
1	$\delta(s)$
$\delta(x)$	1
$\cos(2\pi s_0 x)$ $\sin(2\pi s_0 x)$	$\frac{1}{2}\left[\delta(s-s_0)+\delta(s+s_0)\right]$ $\frac{1}{2i}\left[\delta(s-s_0)-\delta(s+s_0)\right]$
$\mathrm{rect}(x)=\begin{cases}1 & \lvert x\rvert<\frac{1}{2}\\ 0 & \lvert x\rvert\ge\frac{1}{2}\end{cases}$	$\mathrm{sinc}(s)=\frac{\sin(\pi s)}{\pi s}$
$\mathrm{sinc}(x)$	$\mathrm{rect}(s)$
$\mathrm{triangle}(x)=\begin{cases}1-\lvert x\rvert & \lvert x\rvert<1\\ 0 & \lvert x\rvert\ge 1\end{cases}$	$\mathrm{sinc}^2(s)$

$e^{-\pi x^2}$	$e^{-\pi s^2}$
$\mathrm{sgn}(x) = \{\begin{matrix} 1 & x \ge 0 \\ -1 & x < 0 \end{matrix}$	$\dfrac{1}{i\pi s}$
$e^{-\|x\|}$	$\dfrac{2}{1+(2\pi s)^2}$
e^{-x}, for $x > 0$; 0, otherwise	$\dfrac{1-i2\pi s}{1+(2\pi s)^2}$
$\|x\|^{-1/2}$	$\|s\|^{-1/2}$
$J_0(2\pi x)$	$\dfrac{\mathrm{rect}(s/2)}{\pi(1-s^2)^{1/2}}$
$\mathrm{comb}(x)$	$\mathrm{comb}(s)$

The comb function, comb(x). The sampling or "comb" function is a train of delta functions:

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

The Fourier transform of comb(x) is:

$$F\{\mathrm{comb}(x)\} = \mathrm{comb}(s)$$

Proof.

$$F\{\mathrm{comb}(x)\} = F\left\{\sum_{n=-\infty}^{\infty} \delta(x-n)\right\} = \sum_{n=-\infty}^{\infty} e^{i2\pi ns} = F(s)$$

The RHS of the above expression can be viewed as the exponential Fourier series representation of a periodic function $F(s)$ with period 1 and $\alpha_n = 1$ for all n. Recall, the Fourier series expressions are:

$$F(s) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i2\pi ns}, \text{ where } \alpha_n = \int_{-1/2}^{1/2} F(s) e^{-i2\pi ns} ds\,.$$

Now, let $G(s) = \mathrm{rect}(s)F(s)$ (one period of $F(s)$) and thus $F(s) = \sum_{m=-\infty}^{\infty} G(s-m)$. Now observe that

$$\alpha_n = \int_{-1/2}^{1/2} F(s) e^{-i2\pi ns} ds = \int_{-\infty}^{\infty} G(s) e^{-i2\pi ns} ds = F\{G(s)\}\big|_{x=n} = 1$$

One function that satisfies this relationship is $G(s) = \delta(s)$. Thus, one possible Fourier transform of comb(x) is:

$$F(s) = \sum_{m=-\infty}^{\infty} \delta(s-m) = \mathrm{comb}(s)$$

By uniqueness of the Fourier transform, this is the unique Fourier transform of comb(x).

Sampling and replication by comb(*x*). The comb function can be used to sample or extract values of a continuous function $g(x)$. Sampling with period X can be done as:

$$g(x)\mathrm{comb}(\tfrac{x}{X}) = \sum_{n=-\infty}^{\infty} g(x)\delta(\tfrac{x}{X}-n) = X\sum_{n=-\infty}^{\infty} g(x)\delta(x-nX) = X\sum_{n=-\infty}^{\infty} g(nX)\delta(x-nX)\,.$$

By the stretching and sifting properties of the delta function. A function $g(x)$ can be replicated with period X by convolving with a comb function:

$$g(x)*\mathrm{comb}(\tfrac{x}{X}) = \sum_{n=-\infty}^{\infty}\left[g(x)*\delta(\tfrac{x}{X}-n)\right] = X\sum_{n=-\infty}^{\infty}\left[g(x)*\delta(x-nX)\right] = X\sum_{n=-\infty}^{\infty} g(x-nX)$$

By the stretching and convolution properties of the delta function.

Sampling Theory. When manipulating real objects in a computer, we must first sample the continuous domain object into a discretized version that the computer can handle. As described above, we can sample a function $g(x)$ at frequency $f_s = 1/X$ using the comb function:

$$g_s(x) = g(x)\mathrm{comb}(\tfrac{x}{X}) = X\sum_{n=-\infty}^{\infty} g(nX)\delta(x-nX)\,.$$

The Fourier transform is:

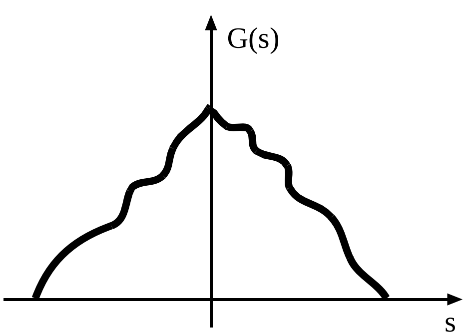
$$G_s(s) = G(s) * X\mathrm{comb}(Xs)$$

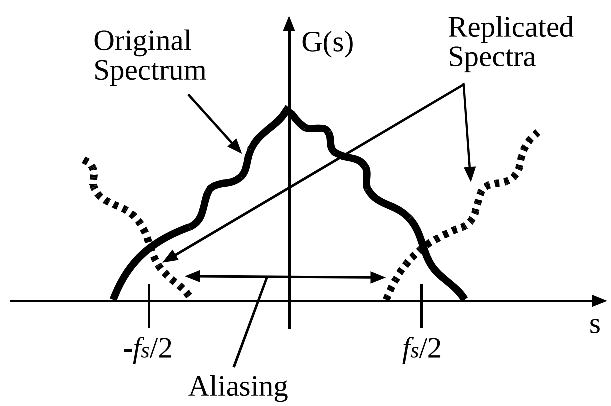
$$= G(s) * \sum_{m=-\infty}^{\infty} X\delta(Xs-m)$$

$$= G(s) * \sum_{m=-\infty}^{\infty} \delta(s-mf_s)$$

$$= \sum_{m=-\infty}^{\infty} G(s-mf_s)$$

Thus, sampling in one domain leads to replication of the spectrum in the other domain. The spectrum is periodic with period f_s. Typically, only frequencies less than $f_s/2$ can be represented in the discrete domain signal. Any components that lie outside of this spectral region ($-f_s/2 \le s \le f_s/2$) results in "aliasing" – the mis-assignment of spectral information.





The Whittaker-Shannon sampling theorem states that a band limited function with maximum frequency s_{max} can be fully represented by a discrete time equivalent provided the sampling frequency satisfies the Nyquist sampling criterion:

$$f_s = \frac{1}{X} \geq 2s_{\max}$$

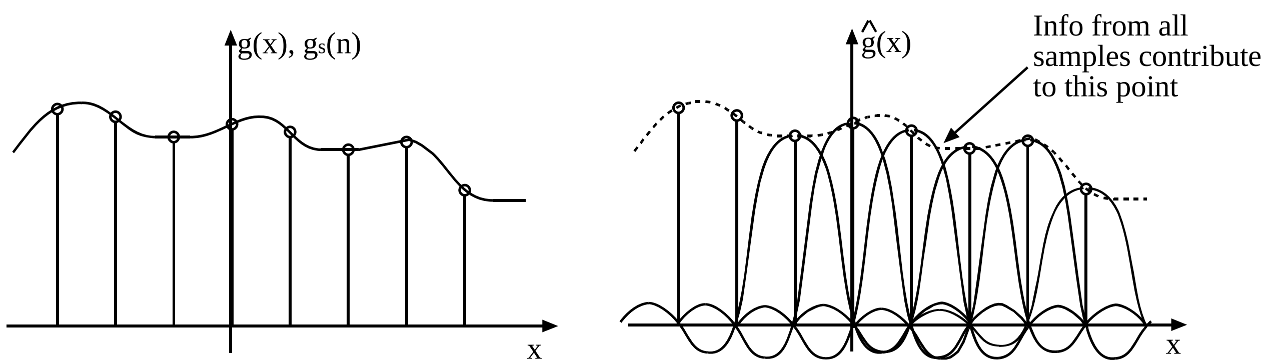
If this is the case, then the original spectrum can be extracted (by filtering) and by uniqueness of the FT, the original signal can be reconstructed. To reconstruct the original signal, we apply a reconstruction filter $H(s) = \mathrm{rect}(s/f_s) = \mathrm{rect}(Xs)$:

$$\begin{aligned}\hat{G}(s) &= G_s(s)H(s) = G_s(s)\mathrm{rect}(Xs) \\ &= G(s), \text{if there is no aliasing}\end{aligned}$$

In the x domain, this results in "sinc" interpolation:

$$\begin{aligned}\hat{g}(x) &= g_s(x) * \tfrac{1}{X}\mathrm{sinc}(\tfrac{x}{X}) \\ &= \left[\sum_{n=-\infty}^{\infty} g(nX)\delta(x-nX)\right] * \mathrm{sinc}(\tfrac{x}{X}) \\ &= \sum_{n=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-nX}{X})g(nX)\end{aligned}$$

If the Nyquist criterion is met, then $\hat{g}(x) = g(x)$.



1D Linear Systems

Consider a system $S[\cdot]$ with an input function $f(x)$ and an output or response function $g(x) = S[f(x)]$. This system is linear if and only if:

$$S[\alpha f_1(x) + \beta f_2(x)] = \alpha S[f_1(x)] + \beta S[f_2(x)] = \alpha g_1(x) + \beta g_2(x)$$

for all α, β, f_1 and f_2. More generally, the superposition of an arbitrary set of input functions will yield a net response that is the superposition of responses to each input function alone.

Additionally, if any input is scaled (e.g. by α) then the output will also be scaled by the same amount.

Based on the sifting properties of the delta function, we know that

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi = f(x) * \delta(x)$$

which is the superposition of an infinite number of weighted and shifted delta functions. Based on linearity the output of this system $g(x) = S[f(x)]$ is:

$$g(x) = S\left[\int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi\right] = \int_{-\infty}^{\infty} f(\xi)S\left[\delta(x-\xi)\right]d\xi$$

(The system operates on functions of x and $f(\xi)$ is a constant scaling factor.) $S[\delta(x- \xi)]$ is a special function know as the impulse response and can is defined as:

$$h(x;\xi) = S\left[\delta(x-\xi)\right]$$

is the response to an impulse located at x = ξ and

$$g(x) = \int_{-\infty}^{\infty} f(\xi)h(x;\xi)d\xi$$

is known as the superposition integral. This representation for the output is valid for any linear system.

Now, consider a system that is shift invariant (or time invariant for functions of time). We define a system as being shift invariant if and only if:

$$g(x-a) = S[f(x-a)]$$

for all g and a. For linear, shift invariant systems, the impulse response can be written as:

$$h(x;\xi) = S\left[\delta(x-\xi)\right] = h(x-\xi;0) = h(x-\xi)$$

The superposition integral then becomes:

$$g(x) = \int_{-\infty}^{\infty} f(\xi)h(x-\xi)d\xi = f(x) * h(x)$$

the convolution of the input function with the impulse response, $h(x)$. For linear, shift invariant systems (or linear, time-invariant systems) only, we can then consider the Fourier domain equivalent:

$$G(s) = F(s)H(s)$$

Where $H(s)$ is known as the transfer function or system spectral response.

Notes on the 2D Fourier Transform

Definition. The 2D Fourier Transform (FT) relates a function to its frequency domain equivalent. The FT of a function *g*(*x*,*y*) is defined by the 2D Fourier integral:

$$G(u,v) = F\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i2\pi(xu+vy)}dxdy$$

There is also an inverse FT relationship:

$$g(x,y) = F^{-1}\{G(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u.v)e^{i2\pi(xu+vy)}dudv$$

Uniqueness: Given the existence of the inverse FT, it follows that if the FT exists, it must be unique. That is, for a function forms a unique pair with its FT:

$$g(x,y) \leftrightarrow G(u,v)$$

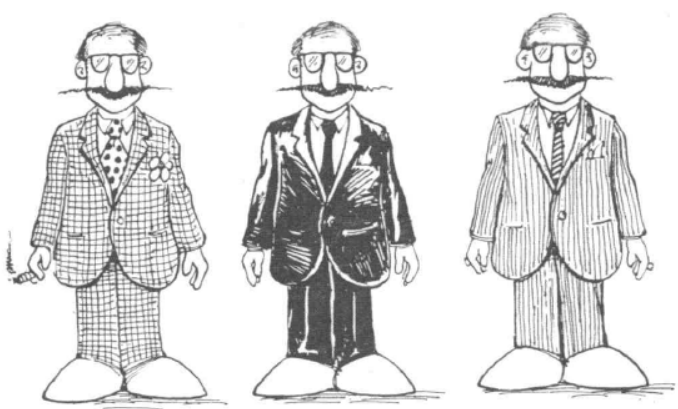


Figure 9-7. Three entrepreneurs and their working attire demonstrate the concept of spatial frequency. The used-car salesman's plaid jacket contains high spatial frequencies both horizontally and vertically. The undertaker's plain black jacket has zero spatial frequency. The banker's pinstripe suit has zero vertical spatial frequency but higher horizontal spatial frequency.

(Given to me by a student, I'm sure I should cite where this came from, but I don't know – if you know, please tell me)

2D FT in Polar Coordinates. We consider a special case where the functional form of *g*(*x*,*y*) is separable in polar coordinates, that is, $g(r,\theta) = g_R(r)g_\Theta(\theta)$. Since $g_\Theta(\theta)$ is periodic in θ, it has a Fourier series representation:

$$g_\Theta(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} .$$

It can be shown that

$$F_{2D}\left\{g_R(r)e^{in\theta}\right\} = (-i)^n e^{in\phi} \cdot \int_0^\infty 2\pi g_R(r)J_n(2\pi r\rho)rdr$$

where the part under the integral in known as the Hankel transform of order *n*, and $J_n(\cdot)$, is the n^{th} order Bessel function of the first kind:

$$J_n(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(a\sin\varphi - n\varphi)}d\varphi .$$

(Derivation of the Hankel transform relationship relies on $e^{-i2\pi(xu+yv)} = e^{-i2\pi r\rho\cos(\theta-\phi)}$.) Thus, the 2D FT in polar form is:

$$G(\rho,\phi) = F\{g_R(r)g_\Theta(\theta)\} = \sum_{n=-\infty}^{\infty} a_n(-i)^n e^{in\phi}\cdot\int_0^\infty 2\pi g_R(r)J_n(2\pi r\rho)rdr$$

For the special case of circular symmetry of g, that is, $g(r,\theta) = g_R(r)$, then:

$$G(\rho,\phi) = G(\rho) = 2\pi\int_0^\infty g_R(r)J_0(2\pi r\rho)rdr$$

which is also a circularly symmetric function. The inverse relationship is the same:

$$g_R(r) = 2\pi\int_0^\infty G(\rho)J_0(2\pi r\rho)\rho d\rho$$

Symmetry Properties of the FT. If $g(x,y)$ is real, then $G(u,v)$ is Hermitian Symmetric, that is, $G(u,v) = G^*(-u,-v)$. If $g(x,y)$ is real and even, that is, $g(x,y)=g(-x,-y)$, then $G(u,v)$ is also real and even. Finally, as described above, if we have a real and circularly symmetric function $g(r,\theta) = g_R(r)$, the $G(\rho,\phi) = G(\rho)$, a real and circularly symmetric function.

The delta function, $\delta(x,y)$. The delta function in two is equal the to product of two 1D delta functions $\delta(x,y) = \delta(x)\delta(y)$. In a manner similar to the 1D delta function, the 2D delta function has the following definition:

$$\delta(x,y) = \{\begin{matrix}\infty, x = 0 \text{ and } y = 0 \\ 0, \text{ otherwise}\end{matrix} \quad \text{and} \quad \int\int \delta(x,y)dxdy = 1$$

Most properties of $\delta(x,y)$ can be derived from the 1D delta function. There is also a polar coordinate version of the 2D delta function: $\delta(x,y) = \delta(r)/\pi r$.

Fourier Transform Theorems. Let a and b are non-zero constants and $F\{g(x,y)\} = G(u,v)$ and $F\{h(x,y)\} = H(u,v)$.

Linearity	$F\{ag(x,y)+bh(x,y)\} = aG(u,v)+bH(u,v)$
Magnification	$F\{g(ax,by)\} = \frac{1}{\lvert ab\rvert} G(\frac{u}{a},\frac{v}{b})$
Shift	$F\{g(x-a,y-b)\} = G(u,v)e^{-i2\pi(ua+vb)}$
Complex Modulation	$F\{g(x,y)e^{i2\pi(xa+yb)}\} = G(u-a,v-b)$
Convolution/Multiplication	$g(x,y)**h(x,y) = \int\int g(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$ $F\{g(x,y)**h(x,y)\} = G(u,v)H(u,v)$ $F\{g(x,y)h(x,y)\} = G(u,v)**H(u,v)$
Separability	$g(x,y) = g_X(x)g_Y(y)$ $F\{g(x,y)\} = F_{1D,x}\{g_X(x)\}F_{1D,y}\{g_Y(y)\}$ $= G_X(u)G_Y(v)$
Power	$\iint \lvert g(x,y)\rvert^2 dxdy = \iint \lvert G(u,v)\rvert^2 dudv$ $\iint g(x,y)h^*(x,y)dxdy = \iint G(u,v)H^*(u,v)dudv$
Axis Reversal	$F\{g(-x,-y)\} = G^*(u,v)$

Some common 2D FT pairs:

$g(x,y)$	$G(u,v)$
1	$\delta(u,v)$
$\delta(x,y)$	1
$\delta(x-a,y-b)$	$e^{-i2\pi(ua+vb)}$
$e^{-\pi r^2} = e^{-\pi x^2}e^{-\pi y^2}$	$e^{-\pi\rho^2} = e^{-\pi u^2}e^{-\pi v^2}$
$\cos(2\pi x) = \cos(2\pi x)\cdot 1$	$\frac{1}{2}[\delta(u-1)+\delta(u+1)]\delta(v)$
$\mathrm{rect}(y) = 1\cdot\mathrm{rect}(y)$	$\delta(u)\mathrm{sinc}(v)$
$\mathrm{rect}(ax)\mathrm{rect}(by)$	$\frac{1}{\lvert ab\rvert}\mathrm{sinc}(\frac{u}{a})\mathrm{sinc}(\frac{v}{b})$
$\mathrm{rect}(r) = \{\begin{matrix}1, r\le\frac{1}{2}\\ 0, r>\frac{1}{2}\end{matrix}$	$\frac{J_1(\pi\rho)}{2\rho} = \mathrm{jinc}(\rho))$
$\mathrm{circ}(r) = \mathrm{rect}(\frac{r}{2}) = \{\begin{matrix}1, r\le 1\\ 0, r>1\end{matrix}$	$\frac{J_1(2\pi\rho)}{\rho} = 4\,\mathrm{jinc}(2\rho))$
$1/r$	$1/\rho$
comb(x,y) = comb(x)comb(y)	comb(u,v) = comb(u)comb(v)

The comb function in 2D, comb(x,y). The 2D sampling or comb function is defined as comb(x,y)=comb(x)comb(y) and has the 2D FT F\{comb(x,y)\}= comb(u,v). Formally, the 2D comb function is defined as:

$$\mathrm{comb}(x,y) = \sum_{n,m=-\infty}^{\infty} \delta(x-n, y-m)$$

In a manner similar to the 1D case, we can prove that Fourier transform of the 2D comb function is also a 2D comb function as given in the above table.

Sampling Theory in 2D. In a manner similar to sampling in 1D, sampling in 2D can be modeled as multiplying a function times the 2D comb function. With sample spacing of X and Y, in the x and y directions, the sampled function is:

$$g_s(x,y) = g(x,y)\mathrm{comb}(\tfrac{x}{X},\tfrac{y}{Y}) = g(x,y)\sum_{n,m=-\infty}^{\infty} \delta(\tfrac{x}{X}-n, \tfrac{y}{Y}-m)$$

$$= XY \sum_{n,m=-\infty}^{\infty} \delta(x-nX, y-mY) g(x,y)$$

$$= XY \sum_{n,m=-\infty}^{\infty} \delta(x-nX, y-mY) g(nX,mY)$$

The discrete domain equivalent is $g_d(n,m) = g(nX, mY) = g_s(nX, mY)$. In the Fourier domain, the result is:

$$G_s(u,v) = G(u,v) ** XY\mathrm{comb}(Xu,Yv)$$

$$= G(u,v) ** \sum_{n,m=-\infty}^{\infty} \delta(u-\tfrac{n}{X}, v-\tfrac{m}{Y})$$

$$= \sum_{n,m=-\infty}^{\infty} G(u-\tfrac{n}{X}, v-\tfrac{m}{Y})$$

Thus, sampling in one domain leads to replication of the spectrum in the other domain. Spacing of the replicated spectra is ($1/X$,$1/Y$). The Whittaker-Shannon sampling theorem in 2D states that a band limited function with maximum frequencies $s_{max,x}$ and $s_{max,y}$ can be fully represented by a discrete time equivalent provided the sampling frequency satisfies the Nyquist sampling criterion:

$$\frac{1}{X} \geq 2s_{\max,x} \text{ and } \frac{1}{Y} \geq 2s_{\max,y}$$

Under these circumstances, there is no spectral overlap (or aliasing) the original spectrum and by uniqueness of the FT, the original signal can be reconstructed.

To reconstruct the original signal, we apply a reconstruction filter $H(u,v) = \mathrm{rect}(Xu)\mathrm{rect}(Yv)$.

$$\hat{G}(u,v) = G_s(u,v)H(u,v) = G_s(u,v)\mathrm{rect}(Xu)\mathrm{rect}(Yv)$$

$$= G(u,v), \text{if there is no aliasing}$$

In the (x,y) domain, this corresponds to "sinc" interpolation in 2D (sinc(x) = $\frac{\sin \pi x}{\pi x}$):

$$\hat{g}(x, y) = g_s(x, y) ** \tfrac{1}{XY}\,\mathrm{sinc}(\tfrac{x}{X})\mathrm{sinc}(\tfrac{y}{Y})$$

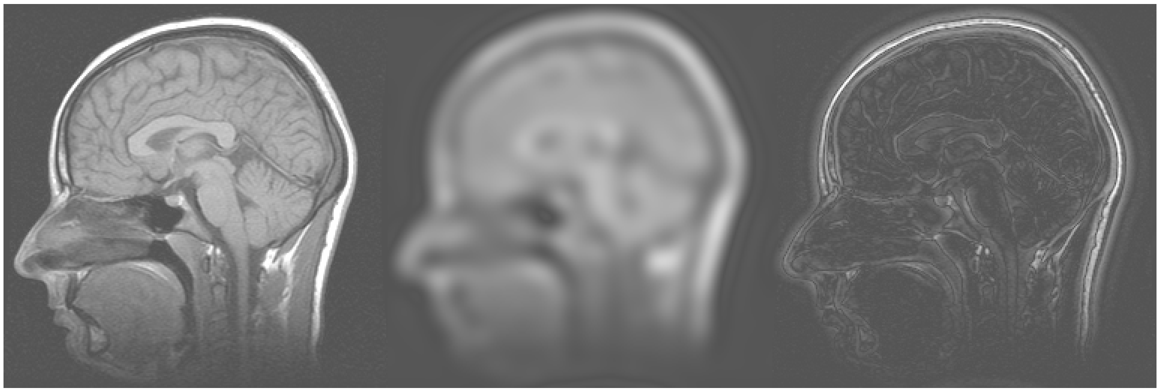
$$= \left[\sum_{n,m=-\infty}^{\infty} \delta(x - nX, y - mY) g(nX, mY) \right] ** \mathrm{sinc}(\tfrac{x}{X})\mathrm{sinc}(\tfrac{y}{Y})$$

$$= \sum_{n,m=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-nX}{X})\mathrm{sinc}(\tfrac{y-mY}{Y}) g(nX, mY)$$

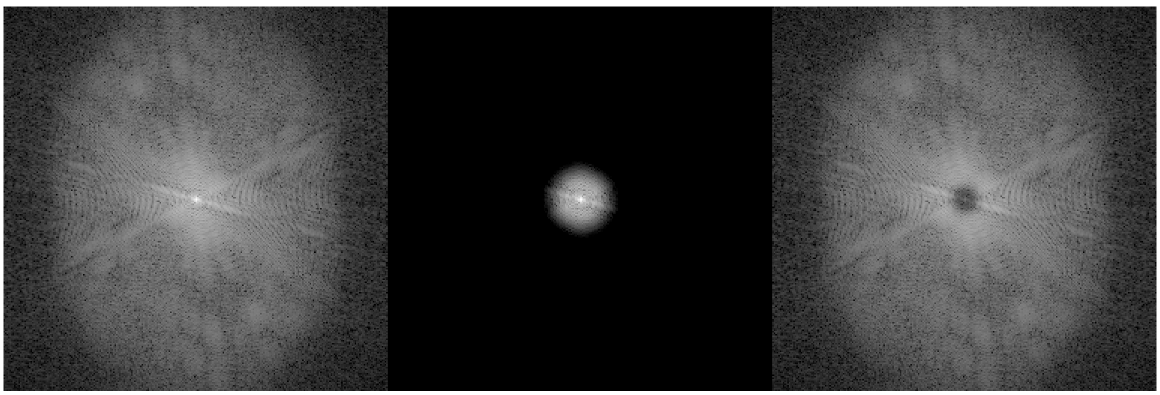
$$= \sum_{n,m=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-nX}{X})\mathrm{sinc}(\tfrac{y-mY}{Y}) g_d(n, m)$$

The last line demonstrates how the original continuous signal can be retrieved from the discrete sampled version of $g(x,y)$.

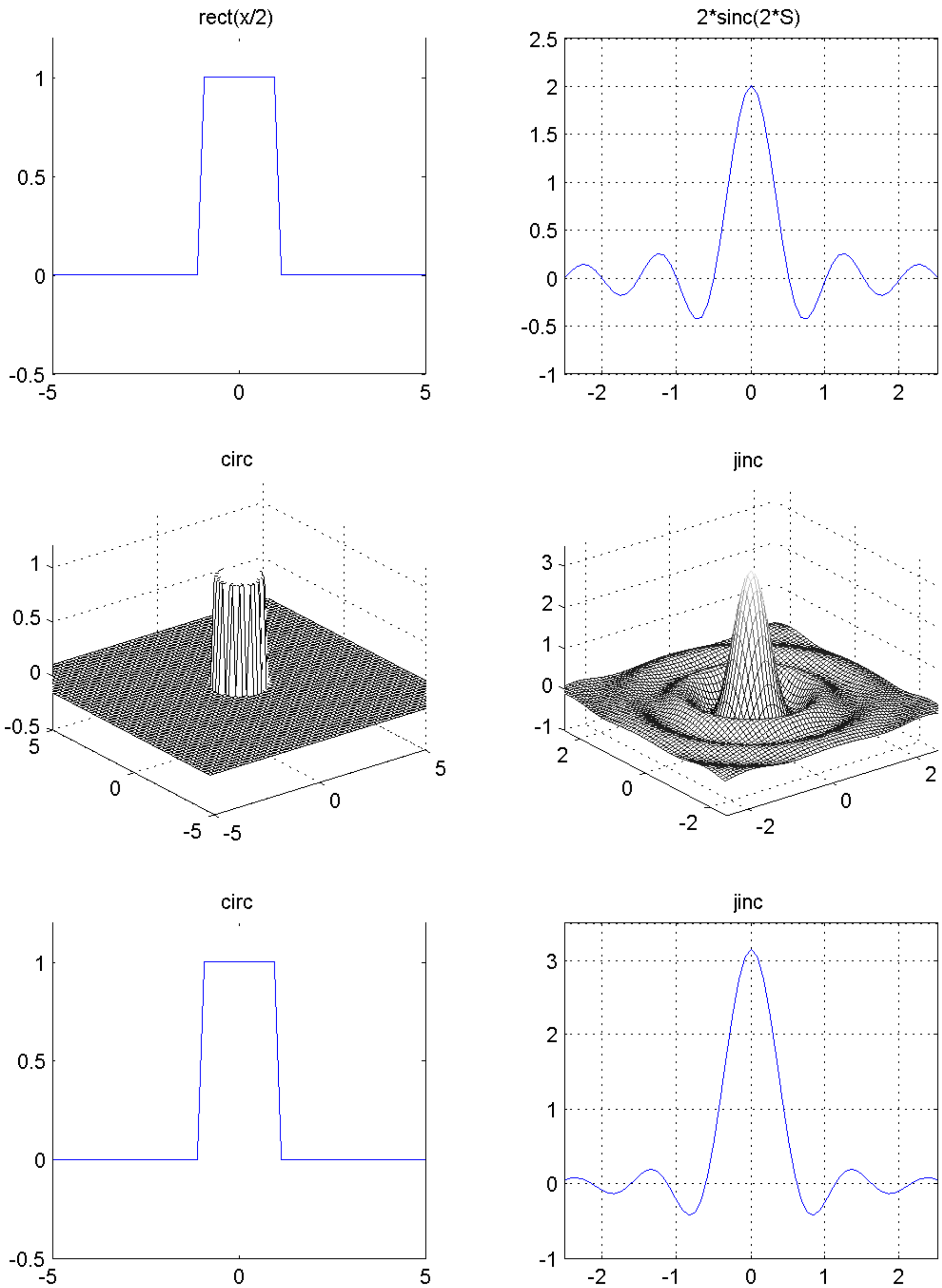
Examples of Fourier Transforms:

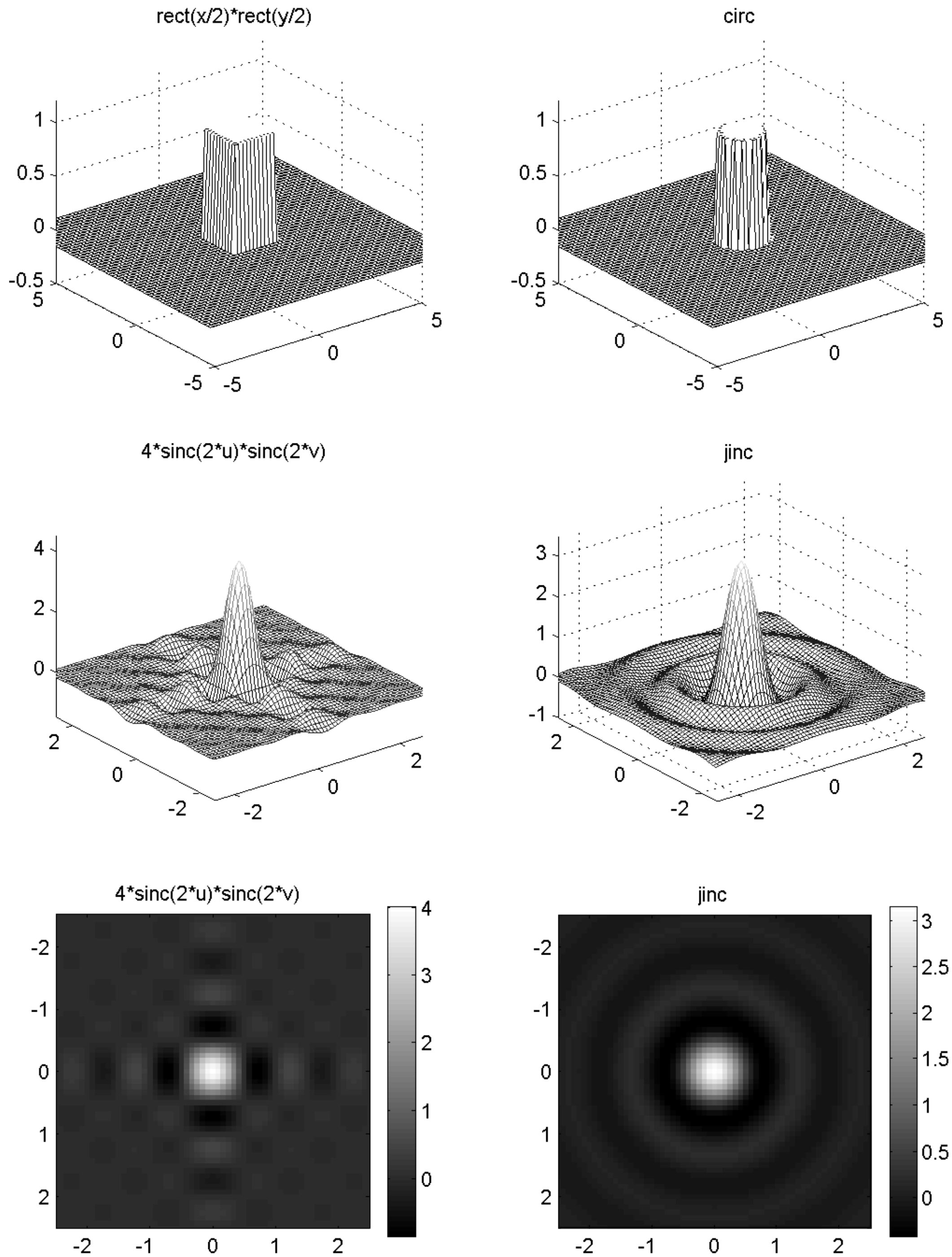


2D data in image domain | Low spatial freq data (image domain) | High spatial freq data (image domain)



2D data in Fourier domain | Low spatial freq data (Fourier domain) | High spatial freq data (Fourier domain)


rect(x/2)
2*sinc(2*S)
circ
jinc
circ
jinc


rect(x/2)*rect(y/2)
circ
4*sinc(2*u)*sinc(2*v)
jinc
4*sinc(2*u)*sinc(2*v)
jinc

g(x,y) = rect(x)rec(y)
G(u,v) = sinc(u)sinc(v)

scaling (magnification) property

scaling (magnification) property

shifting property

modulation

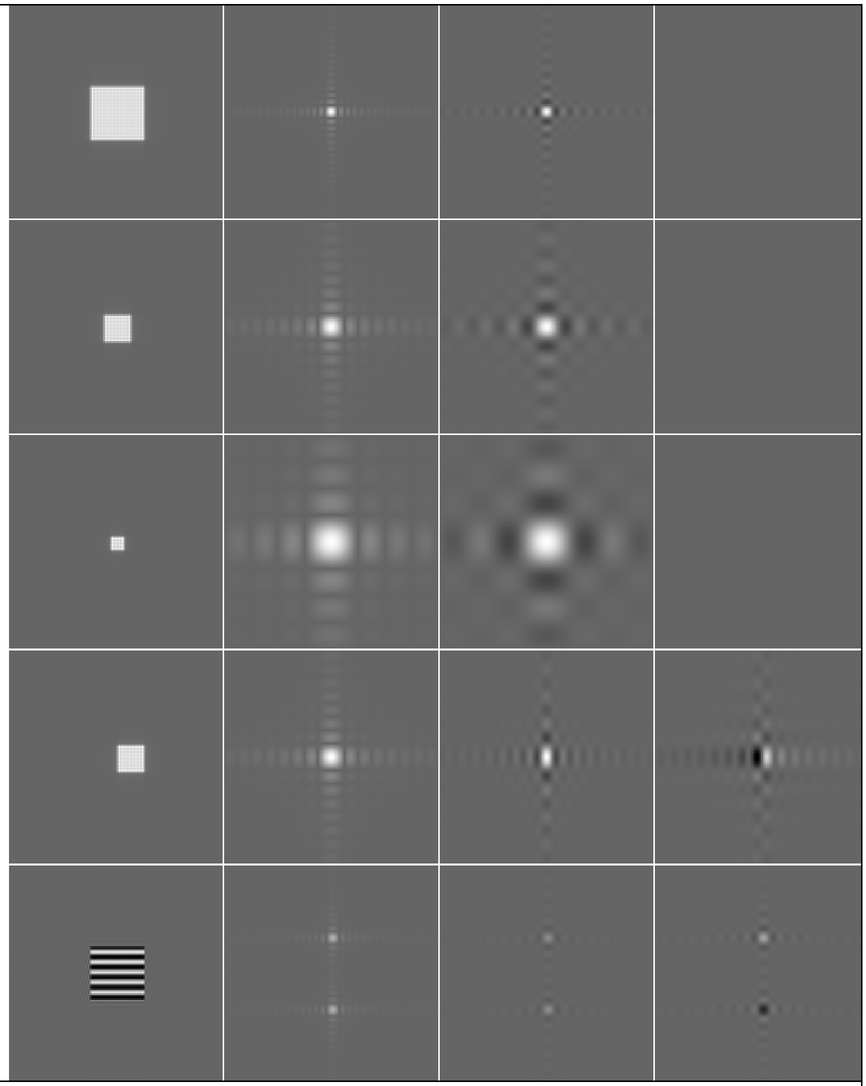
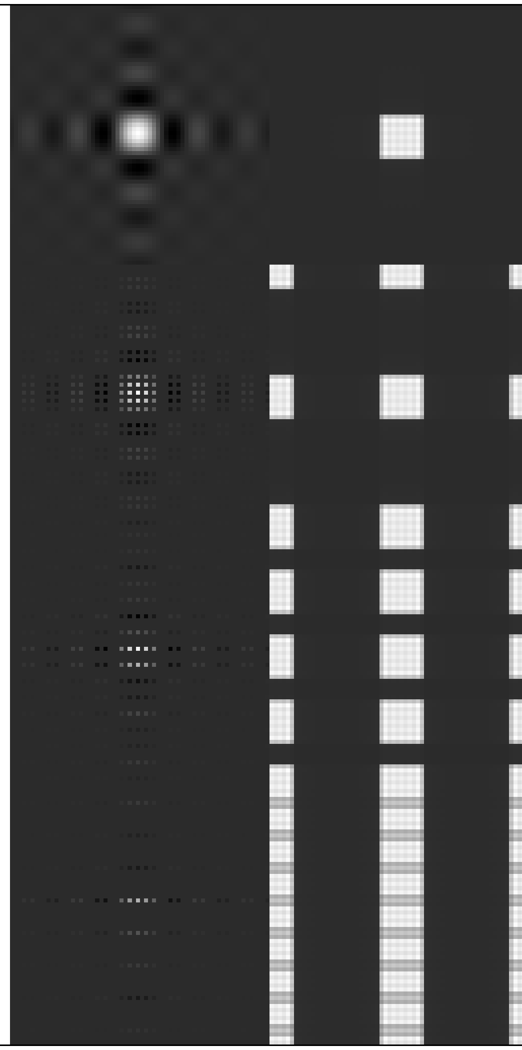
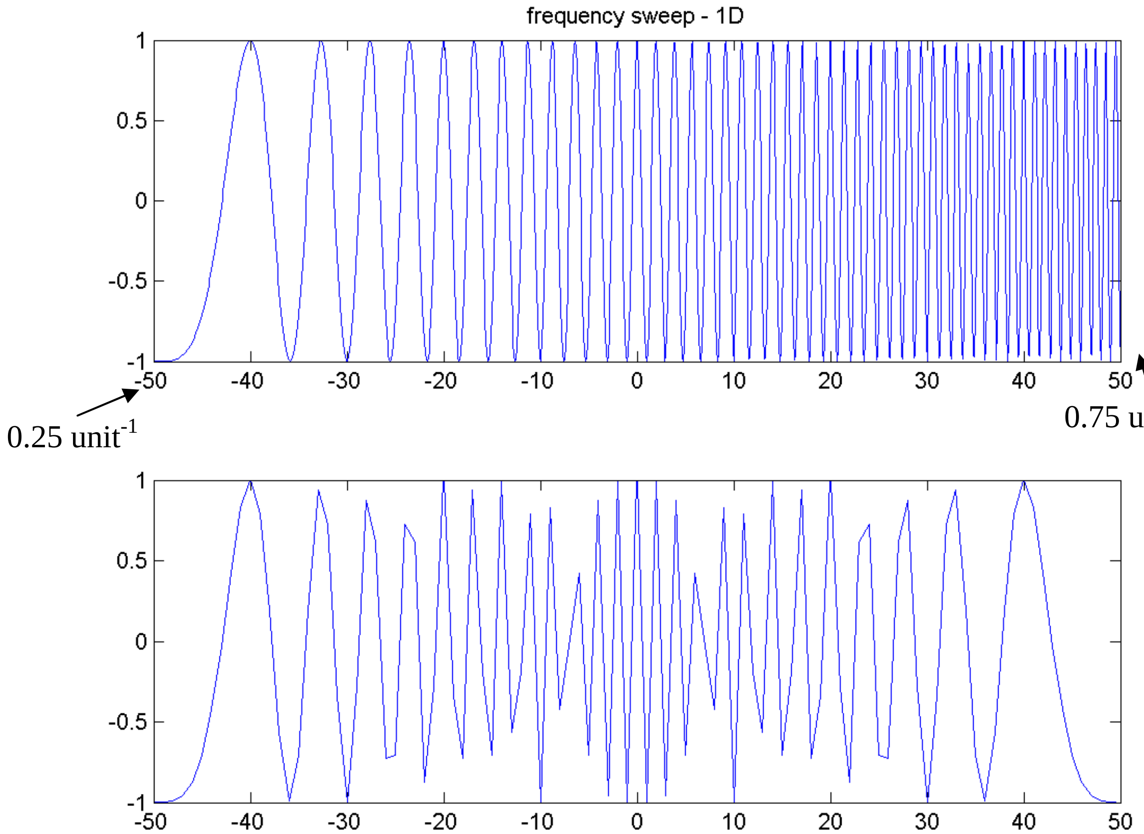


Image	Abs(Fourier)	Real(Fourier)	Imag(Fourier)

$g(x,y) = sinc(x)sinc(y)$ $G(u,v) = rect(u)rect(v)$ sampling pattern with $\Delta x = \Delta y$ In the Fourier transform we have the replication pattern with spacing $1/\Delta x = 1/\Delta y$ sampling pattern with $\Delta x < \Delta y$ In the Fourier transform we have the replication pattern with spacing $1/\Delta x > 1/\Delta y$ sampling pattern with $\Delta x << \Delta y$ This has aliasing in the y (v) direction	
	Image Data Fourier Data

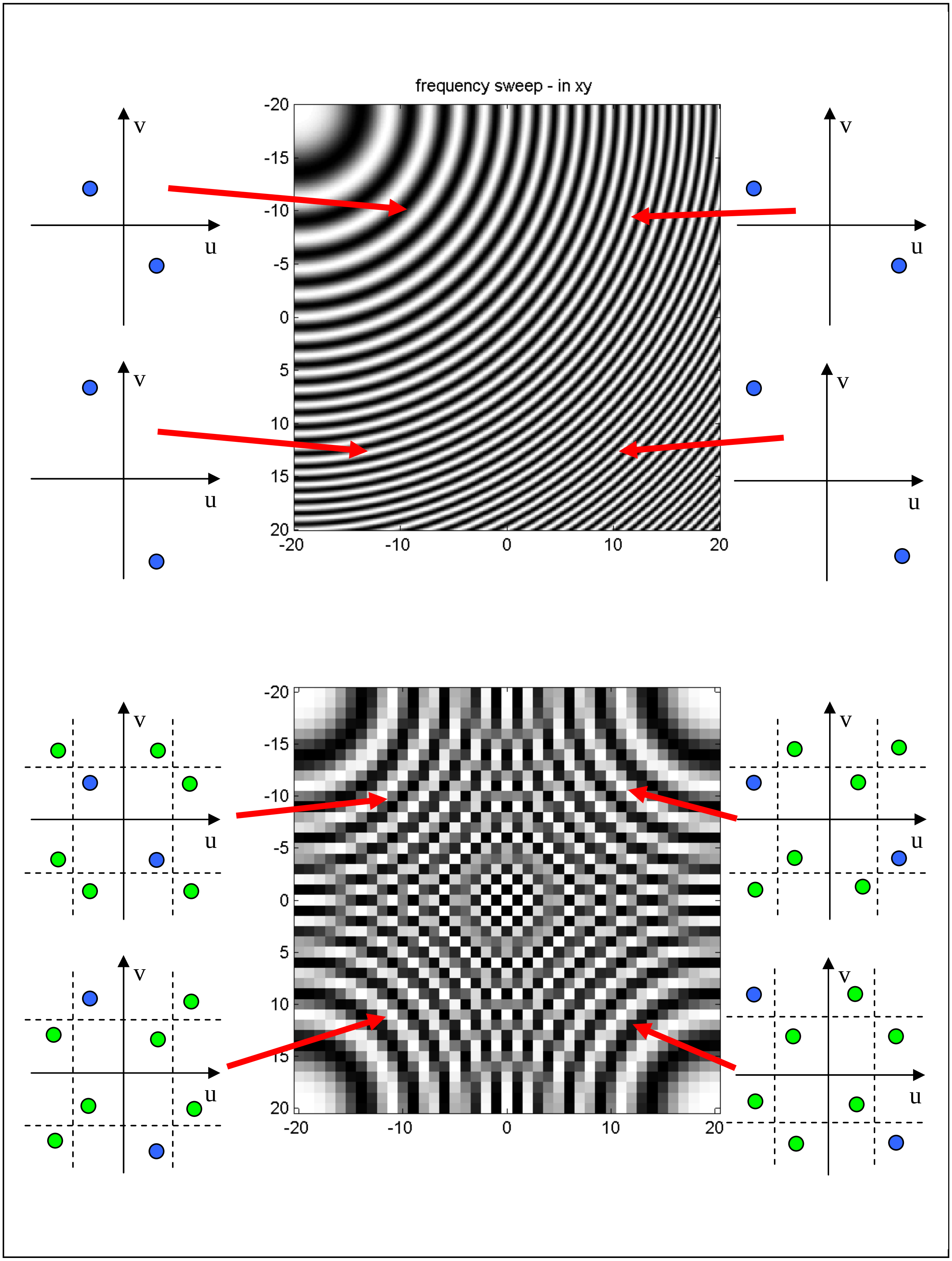
Aliasing in the Spatial Domain



In this example, the frequency is swept from a 0.25 to 0.75 cycles/unit distance as x goes from -50 to 50. At location zero, the frequency is 0.5 cycles/unit. In the upper plot, we see the original signal. In the lower plot, the signal is sampled with a space in of $\Delta x = 1$ unit ($f_s = 1$ unit^{-1})which means that all frequencies higher than $f_s/2 = 0.5$ unit^{-1} (anything to the right of 0) will be aliased. Indeed, as the frequency continues to go higher, the apparent frequency gets lower. The apparent frequency is $(f_s - f_i)$, where f_i is the local frequency.

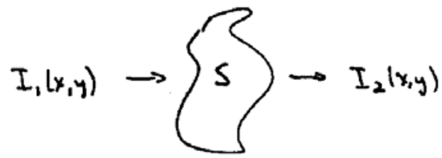
In the second example (below), we extend this to two dimensions. Here we have a linear variation of frequencies in both x and y, that is, the x component of the frequencies varies from 0.25 to 0.75 cycles/unit and the y component of the frequencies varies from 0.25 to 0.75 cycles/unit. The upper image is the original signal and the lower image is the signal sampled at $\Delta x = \Delta y = 1$ unit. The dashed lines mark the +/- 0.5 unit^{-1} line the corresponds the Nyquist limit. Only the spectral components in the central box can be represented.

In the attached spectral plots, the blue dots correspond to delta functions of the true frequencies. The green dots are the spectral replicants in the case of sampling. Only the upper left quadrant has no aliasing. The upper left and lower right appear to have the same frequency, but the latter is aliased. The upper right and lower left also appear to have the same frequency, but from different aliasing mechanisms – aliasing of the x component and y-component, respectively.


frequency sweep - in xy
v
u
-20
-15
-10
-5
0
5
10
15
20
-20
-10
0
10
20

2D Linear Systems

Consider an imaging system with an input image $I_1(x,y)$ that goes through some system $S[\cdot]$ and produces an output image $I_2(x,y)$, e.g. $I_2(x,y) = S[I_1(x,y)]$.



Several properties that we are interested in are:

- **Linearity**, which as two parts superposition and scaling. Thus, a system is linear, if and only if:
  $$S[\alpha g(x,y) + \beta h(x,y)] = \alpha S[g(x,y)] + \beta S[h(x,y)]$$
  For a linear system, we can define an impulse response as the output of a system for an impulse located at position (ξ, η):
  $$h(x,y;\xi,\eta) = S[\delta(x-\xi, y-\eta)]$$
  and in general, we can define the output, given some input image $I_1(x,y)$, using the superposition integral:
  $$I_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\xi,\eta) h(x,y;\xi,\eta) d\xi d\eta$$

- **Space Invariance**. A system is space invariant if and only if:
  $$I_2(x-a,y-b) = S[I_1(x-a,y-b)]$$
  for all a, b, and I_1. Alternately, a system is space invariant if and only if the impulse response can be expressed in terms of the shifts of the delta function:
  $$h(x-\xi, y-\eta) = S[\delta(x-\xi, y-\eta)].$$
  Thus, $h(x,y) = S[\delta(x,y)]$ is all that is needed to specify the system. The superposition integral becomes:
  $$\begin{aligned} I_2(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\xi,\eta) h(x-\xi, y-\eta) d\xi d\eta \\ &= I_1(x,y) ** h(x,y) \end{aligned}$$
  where ** indicates 2D convolution.

<u>Example of a 2D Imaging System.</u> Here we consider a pinhole imaging system:

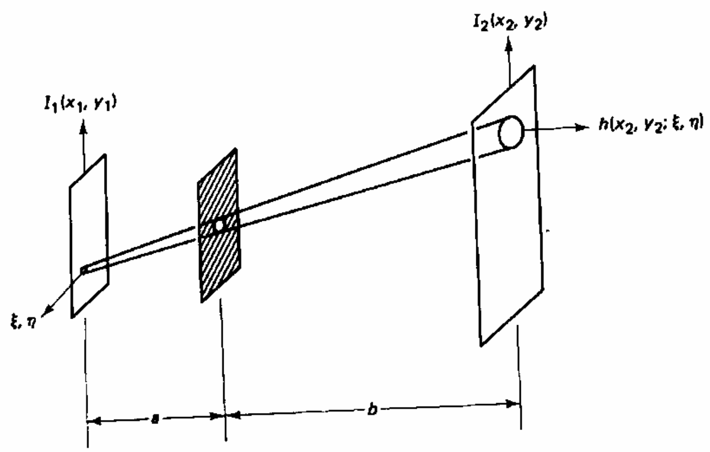
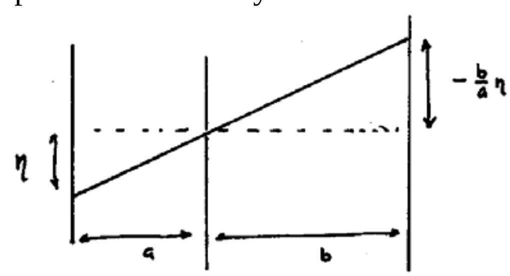


FIG. 2.2 Pinhole imaging system with magnification.

<u>Is this system linear?</u> Assuming that the aperture is open or close and that light always travels in straight lines through the hole, then yes, the system is linear.

We should then be able to determine the impulse response. Let's first consider two different magnifications factors – one for the object and one for the pinhole aperture. For object magnification, we imagine the pinhole in infinitely small:



For a shift of η, in the input, we will get a shift of $-\frac{b}{a}\eta$ in the output plane. Thus, we define an input (source) magnification term as $M = -\frac{b}{a}$.

Consider then a system for a delta function at position (ξ, η):

$$\begin{aligned} h(x, y; \xi, \eta) &= S[\delta(x - \xi, y - \eta)] \\ &= C\delta(x - M\xi, y - M\eta) \end{aligned}$$

where the delta function appears scaled by C and at location $(M\xi, M\eta)$.

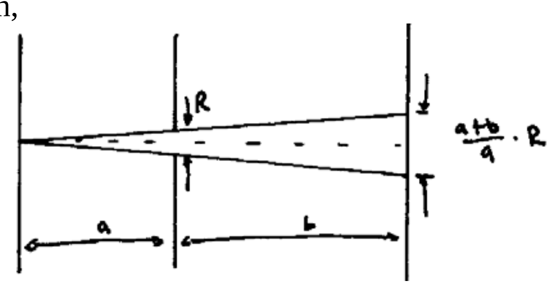
<u>Is this system space invariant?</u> No – the above expression cannot be written as a function of $(x - \xi)$ and $(y - \eta)$.

Now, given an input image $I_1(x,y)$, we can still determine the output image using the superposition integral (remember the system is still linear):

$$I_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\xi,\eta)h(x,y;\xi,\eta)d\xi d\eta$$
$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\xi,\eta)\delta(x-M\xi, y-M\eta)d\xi d\eta$$
$$= \frac{C}{M^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\frac{\xi'}{M},\frac{\eta'}{M})\delta(x-\xi', y-\eta')d\xi' d\eta'$$
$$= \frac{C}{M^2} I_1(\frac{x'}{M},\frac{y'}{M})$$

The output is a scaled and magnified version of the input image.

For the pinhole magnification,



imagine that the pinhole had a radius of R then the radius in output plane would be $\frac{a+b}{a}R$, yielding an aperture magnification function of $m = \frac{a+b}{a}$. Now, suppose we had some aperture function $a(x,y)$, this will now be magnified in the output plane. Thus, the impulse response will take on a form similar to:

$$h(x,y;\xi,\eta) = Ca(\frac{x-M\xi}{m},\frac{y-M\eta}{m})$$

and the ouput image will take on the form:

$$I_2(x,y) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\xi,\eta)a(\frac{x-M\xi}{m},\frac{y-M\eta}{m})d\xi d\eta$$
$$= \frac{C}{M^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(\frac{\xi'}{M},\frac{\eta'}{M})a(\frac{x-\xi'}{m},\frac{y-\eta'}{m})d\xi' d\eta'$$
$$= \frac{C}{M^2} I_1(\frac{x'}{M},\frac{y'}{M}) ** a(\frac{x'}{m},\frac{y'}{m})$$

Here the output image is the convolution of the scaled and magnified versions of the input image and the pinhole function. Though this system is *not* space invariant, we were can still able to write the output in the form of a convolution (though not a convolution of the input image, but with a magnified version of the input image).