Notes on MRI, Part III

1D Imaging – Frequency Encoding

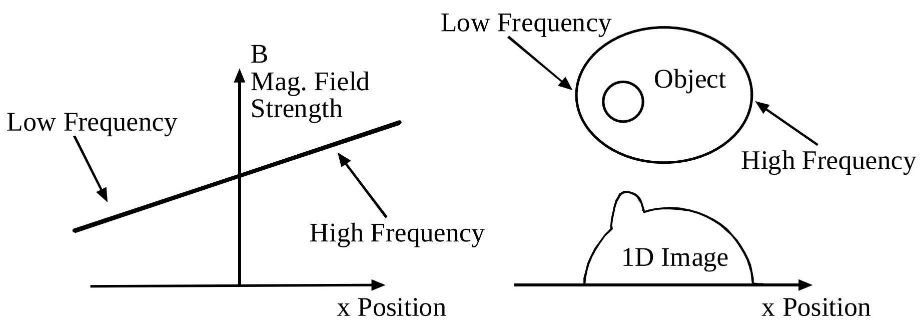
Gradients act to setup a one-to-one correspondence between frequency and spatial position. This is known as *frequency encoding*. For example:

$$\Delta w(x) = gG_x x$$

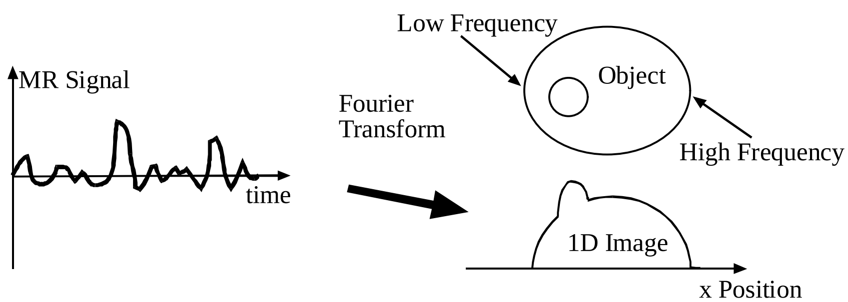
or

$$\Delta f\,(x) = \frac{g}{2p} G_x x$$

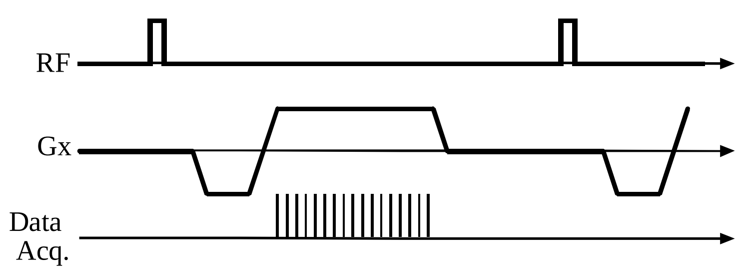
Graphically this is:



As previously described, to create an image, we need to take the 1D (forward or inverse) FT of the received signal:



The pulse sequence to produce this might look like this:



Because we have a constant gradient on while we are acquiring data, the frequency spatial location relationship holds. The negative gradient at the beginning is due to the necessity to "reverse time." Previously, when we defined the relationship:

$$s(t) = \int m(x)\exp(-i2psx)dx\Big|$$

$$= F\{m(x)\}\Big|_{s=gG_x t/2p}$$

$$= M(s = gG_x t/2p)$$

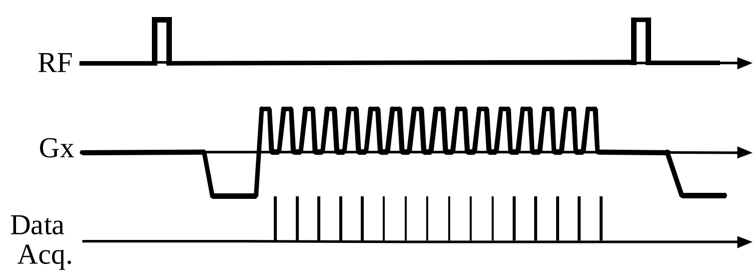
We ignored the fact that we where only acquiring the FT, $M(s)$, for positive time in $gG_x t = 2ps$ (time begins at the point of tipping the magnetization into the transverse plane). If we have a negative gradient for a period of time (leading to negative accumulation of phase), then we have effectively turned back the clock to a negative starting time for phase accumulation.

Alternate Methods for 1D Localization – Phase Encoding

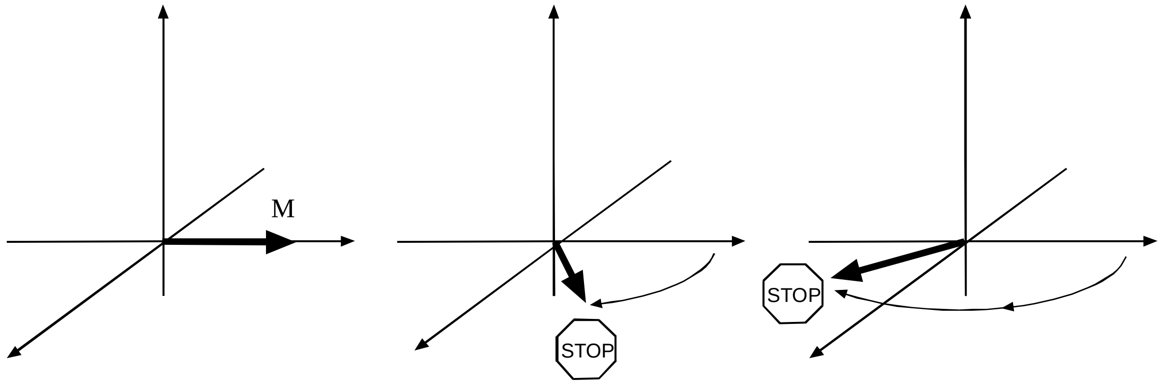
While the constant gradient is an excellent method for 1D localization, it is not the only method. It is important to note that it is the instantaneous phase of the magnetization at each sample that is important and not the instantaneous frequency – that, where the magnetization is pointing the complex plane, e.g. $m_{xy,rot}(x,y,t)$, is what is really important and not how fat it is moving. Recall, we are sampling the received signal:

$$s(t) = \iint m_{xy,rot}(x,y,t)dxdy$$

Thus, we can look at other methods that will produce the same distribution of magnetization in the rotating frame. Consider the following pulse sequence:

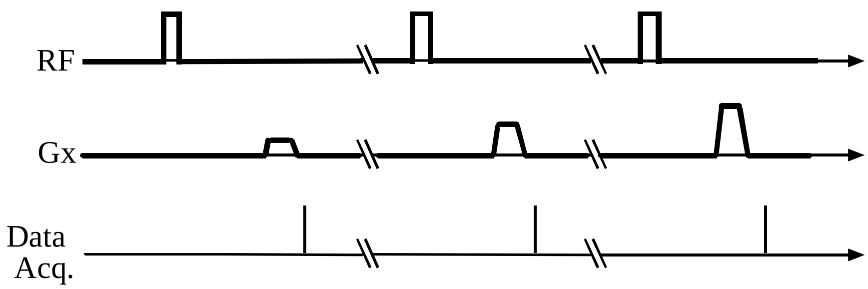


In this pulse sequence the same amount of phase accumulation occurs between samples as in the previous pulse sequence. When the gradient amplitude is zero, there is zero frequency offset ($\Delta B = \Delta w = 0$) and thus not phase accumulation during these periods (there is, of course T2 decay, but we're ignoring that for now). If you looked at a particular spin, it would have a stop-action kind of appearance:

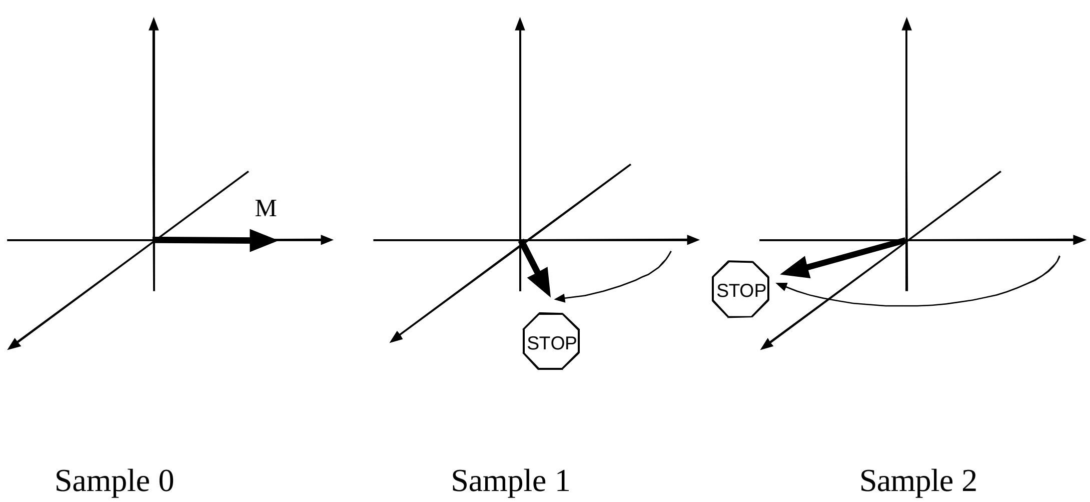


We could imaging putting a whole bunch of these frames together and it might have the appearance of a continuous movement at a frequency Δw, but in this case, it is really occurring in little jumps. If we take the FT of these sample, we will still get our 1D image.

Consider yet an alterative pulse sequence:



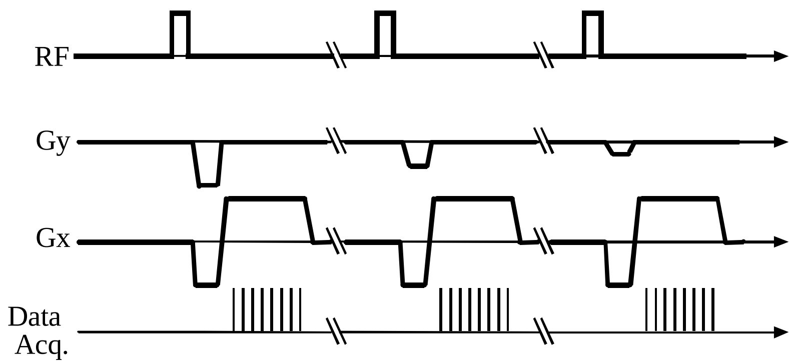
Here, there is a differing amount of phase accumulation after each RF pulse. In this case, it comes from having different G values for the same amount of time, but the impact is the same:



Again, if we take the FT of these sample, we will still get our 1D image. In both cases, the samples are collected along a dimension that looks like time, but isn't quite. Some in MRI call this dimension pseudo-time. In these examples, the spatial location is not encoded in the frequency, but rather is encoded into a specific sequence of phase accumulation between samples. This is known as *phase encoding*.

2D Imaging

One way to do 2D imaging is to combine frequency encoding in one spatial dimension (e.g. x) with phase encoding in another spatial dimension (e.g. y). Take the following pulse sequence:



For a give phase encoding value, if we 1D FT the group of samples we'll get a 1D image in x. If, on the other hand, we take the n^{th} sample from each group of samples (that is, each of these differs only by the phase encoding amount), and take the 1D FT of this data, we'll get a 1D image of the object in y.

Thus, it we put the data in an array (each group along a row) and take the 2D FT of this data, we will get a 2D image of the object! The above pulse sequence is known as the spin-warp pulse sequence is it is the most used pulse sequence (imaging method) in MRI.

k-Space

We will now develop the general theory for 2D imaging. The received signal is:

$$s(t) = \iint m_{xy,rot}(x,y,t)\,dx\,dy$$

Again, please bear in mind that we are mixing coordinate systems here. For example, in $m_{xy,rot}(x,y,t)$, the *x*,*y* in the argument refers to physical (*x*,*y*) locations in space, whereas the *xy* in the subscript refers to a mini-coordinate frame to describe direction of the magnetization vector at each point in space.

Let's consider a spatially and temporally varying applied magnetic fields introduced by time-varying gradient fields:

$$B(x,y,t) = B_0 + G_x(t)\cdot x + G_y(t)\cdot y$$

The instantaneous frequency at each point is space is then:

$$gB(x,y,t) = g\left(B_0 + G_x(t)\cdot x + G_y(t)\cdot y\right)$$

which in the rotating frame is:

$$\Delta w(x,y,t) = g\left(G_x(t)\cdot x + G_y(t)\cdot y\right)$$

and the spatially variant phase distribution is:

$$f(x,y,t) = \int_0^t g\left(G_x(t)\cdot x + G_y(t)\cdot y\right)dt$$

where time, *t*, begins with each RF pulse the bring magnetization from the longitudinal axis into the transverse plane where it is observable.

The Signal Equation, revisited. For convenience, we will let $C = 1$ and we will define $m(x,y) = m_0(x,y) = m_0(\mathbf{r})$. We now get a revised version of the signal equation:

$$\begin{aligned} s(t) &= \iint m_{xy,rot}(x,y,t)\,dxdy \\ &= \iint m(x,y)\exp(-if(x,y,t))\,dxdy \\ &= \iint m(x,y)\exp\left(-i\int_0^t g\left(G_x(t)\cdot x + G_y(t)\cdot y\right)dt\right)dxdy \\ &= \iint m(x,y)\exp\left(-ig\left(\int_0^t G_x(t)dt\cdot x + \int_0^t G_y(t)dt\cdot y\right)\right)dxdy \end{aligned}$$

Finally, we define two quantities:

$$k_x(t) = \frac{g}{2p}\int_0^t G_x(t)\,dt$$

$$k_y(t) = \frac{g}{2p}\int_0^t G_y(t)\,dt$$

And substituting into the above signal equation:

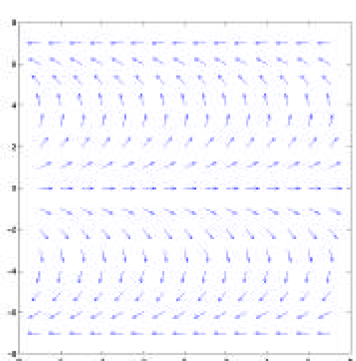
$$s(t) = \iint m(x,y)\exp\left(-i2p\left(xk_x(t) + yk_y(t)\right)\right)dxdy$$
$$= F_{2D}\{m(x,y)\}|_{u=k_x(t),v=k_y(t)} = M\left(k_x(t), k_y(t)\right)$$

That is, the signal is equal to the Fourier transform of the initial magnetization evaluated at locations defined by k_x and k_y above. Reminder:

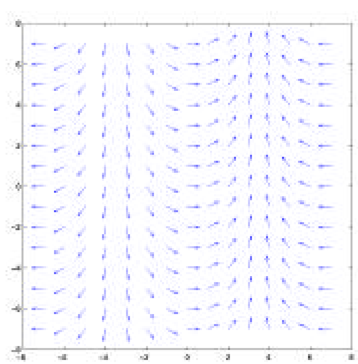
$$G(u,v) = F_{2D}\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i2p(xu+vy)}dxdy$$

The signal equation says that samples of the received signal are equal to samples of the 2D Fourier transform of the object. This make sense if we think about what exactly the expression for the 2D FT means – the FT at any point (u,v) is the integral over the object modified by a spatially variant (linear) rotation in the complex plane.

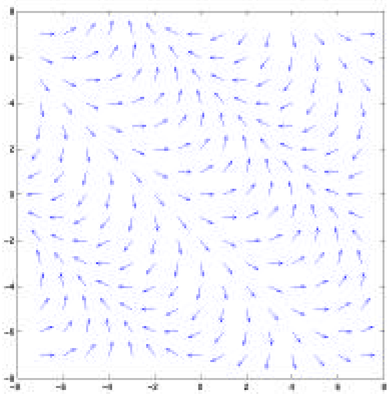
exp(-i2py)



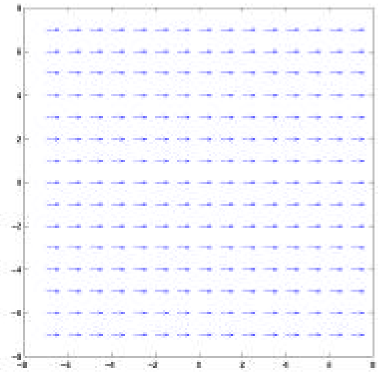
exp(-i2px)



exp(-*i*2*p*(*x*+*y*))



exp(-*i*0)



In MRI, the integration is performed by the integration of voltages in the RF coil. The phase variation is performed by the gradients – by shifting the field (and thus frequency) in a spatially linear fashion for a period of time, the magnetization will rotate to a new orientation (in the complex plane). Thus MRI has exactly the same mechanisms as the FT operation.

K-space always begins at the origin (0,0). Why? After the excitation pulse, all spins across the object are pointing in the same direction (e.g. exp(-*i*0)) and the integral of this is the DC value of the FT.

If we want to determine the object, we must fully sample its Fourier transform. A sequence of samples can be viewed as samples along a pathway determined by "running integral" under the gradient waveforms as defined by k_x and k_y above. The final object $m(x,y)$ can be reconstructed simply by taking the inverse 2D FT of the sampled Fourier data (k-space data):

$$m(x,y) = F_{2D}^{-1}\left\{M\left(k_x(t), k_y(t)\right)\right\}$$

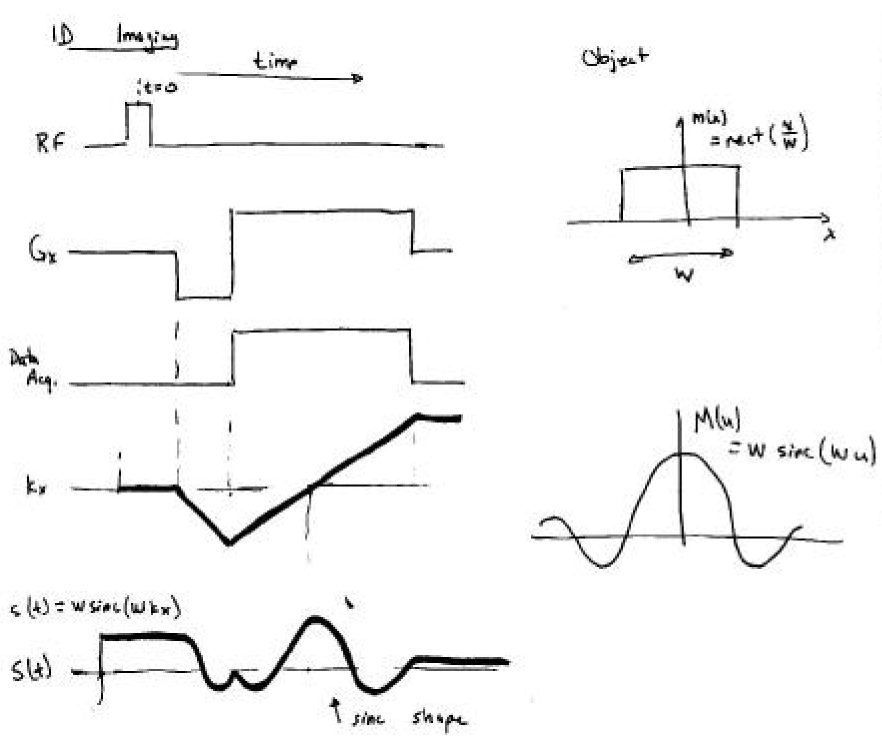
Does it make sense that our samples in time are actually samples of spatial frequency data? Remember the 1D case – we sampled time, FT'ed to get a spectrum. Since there was a 1-1 correspondence between frequency and spatial position, the FT of the frequency data produces time-domain data thus there is 1-1 correspondence between time and spatial-frequency.

Comments:

- Fourier space is called "k-space" in the MRI literature
- $G_x(t)$ and $G_y(t)$ control the k-space "trajectories" or paths on which sample locations fall.
- To create an image, we must sample $M(u,v)$ densely enough to prevent aliasing and of a large enough extent to have sufficient spatial resolution.

1D Imaging

We first examine the case of a 1D object $m(x) = \text{rect}(x/W)$. The received signal will be the Fourier transform $M(u) = W\,\text{sinc}(Wu)$ evaluated at particular k_x locations as dictated by the integration of the G_x gradient waveform. Presented here is "pulse sequence" for a 1D imaging experiment along with the k-space values and the received signal:

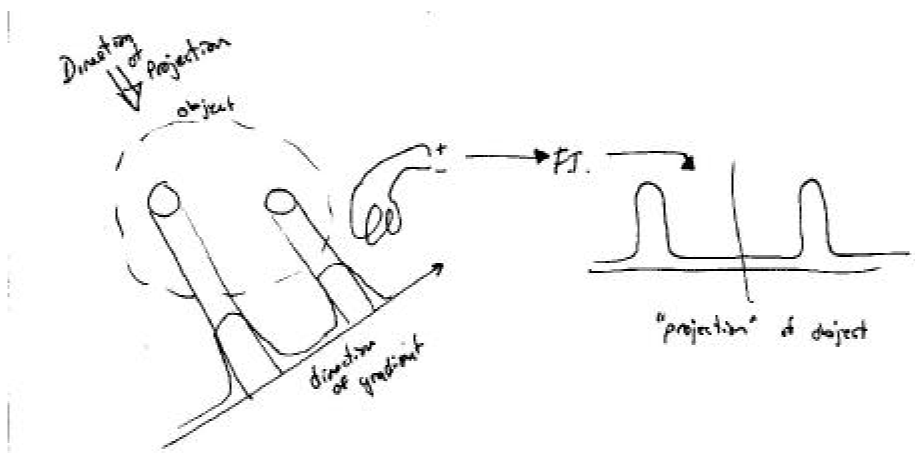


Recall the in the first MRI lecture we talked about taking the FT of the received signal to get a 1D view of the object. In this case, the received signal is a sinc function and thus the 1D FT of the sinc function is a rect function, which is in fact the object. During the entire time that data is acquired (see the Data Acq. line in the pulse sequence) the gradient is constant – during this situation there is 1-1 correspondence between frequency and spatial position. This is know as "frequency encoding" since spatial location is encoded as frequency.

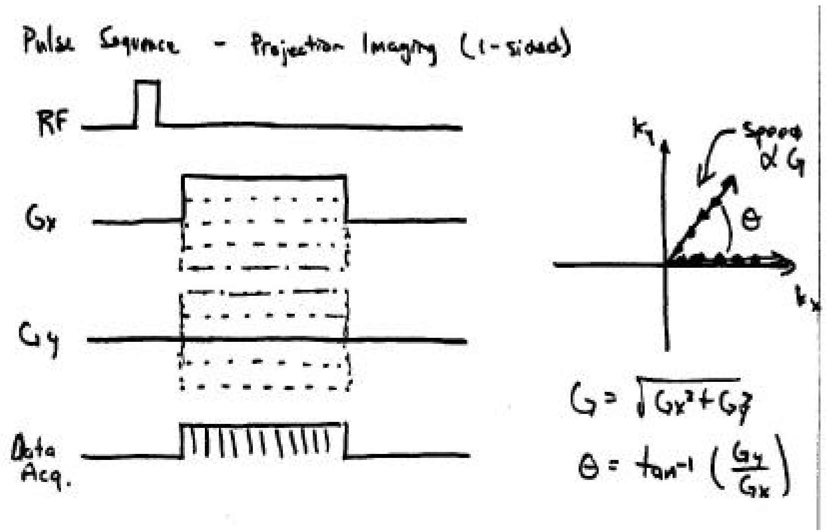
Notice also that we use a negative gradient before the positive gradient. Without the negative gradient, we can only acquire the positive spatial frequencies – or only ½ of the FT of the object. This is the difference from the previous example – there we took the FT of $s(t)$, but assumed that we knew $s(t)$ for all time – really we only know if for positive time.

2D Imaging using Projections

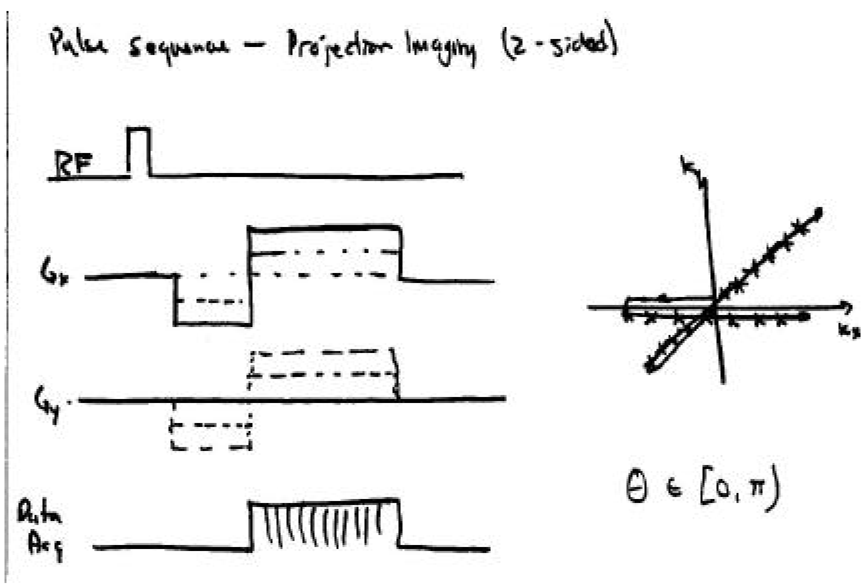
The first 2D imaging method implemented in MRI (by Paul Lauterbur while at SUNY Stony Brook), used a series of 1D acquisition with the gradients in different directions. Please not that by applying 1D gradients in x and y simultaneously, we get a single 1D gradient at an angle $q=\tan^{-1}(G_y/G_x)$. Thus we can get 1D views of the object (or projections) from many different angles.



We'll discuss (in the section on computed tomography) the methods for reconstructing images from 1D projections. For now, suffice it to say that if we acquire enough projections, we can fully determine the underlying object. The pulse sequence used by Lauterbur is given here:

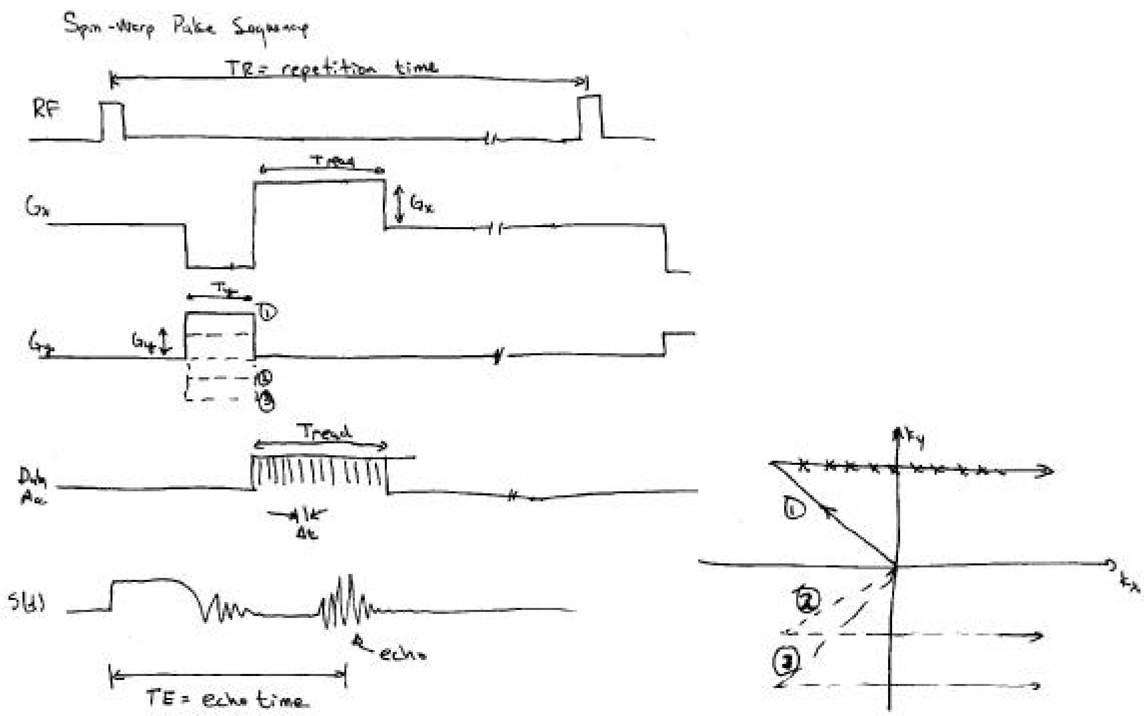


By sweeping through angles [0,2π) we acquire the full k-space (Fourier) data for the object. There is also a variant on projection imaging in which the positive gradient is preceded by a negative gradient – this will allow both positive and negative frequencies to be acquired along a particular line in k-space. One advantage to this approach is that one only needs to sweep through angles [0,π) in order to fully acquire the k-space data.



2D Spin-Warp Imaging

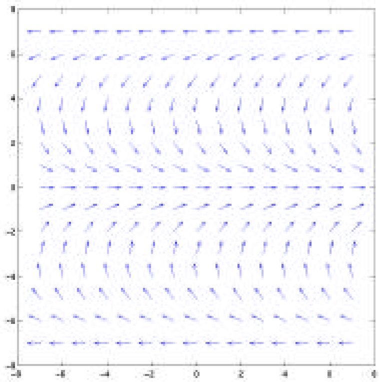
The most common acquisition used in MRI today is known as the "spin-warp" acquisition. The pulse sequence is given here along with the corresponding k-space trajectory:



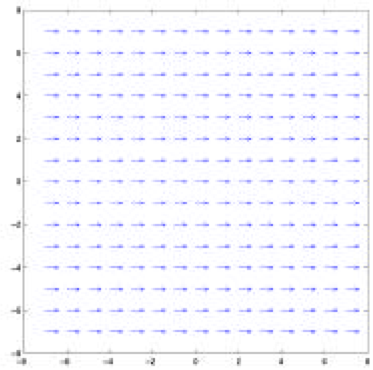
This is a repeated pulse sequence with a different y-gradient value for each RF excitation (TR interval). Let's look at the x-gradient – as in the case of 1D imaging, above, this gradient encodes the *x* spatial position into frequency. This is often called the "freq gradient or *x*, in this case, is known as the "frequency direction."

The y-gradient is on briefly before each acquisition but is not on during data acquisition. Thus, whatever encoding performed by the y-gradient is done. In this case, the y-gradient sets up a spatially dependent phase distribution that remains fixed during the frequency encoding process. In other words, the y-gradient encodes spatial position into the phase of the magnetization (direction of the **m** vector), which is known as "phase encoding." *y*, in this case, is known as the "phase direction." Below are depictions of the phase distribution set up for the 1, 0, +1 and +2 phase encoding steps. These correspond to –1, 0, +1 and +2 cycles of phase across the field of view, respectively.

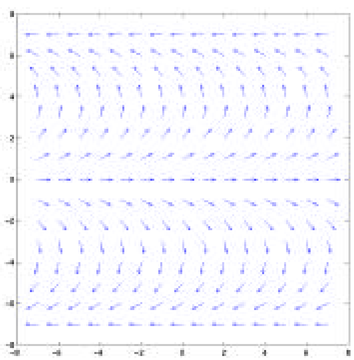
exp(-*ig*(-1)Δ$G_y T_y$)



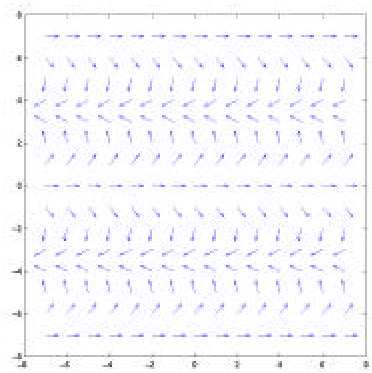
exp(-*ig*(0)Δ$G_y T_y$)



exp(-*ig*(1)Δ$G_y T_y$)



exp(-*ig*(2)Δ$G_y T_y$)



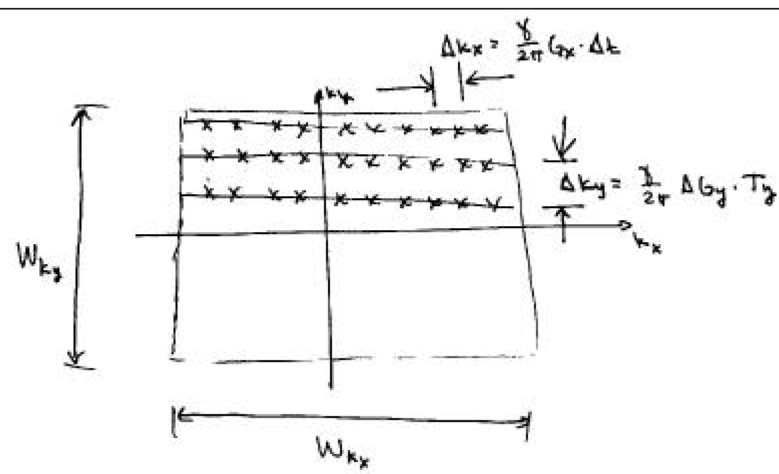
In terms of parameters described in the above pulse sequence, we can define several parameters of interest in the acquired space. The sample spacing and width of the k-space are:

$$\Delta k_x = \frac{g}{2p} G_x \Delta t$$

$$\Delta k_y = \frac{g}{2p} \Delta G_y T_y$$

$$W_{kx} = N_x \Delta k_x = \frac{g}{2p} G_x T_{read}$$

$$W_{ky} = N_y \Delta k_y = \frac{g}{2p} 2 G_{y,\max} T_y$$



Sampling in k-space (spatial frequency domain). Previously we discussed sampling of the object and its effect on the spectrum. Here we have the reverse – sampling in Fourier domain and its effect on the reconstructed object. Again, we will perform our sampling by multiplying a function times the 2D comb function. With sample spacing of Dk_x and Dk_y, in the k_x and k_y directions, the sampled Fourier data is:

$$\tilde{M}(u,v) = M(u,v)\mathrm{comb}\left(\frac{u}{\Delta k_x}, \frac{v}{\Delta k_y}\right)$$

$$= \Delta k_x \Delta k_y \sum_{n,m=-\infty}^{\infty} d(u - n\Delta k_x, v - m\Delta k_y) M(n\Delta k_x, m\Delta k_y)$$

The image (space) domain equivalent is:

$$\tilde{m}(x,y) = m(x,y) ** \Delta k_x \Delta k_y \mathrm{comb}(\Delta k_x u, \Delta k_y v)$$

$$= m(u,v) ** \sum_{n,m=-\infty}^{\infty} d\left(u - \frac{n}{\Delta k_x}, v - \frac{m}{\Delta k_y}\right)$$

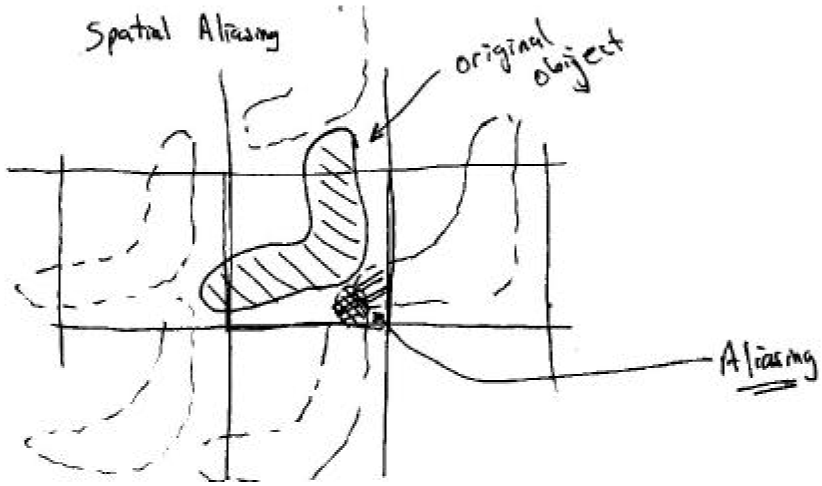
$$= \sum_{n,m=-\infty}^{\infty} m\left(u - \frac{n}{\Delta k_x}, v - \frac{m}{\Delta k_y}\right)$$

Thus, sampling in the Fourier domain leads to replication in the image domain. Spacing of the replicated image (object) is $(1/Dk_x, 1/Dk_y)$. The replicated images will not overlap the original image if the highest spatial position in x is $x_{\max} \le \frac{1}{2\Delta kx}$ and the highest spatial position in y is $y_{\max} \le \frac{1}{2\Delta ky}$. If this is not satisfied, then there will be spatial overlap in the images (or aliasing).

The field of view of an acquisition is typically defined as one over the k-space sample spacing:

$$\mathrm{FOV}_x = 1/\Delta k_x \qquad \text{and} \qquad \mathrm{FOV}_y = 1/\Delta k_y$$

and aliasing will not occur if $x_{max} < ½\ \mathrm{FOV}_x$ and $y_{max} < ½\ \mathrm{FOV}_y$.



Point Spread Function.

Observe that the practical k-space is not of infinite extent, but rather is limited to W_{kx} and W_{ky}. The sampled k-space can be written as:

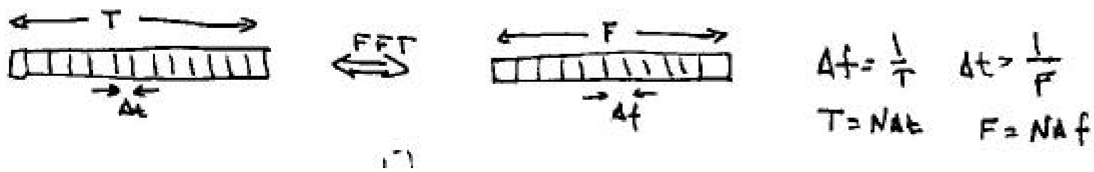
$$\tilde{M}(u,v) = M(u,v)\,\mathrm{rect}\left(\frac{u}{W_{kx}},\frac{v}{W_{ky}}\right)\mathrm{comb}\left(\frac{u}{\Delta k_x},\frac{u}{\Delta k_y}\right)$$

which results in an image of the following form:

$$\tilde{m}(x,y) = m(x,y) ** W_{kx}W_{ky}\,\mathrm{sinc}(W_{kx}x)\,\mathrm{sinc}(W_{ky}y) ** \Delta k_x \Delta k_y\,\mathrm{comb}(\Delta k_x u, \Delta k_y v)$$

The sinc functions are the point spread function and the comb function generates replicated versions of the object (does the aliasing). Observe that the sinc functions have approximate widths (in x and) of $\Delta x = 1/W_{kx}$ and $\Delta y = 1/W_{ky}$. This defines, in essence, the spatial resolution of an MRI acquisition – in order to get better (finer) spatial resolution, we need to acquire a larger area in k-space.

Resolution of the FFT. Most forms of the FFT work this way – for an N point input function the FFT will produce an N point output. Each output point corresponds to an integer number, n, of cycles in exp(-i2pnx) across the object and go from n = [-N/2:N/2-1]. Observe the DFT of -N/2 and N/2 are the same and thus this represents the entire, unaliased frequency domain of the object. Thus F/2 (see below) = ½ of $1/\Delta t$ or F = $1/\Delta t$. A similar argument can be made in reverse to get T = $1/\Delta f$.



Resolution and Object and Sample Spacing in MRI. For data acquired on a 2D rectilinear grid in k-space and reconstructed with a 2D FFT the spatial resolution and Field of View relationships are:

$$FOV_x = 1/\Delta k_x \quad \text{and} \quad FOV_y = 1/\Delta k_y$$

$$\Delta x = 1/W_{kx} \quad \text{and} \quad \Delta y = 1/W_{ky}$$