Supplemental: A Rank-based Approach to Active Diagnosis

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APPENDIX A
PROOFS

A.1 Proof of Proposition 1

We will show that the estimates for the area above the ROC curve, $\overline{\mathbf{A}}_{lr}(\mathbf{z}_\mathcal{A})$, $\overline{\mathbf{A}}_{l}(\mathbf{z}_\mathcal{A})$ and $\overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A})$ defined in (6) in the paper can be equivalently expressed as

$$\overline{\mathbf{A}}_{lr}(\mathbf{z}_\mathcal{A}) = \frac{1}{2} + \frac{\mathbf{U}(\mathbf{z}_\mathcal{A}) + \mathbf{V}(\mathbf{z}_\mathcal{A})}{2\mathbf{W}(\mathbf{z}_\mathcal{A})}$$
$$\overline{\mathbf{A}}_{l}(\mathbf{z}_\mathcal{A}) = \frac{1}{2} + \frac{\mathbf{U}(\mathbf{z}_\mathcal{A})}{2\mathbf{W}(\mathbf{z}_\mathcal{A})}$$
$$\overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A}) = \frac{1}{2} + \frac{\mathbf{U}(\mathbf{z}_\mathcal{A}) - \mathbf{V}(\mathbf{z}_\mathcal{A})}{2\mathbf{W}(\mathbf{z}_\mathcal{A})}$$

where

$$\mathbf{U}(\mathbf{z}_\mathcal{A}) = \sum_{i=1}^{M}(2i - M - 1)\Pr(X_{r(i)} = 1|\mathbf{z}_\mathcal{A}) \tag{1a}$$
$$\mathbf{V}(\mathbf{z}_\mathcal{A}) = \sum_{i=1}^{M}\Pr(X_i = 1|\mathbf{z}_\mathcal{A})\Pr(X_i = 0|\mathbf{z}_\mathcal{A}) \tag{1b}$$
$$\mathbf{W}(\mathbf{z}_\mathcal{A}) = \sum_{i=1}^{M}\Pr(X_i = 1|\mathbf{z}_\mathcal{A})\sum_{i=1}^{M}\Pr(X_i = 0|\mathbf{z}_\mathcal{A}).$$

The result in Proposition 1 will then follow by observing that under a single fault approximation, $\mathbf{W}(\mathbf{z}_\mathcal{A}) = M - 1$.

To prove the above equivalences, we will first show this result for $\overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A})$, and the other two results follow by observing that

$$\overline{\mathbf{A}}_{lr}(\mathbf{z}_\mathcal{A}) = \overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A}) + \frac{\mathbf{V}(\mathbf{z}_\mathcal{A})}{\mathbf{W}(\mathbf{z}_\mathcal{A})}$$
$$\overline{\mathbf{A}}_{l}(\mathbf{z}_\mathcal{A}) = \overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A}) + \frac{\mathbf{V}(\mathbf{z}_\mathcal{A})}{2\mathbf{W}(\mathbf{z}_\mathcal{A})}.$$

We will now show the equivalence result for $\overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A})$. Let $\mathbf{N}(\mathbf{z}_\mathcal{A}) := \sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\Pr(X_{r(i)} = 0|\mathbf{z}_\mathcal{A})\Pr(X_{r(j)} =$

$1|\mathbf{z}_{\mathcal{A}})$ denote its numerator. Then, the result follows by observing that

$$\begin{aligned}
&\sum_{i=1}^{M} \Pr(X_i = 0|\mathbf{z}_{\mathcal{A}}) \sum_{i=1}^{M} \Pr(X_i = 1|\mathbf{z}_{\mathcal{A}}) \\
&= \sum_{i=1}^{M} \Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) \sum_{i=1}^{M} \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&= \mathbf{N}(\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M} \Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) \sum_{j=1}^{i} \Pr(X_{r(j)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&= \mathbf{N}(\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M} \Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&\quad + \sum_{i=2}^{M} \sum_{j=1}^{i-1} \Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(j)} = 1|\mathbf{z}_{\mathcal{A}}),
\end{aligned} \tag{2}$$

where the last term in the above expression can be expressed in terms of $\mathbf{N}(\mathbf{z}_{\mathcal{A}})$ using the relation $\Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) = 1 - \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}})$,

$$\begin{aligned}
&\sum_{i=2}^{M} \sum_{j=1}^{i-1} \Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(j)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&= \sum_{i=2}^{M} \sum_{j=1}^{i-1} \Big[1 - \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) - \Pr(X_{r(j)} = 0|\mathbf{z}_{\mathcal{A}}) + \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(j)} = 0|\mathbf{z}_{\mathcal{A}}) \Big] \\
&= \sum_{i=2}^{M} \sum_{j=1}^{i-1} \Big[- \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) + \Pr(X_{r(j)} = 1|\mathbf{z}_{\mathcal{A}}) + \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(j)} = 0|\mathbf{z}_{\mathcal{A}}) \Big] \\
&= \sum_{i=2}^{M} -(i-1) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M-1} (M-i) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&\quad + \sum_{i=2}^{M} \sum_{j=1}^{i-1} \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(j)} = 0|\mathbf{z}_{\mathcal{A}}) \\
&= \sum_{i=1}^{M} (M - 2i + 1) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&\quad + \sum_{j=1}^{M-1} \sum_{i=j+1}^{M} \Pr(X_{r(j)} = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&= \sum_{i=1}^{M} (M - 2i + 1) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) + \mathbf{N}(\mathbf{z}_{\mathcal{A}}).
\end{aligned}$$

Finally, substituting the above relation in (2), we get

$$\begin{aligned}
&\sum_{i=1}^{M} \Pr(X_i = 0|\mathbf{z}_{\mathcal{A}}) \sum_{i=1}^{M} \Pr(X_i = 1|\mathbf{z}_{\mathcal{A}}) \\
&= 2\mathbf{N}(\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M} \Pr(X_{r(i)} = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M} (M - 2i + 1) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \\
&= 2\mathbf{N}(\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M} \Pr(X_i = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_i = 1|\mathbf{z}_{\mathcal{A}}) + \sum_{i=1}^{M} (M - 2i + 1) \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}})
\end{aligned}$$

from which, the result follows.

A.2 Adaptive Monotonicity Property

As we show below, AUC approximated using lower rectangles or a linear approximation exhibits another interesting property. In particular, it can be shown that these two AUC estimators are adaptive monotone [1], i.e., the accuracy of diagnosis given by $\underline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}})$ or $\underline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}})$ is guaranteed to increase by acquiring more query information (equivalently, the area above the ROC curve given by $\overline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}})$ or $\overline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}})$ is guaranteed to decrease by acquiring more query information).

Theorem 1. *Under a single fault approximation, the quality function $\underline{\mathbf{A}}(\mathbf{Z}_{\mathcal{A}})$ estimated using either lower rectangles or a linear approximation, is adaptive monotone, i.e., $\forall \mathcal{A}' \subseteq \mathcal{A}$*

$$\underline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}'}) \leq \underline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}}) \quad \textit{and} \quad \underline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}'}) \leq \underline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}})$$

Proof: Since $\underline{\mathbf{A}}(\mathbf{z}_{\mathcal{A}}) = 1 - \overline{\mathbf{A}}(\mathbf{z}_{\mathcal{A}})$, the result in Theorem 1 follows by showing that $\forall \mathcal{A}' \subseteq \mathcal{A}$

$$\overline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}}) \leq \overline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}'}) \text{ and } \overline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}}) \leq \overline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}'})$$

Let $\mathbf{z}_{\mathcal{A}}$ denote the responses to queries in the set $\mathcal{A}$. To prove adaptive monotonicity for $\overline{\mathbf{A}}_{lr}(\mathbf{Z}_{\mathcal{A}})$, it suffices to show that for any query $j \notin \mathcal{A}$, $\overline{\mathbf{A}}_{lr}(\mathbf{z}_{\mathcal{A}}) - \mathbb{E}_{Z_j}[\overline{\mathbf{A}}_{lr}(\mathbf{z}_{\mathcal{A}} \cup Z_j)] \geq 0$ [1]. Similarly, for $\overline{\mathbf{A}}_{l}(\mathbf{Z}_{\mathcal{A}})$, we need to show that $\overline{\mathbf{A}}_{l}(\mathbf{z}_{\mathcal{A}}) - \mathbb{E}_{Z_j}[\overline{\mathbf{A}}_{l}(\mathbf{z}_{\mathcal{A}} \cup Z_j)] \geq 0$.

Under a single fault approximation, we have

$$\overline{\mathbf{A}}_{lr}(\mathbf{z}_{\mathcal{A}}) = \frac{1}{2} + \frac{\mathbf{U}(\mathbf{z}_{\mathcal{A}}) + \mathbf{V}(\mathbf{z}_{\mathcal{A}})}{2(M-1)}, \text{ and}$$
$$\overline{\mathbf{A}}_{l}(\mathbf{z}_{\mathcal{A}}) = \frac{1}{2} + \frac{\mathbf{U}(\mathbf{z}_{\mathcal{A}})}{2(M-1)},$$

where $\mathbf{U}(\mathbf{z}_{\mathcal{A}})$ and $\mathbf{V}(\mathbf{z}_{\mathcal{A}})$ are as defined in (1a) and (1b), respectively. Hence, the adaptive monotonicity of $\overline{\mathbf{A}}_{lr}(\mathbf{z}_{\mathcal{A}})$ and $\overline{\mathbf{A}}_{l}(\mathbf{z}_{\mathcal{A}})$ follows by showing that $\forall j \notin \mathcal{A}$

$$\mathbf{U}(\mathbf{z}_{\mathcal{A}}) - \mathbb{E}_{Z_j}[\mathbf{U}(\mathbf{z}_{\mathcal{A}} \cup Z_j)] \geq 0, \text{ and}$$
$$\mathbf{V}(\mathbf{z}_{\mathcal{A}}) - \mathbb{E}_{Z_j}[\mathbf{V}(\mathbf{z}_{\mathcal{A}} \cup Z_j)] \geq 0,$$

which follow from Lemma 1 and 2, below. □

Lemma 1. *Let $\mathbf{z}_{\mathcal{A}}$ denote the observed responses to queries in the set $\mathcal{A}$. Then, for any query $j \notin \mathcal{A}$,*

$$\mathbf{U}(\mathbf{z}_{\mathcal{A}}) - \mathbb{E}_{Z_j}[\mathbf{U}(\mathbf{z}_{\mathcal{A}} \cup Z_j)] \geq 0$$

Proof: Under a single fault approximation, $\mathbf{U}(\mathbf{z}_{\mathcal{A}}) = -(M+1) + \sum_{i=1}^{M} 2i\Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}})$. Hence, the result follows by showing that $\forall\ j \notin \mathcal{A}$,

$$\sum_{i=1}^{M} i \Bigg\{ \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) - \Bigg[\Pr(Z_j = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r_0(i)} = 1|\mathbf{z}_{\mathcal{A}}, 0) + \Pr(Z_j = 1|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r_1(i)} = 1|\mathbf{z}_{\mathcal{A}}, 1) \Bigg] \Bigg\} \geq 0. \tag{3}$$

As mentioned earlier, the rank order depends on the queries chosen $\mathcal{A}$ and their observed responses $\mathbf{z}_{\mathcal{A}}$. Hence, to differentiate the rank orders in the above expression, we use $r(i)$ to denote the rank order of the objects based on the observed responses $\mathbf{z}_{\mathcal{A}}$, and $r_0(i)$, $r_1(i)$ to denote the rank order of the objects based on the observed responses $\mathbf{z}_{\mathcal{A}} \cup 0$ and $\mathbf{z}_{\mathcal{A}} \cup 1$ to queries in $\mathcal{A} \cup \{j\}$.

Note that (3) is equivalent to showing

$$\sum_{i=1}^{M} (M-i+1) \Bigg\{ \Bigg[\Pr(Z_j = 0|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r_0(i)} = 1|\mathbf{z}_{\mathcal{A}}, 0) + \Pr(Z_j = 1|\mathbf{z}_{\mathcal{A}}) \Pr(X_{r_1(i)} = 1|\mathbf{z}_{\mathcal{A}}, 1) \Bigg] - \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}}) \Bigg\} \geq 0. \tag{4}$$

Let $\mathbf{f_t}(\mathbf{r}, \mathbf{z}_{\mathcal{A}}) := \sum_{i=1}^{t} \Pr(X_{r(i)} = 1|\mathbf{z}_{\mathcal{A}})$, i.e., the probability mass of the top t objects in the ranked list given by $\mathbf{r}$. Then,

$$\sum_{i=1}^{M} (M-i+1) \Pr(X_{r(i)}|\mathbf{z}_{\mathcal{A}}) = \sum_{t=1}^{M} \mathbf{f_t}(\mathbf{r}, \mathbf{z}_{\mathcal{A}}),$$

and hence (4) is equivalent to showing

$$\sum_{t=1}^{M}\left[\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\mathbf{f_t}(\mathbf{r}_0,\mathbf{z}_{\mathcal{A}}\cup 0)+\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\mathbf{f_t}(\mathbf{r}_1,\mathbf{z}_{\mathcal{A}}\cup 1)\right]-\mathbf{f_t}(\mathbf{r},\mathbf{z}_{\mathcal{A}})\geq 0.$$

Now, note that

$$\begin{aligned}\mathbf{f_t}(\mathbf{r}_0,\mathbf{z}_{\mathcal{A}}\cup 0)&\geq \mathbf{f_t}(\mathbf{r},\mathbf{z}_{\mathcal{A}}\cup 0)\\&=\sum_{i=1}^{t}\Pr(X_{r(i)}=1|\mathbf{z}_{\mathcal{A}},0).\end{aligned}$$

Since the rank order $\mathbf{r}_0$ corresponds to the decreasing order of the posterior probabilities in $\{\Pr(X_i=1|\mathbf{z}_{\mathcal{A}},0)\}_{i=1}^{M}$, the probability mass of the top t objects in this ranked list is greater than any other t objects. Similarly, $\mathbf{f_t}(\mathbf{r}_1,\mathbf{z}_{\mathcal{A}}\cup 1)\geq \mathbf{f_t}(\mathbf{r},\mathbf{z}_{\mathcal{A}}\cup 1)$. Hence,

$$\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\mathbf{f_t}(\mathbf{r}_0,\mathbf{z}_{\mathcal{A}}\cup 0)+\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\mathbf{f_t}(\mathbf{r}_1,\mathbf{z}_{\mathcal{A}}\cup 1) \tag{5a}$$

$$\geq \Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\mathbf{f_t}(\mathbf{r},\mathbf{z}_{\mathcal{A}}\cup 0)+\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\mathbf{f_t}(\mathbf{r},\mathbf{z}_{\mathcal{A}}\cup 1) \tag{5b}$$

$$=\sum_{i=1}^{t}\left[\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\Pr(X_{r(i)}=1|\mathbf{z}_{\mathcal{A}},0)+\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\Pr(X_{r(i)}=1|\mathbf{z}_{\mathcal{A}},1)\right] \tag{5c}$$

$$=\sum_{i=1}^{t}\Big[\Pr(Z_j=0|X_{r(i)}=1)\Pr(X_{r(i)}=1|\mathbf{z}_{\mathcal{A}}) + \Pr(Z_j=1|X_{r(i)}=1)\Pr(X_{r(i)}=1|\mathbf{z}_{\mathcal{A}})\Big] \tag{5d}$$

$$=\sum_{i=1}^{t}\Pr(X_{r(i)}=1|\mathbf{z}_{\mathcal{A}})=\mathbf{f_t}(\mathbf{r},\mathbf{z}_{\mathcal{A}}). \tag{5e}$$

Thus proving the inequality.

Note that in the above equation, (5d) follows from (5c) by observing that under a single fault approximation, $X_i=1 \iff \mathbf{X}=\mathbf{I}_i$, and hence, using the conditional independence assumption of Section 2, the posterior probability can be expressed as

$$\begin{aligned}\Pr(X_i=1|\mathbf{z}_{\mathcal{A}},z)&=\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\\&=\frac{\Pr(\mathbf{X}=\mathbf{I}_i)\Pr(\mathbf{z}_{\mathcal{A}}|\mathbf{X}=\mathbf{I}_i)\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})\Pr(\mathbf{Z}_{\mathcal{A}}=\mathbf{z}_{\mathcal{A}})}\\&=\frac{\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})}\\&=\frac{\Pr(X_i=1|\mathbf{z}_{\mathcal{A}})\Pr(Z_j=z|X_i=1)}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})}.\end{aligned} \tag{6}$$

□

Lemma 2. *Let $\mathbf{z}_{\mathcal{A}}$ denote the observed responses to queries in the set $\mathcal{A}$. Then, for any query $j\notin\mathcal{A}$,*

$$\mathbf{V}(\mathbf{z}_{\mathcal{A}})-\mathbb{E}_{Z_j}[\mathbf{V}(\mathbf{z}_{\mathcal{A}}\cup Z_j)]\geq 0$$

Proof: Note that under a single fault approximation, $\mathbf{V}(\mathbf{z}_{\mathcal{A}})=1-\sum_{i=1}^{M}\Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}})$. Hence, we need to show that $\forall\ j\notin\mathcal{A}$,

$$\sum_{i=1}^{M}\left\{\left[\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}},0)+\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}},1)\right] - \Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}})\right\}\geq 0. \tag{7}$$

Substituting the expression for posterior probability from (6) in the LHS of (7), we get

$$\begin{aligned}
&\sum_{i=1}^{M}\left\{\Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}})\left[\frac{\Pr^2(Z_j=0|X_i=1)}{\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})}+\frac{\Pr^2(Z_j=1|X_i=1)}{\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})}-1\right]\right\}\\
&\quad=\sum_{i=1}^{M}\left\{\Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}})\left[\frac{\Big(1-\Pr(Z_j=1|X_i=1)\Big)^2}{\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})}+\frac{\Pr^2(Z_j=1|X_i=1)}{\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})}-1\right]\right\},\\
&\quad=\sum_{i=1}^{M}\left\{\Pr^2(X_i=1|\mathbf{z}_{\mathcal{A}})\left[\frac{\Bigg(\Pr(Z_j=1|X_i=1)-\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\Bigg)^2}{\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})}\right]\right\}\\
&\quad\geq 0
\end{aligned}$$

where the last equality follows by using the relation $\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})=1-\Pr(Z_j=1|\mathbf{z}_{\mathcal{A}})$, and completing the square. □

A.3 Proof of Proposition 2

The entropy-based query selection criterion is given by

$$j^*=\underset{j\notin\mathcal{A}}{\operatorname{argmin}}\sum_{z=0,1}\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})H(\mathbf{X}|\mathbf{z}_{\mathcal{A}},z). \tag{8}$$

Since, under a single fault approximation, $X_i=1\iff\mathbf{X}=\mathbf{I}_i$, we need to show that the above query selection criterion reduces to

$$j^*:=\underset{j\notin\mathcal{A}}{\operatorname{argmin}}\sum_{i=1}^{M}\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})H\Big(\Pr(Z_j=0|\mathbf{X}=\mathbf{I}_i)\Big)-H\Big(\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\Big).$$

We show this by first noting that under a single fault approximation, the conditional entropy reduces to

$$H(\mathbf{X}|\mathbf{z}_{\mathcal{A}},z)=-\sum_{i=1}^{M}\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\log\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z).$$

In addition, as noted in (6), under the conditional independence assumption of Section 2, the posterior probability can be expressed as

$$\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)=\frac{\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})}. \tag{9}$$

Substituting the above expression in (8), we get

$$\begin{aligned}
\sum_{z=0,1}\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})H(\mathbf{X}|\mathbf{Z}_{\mathcal{A}},z)=-\sum_{z=0,1}\sum_{i=1}^{M}&\Bigg[\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\\
&\log\frac{\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})}\Bigg].
\end{aligned} \tag{10}$$

This expression can be broken down into 3 different terms. The first term is given by

$$\begin{aligned}
&-\sum_{z=0,1}\sum_{i=1}^{M}\Big[\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\;\log\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\Big]\\
&\qquad=-\sum_{i=1}^{M}\Big[\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\log\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\sum_{z=0,1}\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)\Big]\\
&\qquad=H(\mathbf{X}|\mathbf{z}_{\mathcal{A}}),
\end{aligned}$$

where the second equality follows from (9) and the last equality follows since $\sum_z\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)=1$.

The second term is given by

$$
\begin{aligned}
-\sum_{z=0,1}\sum_{i=1}^{M}&\left[\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\ \log\frac{1}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})}\right]\\
&=-\sum_{z=0,1}\left[\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})\log\frac{1}{\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})}\ \sum_{i=1}^{M}\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\right]\\
&=-H\Big(\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\Big),
\end{aligned}
$$

where the last equality follows since $\sum_{i=1}^{M}\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)=1$.

The last term is given by

$$
\begin{aligned}
-\sum_{z=0,1}\sum_{i=1}^{M}&\left[\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}},z)\ \log\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)\right]\\
&=-\sum_{i=1}^{M}\left[\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})\left(\sum_{z=0,1}\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)\ \log\Pr(Z_j=z|\mathbf{X}=\mathbf{I}_i)\right)\right]\\
&=\sum_{i=1}^{M}\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})H\Big(\Pr(Z_j=0|\mathbf{X}=\mathbf{I}_i)\Big).
\end{aligned}
$$

Substituting these 3 terms back into (10), we get

$$
\begin{aligned}
\sum_{z=0,1}&\Pr(Z_j=z|\mathbf{z}_{\mathcal{A}})H(\mathbf{X}|\mathbf{Z}_{\mathcal{A}},z)\\
&=H(\mathbf{X}|\mathbf{z}_{\mathcal{A}})-H\Big(\Pr(Z_j=0|\mathbf{z}_{\mathcal{A}})\Big)+\sum_{i=1}^{M}\Pr(\mathbf{X}=\mathbf{I}_i|\mathbf{z}_{\mathcal{A}})H\Big(\Pr(Z_j=0|\mathbf{X}=\mathbf{I}_i)\Big),
\end{aligned}
$$

and the result follows since $H(\mathbf{X}|\mathbf{z}_{\mathcal{A}})$ does not depend on the query j.

A.4 Proof of Lemma 1

For any given k and h, let $g(p):=\log[p^h(1-p)^{k-h}]$. It can be easily verified that $g'(p)=0$ when $p=\frac{h}{k}$ and $g''(p)|_{p=\frac{h}{k}}<0$ which implies that $g(p)\le g(\frac{h}{k})$, $\forall\ p$, from which the inequality in (12) shown in the paper follows.

In addition, when $p\le p_2$, we need to show that the bound can be improved to

$$
p^h(1-p)^{k-h}\le\begin{cases}p_2^h(1-p_2)^{k-h} & \text{if } p_2\le\frac{h}{k},\\ \left(\frac{h}{k}\right)^h\left(1-\frac{h}{k}\right)^{k-h} & \text{if } p_2>\frac{h}{k}.\end{cases}
$$

Note that the second part of this result, where $p_2>h/k$ follows from the above result. Hence, it remains to show that $\forall\ p_2\le\frac{h}{k}$, $p^h(1-p)^{k-h}\le p_2^h(1-p_2)^{k-h}$, which is equivalent to showing that $\forall h\ge kp_2$, $g(p_2)-g(p)\ge 0$.

$$
\begin{aligned}
g(p_2)-g(p)&=h\log\frac{p_2(1-p)}{p(1-p_2)}+k\log\frac{1-p_2}{1-p}\\
&\ge kp_2\log\frac{p_2(1-p)}{p(1-p_2)}+k\log\frac{1-p_2}{1-p}\\
&=k\left[p_2\log\frac{p_2}{p}+(1-p_2)\log\frac{1-p_2}{1-p}\right]\ge 0
\end{aligned}
$$

where the first inequality follows from $h\ge kp_2$ (the first log is ≥ 0 since $p\le p_2$) and the last inequality follows from the non-negativity of Kullback-Leibler divergence. The other two cases can be proved in a similar manner.

A.5 Proof of Proposition 3

Let $|\mathcal{A}|=k$. Consider the case where $\exists\ \overline{p}\in(0,\rho/(1+\rho))$ such that $0<p\le\overline{p}$ (The other case where $\exists\ \underline{p}\in(1/(1+\rho),1)$ such that $1>p\ge\underline{p}$ can be proved in a similar manner). Note from the definitions of $r_{wc}(\theta|\mathbf{z}_{\mathcal{A}})$ and $\overline{r}_{wc}(\theta|\mathbf{z}_{\mathcal{A}})$ that the result follows by showing the following relational equivalence between the true probabilities and the estimated probabilities: $\forall i,j$

$$
\pi_i\Pr(\mathbf{z}_{\mathcal{A}}|X_i=1)\ge\pi_j\Pr(\mathbf{z}_{\mathcal{A}}|X_j=1)\Longleftrightarrow\pi_i\overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_i=1)\ge\pi_j\overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_j=1),\tag{11}
$$

where the true likelihood and the estimated likelihood of any object θ_i are given by $\Pr(\mathbf{z}_{\mathcal{A}}|X_i=1) = p^{h_i}(1-p)^{k-h_i}$ and $\overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_i=1) = \varepsilon_i^{h_i}(1-\varepsilon_i)^{k-h_i}$, $h_i = \delta_{i,\mathcal{A}}$ and $\varepsilon_i := \min\{h_i/k, \overline{p}\}$.

The above equivalence follows trivially for any pair of objects θ_i, θ_j whose $h_i = h_j$. To show that the equivalence holds even when $h_i \neq h_j$, we will show that, for any two objects θ_i, θ_j with priors π_i, π_j,

$$\pi_i \Pr(\mathbf{z}_{\mathcal{A}}|X_i=1) > \pi_j \Pr(\mathbf{z}_{\mathcal{A}}|X_j=1) \ \&\ (h_i \neq h_j) \iff h_j > h_i \tag{12a}$$

$$\text{and } \pi_i \overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_i=1) > \pi_j \overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_j=1) \ \&\ (h_i \neq h_j) \iff h_j > h_i. \tag{12b}$$

We will first prove (12a), followed by (12b). Note that $h_j > h_i$ is equivalent to $h_j \geq h_i + 1$. Using the fact that $p < \frac{\rho}{1+\rho}$ and that for any i, j, $\frac{\pi_j}{\pi_i} \leq \frac{\max_k \pi_k}{\min_k \pi_k} = \frac{1}{\rho}$, we can show the converse of (12a) as follows. If $h_j - h_i \geq 1$, then

$$\begin{aligned}
&(h_j - h_i)\log\frac{1-p}{p} \geq \log\frac{1-p}{p} > \log\frac{1}{\rho} \geq \log\frac{\pi_j}{\pi_i} \\
&\implies \log\pi_i + h_i \log\frac{p}{1-p} > \log\pi_j + h_j\log\frac{p}{1-p} \\
&\implies \log\pi_i p^{h_i}(1-p)^{k-h_i} > \log\pi_j p^{h_j}(1-p)^{k-h_j}.
\end{aligned}$$

To prove the forward direction, we need to show that

$$h_j \leq h_i \implies (h_i = h_j) \text{ or } \pi_i\Pr(\mathbf{z}_{\mathcal{A}}|X_i=1) \leq \pi_j\Pr(\mathbf{z}_{\mathcal{A}}|X_j=1).$$

If $h_j < h_i$, then $\pi_i\Pr(\mathbf{z}_{\mathcal{A}}|X_i=1) < \pi_j\Pr(\mathbf{z}_{\mathcal{A}}|X_j=1)$ using the converse result with dummy variables i and j interchanged, thereby proving (12a). Similarly, to prove the converse of (12b), we need to show that $h_j > h_i$ leads to $\pi_i\overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_i=1) > \pi_j\overline{\Pr}(\mathbf{z}_{\mathcal{A}}|X_j=1)$, for which we need to consider three different cases.

Case 1 : Let $h_j > h_i \geq k\overline{p} \implies \varepsilon_i = \varepsilon_j = \overline{p}$. Then,

$$\begin{aligned}
&(h_j - h_i)\log\frac{1-\overline{p}}{\overline{p}} \geq \log\frac{1-\overline{p}}{\overline{p}} > \log\frac{1}{\rho} \geq \log\frac{\pi_j}{\pi_i} \\
&\implies \log\pi_i + h_i\log\frac{\overline{p}}{1-\overline{p}} > \log\pi_j + h_j\log\frac{\overline{p}}{1-\overline{p}} \\
&\implies \log\pi_i\overline{p}^{h_i}(1-\overline{p})^{k-h_i} > \log\pi_j\overline{p}^{h_j}(1-\overline{p})^{k-h_j} \\
&\implies \log\pi_i\varepsilon_i^{h_i}\left(1-\varepsilon_i\right)^{k-h_i} > \log\pi_j\varepsilon_j^{h_j}\left(1-\varepsilon_j\right)^{k-h_j}.
\end{aligned}$$

Case 2 : Let $h_j \geq k\overline{p} > h_i \implies \varepsilon_i = h_i/k$ and $\varepsilon_j = \overline{p}$. Then, following along the same lines as above, we have

$$\begin{aligned}
&\log\pi_i\overline{p}^{h_i}(1-\overline{p})^{k-h_i} > \log\pi_j\overline{p}^{h_j}(1-\overline{p})^{k-h_j} \\
&\implies \log\pi_i\left(\frac{h_i}{k}\right)^{h_i}\left(1-\frac{h_i}{k}\right)^{k-h_i} > \log\pi_j\overline{p}^{h_j}(1-\overline{p})^{k-h_j} \\
&\implies \log\pi_i\varepsilon_i^{h_i}\left(1-\varepsilon_i\right)^{k-h_i} > \log\pi_j\varepsilon_j^{h_j}\left(1-\varepsilon_j\right)^{k-h_j}
\end{aligned}$$

where the second statement follows from (12) in Lemma 1.

Case 3 : Let $k\overline{p} > h_j > h_i$, which implies $\varepsilon_i = h_i/k$ and $\varepsilon_j = h_j/k$. Defining $g_1(h) = \log[(h/k)^h(1-h/k)^{k-h}]$ and $g_2(h) = \log\overline{p}^h(1-\overline{p})^{k-h}$, we have,

$$\frac{dg_1}{dh} = \log\frac{h/k}{1-\frac{h}{k}} < \frac{dg_2}{dh} = \log\frac{\overline{p}}{1-\overline{p}} < 0,$$

when $h < k\overline{p}$. This implies that $g_1(h)$ has a larger slope than $g_2(h)$ when $h \in [0, k\overline{p})$, and hence

$$\begin{aligned}
&\log\left(\varepsilon_i\right)^{h_i}\left(1-\varepsilon_i\right)^{k-h_i} - \log\left(\varepsilon_j\right)^{h_j}\left(1-\varepsilon_j\right)^{k-h_j} \\
&> \log\overline{p}^{h_i}(1-\overline{p})^{k-h_i} - \log\overline{p}^{h_j}(1-\overline{p})^{k-h_j} \\
&= (h_j - h_i)\log\frac{1-\overline{p}}{\overline{p}} > \log\frac{\pi_j}{\pi_i} \\
&\implies \log\pi_i\varepsilon_i^{h_i}\left(1-\varepsilon_i\right)^{k-h_i} > \log\pi_j\varepsilon_j^{h_j}\left(1-\varepsilon_j\right)^{k-h_j},
\end{aligned}$$

thus proving the converse of (12b). The forward direction can be proved using the converse result in the same way as it is done for (12a).

APPENDIX B
MISCELLANIES

B.1 Choice of upper rectangles

As mentioned in the paper, query selection based on AUC approximated using the upper rectangles performs better than the other two. We will now provide an intuitive explanation for this phenomenon.

Using the result in Proposition 1, and noting that $\Pr(X_i = 0|\mathbf{z}_\mathcal{A}) = 1 - \Pr(X_i = 1|\mathbf{z}_\mathcal{A})$, we can re-write the expressions for the area above the ROC curve given by (9) in the paper as

$$\overline{\mathbf{A}}_{lr}(\mathbf{z}_\mathcal{A}) = \frac{\sum_{i=1}^{M} 2i\Pr(X_{r(i)} = 1|\mathbf{z}_\mathcal{A}) - \Pr^2(X_i = 1|\mathbf{z}_\mathcal{A})}{2(M-1)} + c_l,$$

$$\overline{\mathbf{A}}_{l}(\mathbf{z}_\mathcal{A}) = \frac{\sum_{i=1}^{M} 2i\Pr(X_{r(i)} = 1|\mathbf{z}_\mathcal{A})}{2(M-1)} + c_m,$$

$$\overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A}) = \frac{\sum_{i=1}^{M} 2i\Pr(X_{r(i)} = 1|\mathbf{z}_\mathcal{A}) + \Pr^2(X_i = 1|\mathbf{z}_\mathcal{A})}{2(M-1)} + c_u,$$

where c_l, c_m and c_u are constants that do not contribute to query selection.

Now note that all three approximations have the same first term, which corresponds to the expected rank of the faults in the ranked list. However, they differ with respect to the second term, which makes the crucial difference in terms of the query selected. More specifically, given two or more queries with the the same expected rank value (i.e., same value for the first term), query selected using $\overline{\mathbf{A}}_{ur}(\mathbf{z}_\mathcal{A})$ chooses the one that most evenly distributes the posterior probability mass of 1 among all the objects, while query selected using $\overline{\mathbf{A}}_{lr}(\mathbf{z}_\mathcal{A})$ chooses the one that assigns most of the probability mass to one object, and the query selected using $\overline{\mathbf{A}}_{l}(\mathbf{z}_\mathcal{A})$ just picks one at random. Hence, the queries selected using $\overline{\mathbf{A}}_{lr}(\mathbf{z}_\mathcal{A})$ and $\overline{\mathbf{A}}_{l}(\mathbf{z}_\mathcal{A})$ are more prone to increasing the posterior fault probability of one (or few) object(s), thereby creating a bias towards those objects in the queries selected there after. However, this is overcome by the AUC-based query selection criterion approximated using the upper rectangles.

B.2 GBS as a special case

As shown in the paper, in the single fault scenario, the rank-based greedy strategy reduces to

$$j^* = \operatorname*{argmin}_{j \notin \mathcal{A}} \sum_{z=0,1} \sum_{i=1}^{M} \pi_i \Pr(\mathbf{z}_\mathcal{A}, z|X_i = 1) r_{wc}(i|\mathbf{z}_\mathcal{A} \cup z). \tag{13}$$

In the noise-free case with uniform prior on the objects (i.e., $\pi_i = 1/M$, $\forall i$), the above strategy can be shown to be equivalent to GBS [2], [3].

We begin by noting that in the noise-free case, the likelihood values are binary with $\Pr(\mathbf{z}_\mathcal{A}|X_i = 1) = 1$ for all those objects whose true responses to queries in $\mathcal{A}$ are equal to the observed responses $\mathbf{z}_\mathcal{A}$, and 0 otherwise. Given the responses $\mathbf{z}_\mathcal{A}$ to queries in $\mathcal{A}$, let $M(\mathbf{z}_\mathcal{A})$ be defined as follows,

$$M(\mathbf{z}_\mathcal{A}) := \sum_{i=1}^{M} \mathbf{I}\{\Pr(\mathbf{z}_\mathcal{A}|X_i = 1) = 1\}.$$

Then, the worst case rank of all those objects with a likelihood value equal to 1 is given by $M(\mathbf{z}_\mathcal{A})$, and that of the remaining objects is equal to M.

Under a uniform prior, the greedy query selection criterion in (13) then reduces to

$$\begin{aligned} j^* &= \operatorname*{argmin}_{j \notin \mathcal{A}} \frac{1}{M} \sum_{z=0,1} \sum_{i=1}^{M(\mathbf{z}_\mathcal{A} \cup z)} M(\mathbf{z}_\mathcal{A} \cup z) \\ &= \operatorname*{argmin}_{j \notin \mathcal{A}} \frac{1}{M} \Big[M^2(\mathbf{z}_\mathcal{A} \cup 0) + M^2(\mathbf{z}_\mathcal{A} \cup 1) \Big], \end{aligned}$$

where $M(\mathbf{z}_\mathcal{A} \cup 0) + M(\mathbf{z}_\mathcal{A} \cup 1) = M(\mathbf{z}_\mathcal{A})$, and $\mathbf{z}_\mathcal{A} \cup z$ corresponds to the observed responses to queries in $\mathcal{A} \cup j$. The solution to this constrained optimization problem is to choose a query that most evenly divides the $M(\mathbf{z}_\mathcal{A})$ objects, which is the standard splitting algorithm or GBS.

B.3 Details of networks generated for experiments

We will now briefly describe how the networks used in the experiments were generated.

- *Random Networks*: The Erdös-Rényi random networks were generated using an edge density value (p) between 0.02 and 0.2, where p corresponds to the probability that a particular object and query are connected. The Preferential Attachment random network model consists of two parameters, α and ν, where α corresponds to the probability with which an edge is generated uniformly at random, and ν corresponds to the maximum edge degree of the objects in the bipartite diagnosis graph. For more details, refer to [4]. In the networks we generated, we used α values in the range of $[0.1, 0.3]$ and ν was chosen to be less than 10% of the maximum possible edge degree.
- *Computer Networks*: The computer networks used in this paper were generated in a two-stage process consisting of (1) network topology generation and (2) probe set selection. In the first stage, network topologies were created using the BRITE [5] and the INET 3.0 [6] generators, which simulate an Internet like topology at the Autonomous Systems (AS) level. More specifically, the BRITE networks were generated using the AS Waxman model under default parameters, where the plane dimensions were scaled based on the number of components. The INET network was also generated using an AS model with default parameters.

  Given this network topology, a random set of K network components were chosen to be designated as probe stations. Probes were then generated by computing the shortest path from each probe station to every component. This set is then decreased in size using a greedy process known as Subtractive search [7], where the probes were selected passively such that the resulting probe set guarantees single fault diagnosis. Once this set has been created, additional probes were added greedily to allow for multiple fault diagnosis. In the INET network we generated, Subtractive search was slow, and hence the probes were selected based on greedy covering.

B.4 Experiments

In this section, we provide more experimental evidence to support our argument that AUC-based query selection under single-fault approximation (AUC+SF) is a reliable, practical alternative to BPEA in large scale diagnosis problems.

We compare the performance of the three query selection criteria, i.e., BPEA, AUC-based query selection under single fault approximation (AUC+SF), and entropy-based query selection under single fault approximation (Entropy+SF), on two different datasets. The first dataset is a random bipartite diagnosis graph generated using the standard Preferential Attachment (PA) random network model. The second dataset is a network topology built using the BRITE generator, which simulates an Internet-like topology at the Autonomous Systems level.

Figures 1 and 2 compare the performance of the three query selection criteria on the two datasets, for different values of prior fault probability α, leak and inhibition probabilities ρ_l and ρ_i. In these figures, the area under the ROC curve (AUC) is obtained by ranking the objects based on their posterior probabilities, which in turn are computed using a single-fault approximation. Alternatively, note that these posterior probabilities could be estimated using belief propagation on these networks (as the networks are small in size), and the ranking obtained there after could be used to compute the AUC. Figures 3 and 4 compare the three query selection criteria using AUC computed through BP based ranking. In all the experiments, the information gain is computed using BPEA as described in Zheng et al. [8]. Finally, note from Figure 5 that the time complexity of selecting a query grows exponentially for BPEA, whereas for AUC+SF, it grows near quadratically ($O(NM \log M)$) with the time taken to select a probe being less than 2 seconds even in networks with 2000 components.

These experiments demonstrate that AUC+SF invariably performs better than Entropy+SF, and often comparable to BPEA, while having a computational complexity that is orders less than that of BPEA, thereby making it a robust, practical alternative to BPEA in large scale diagnosis problems.

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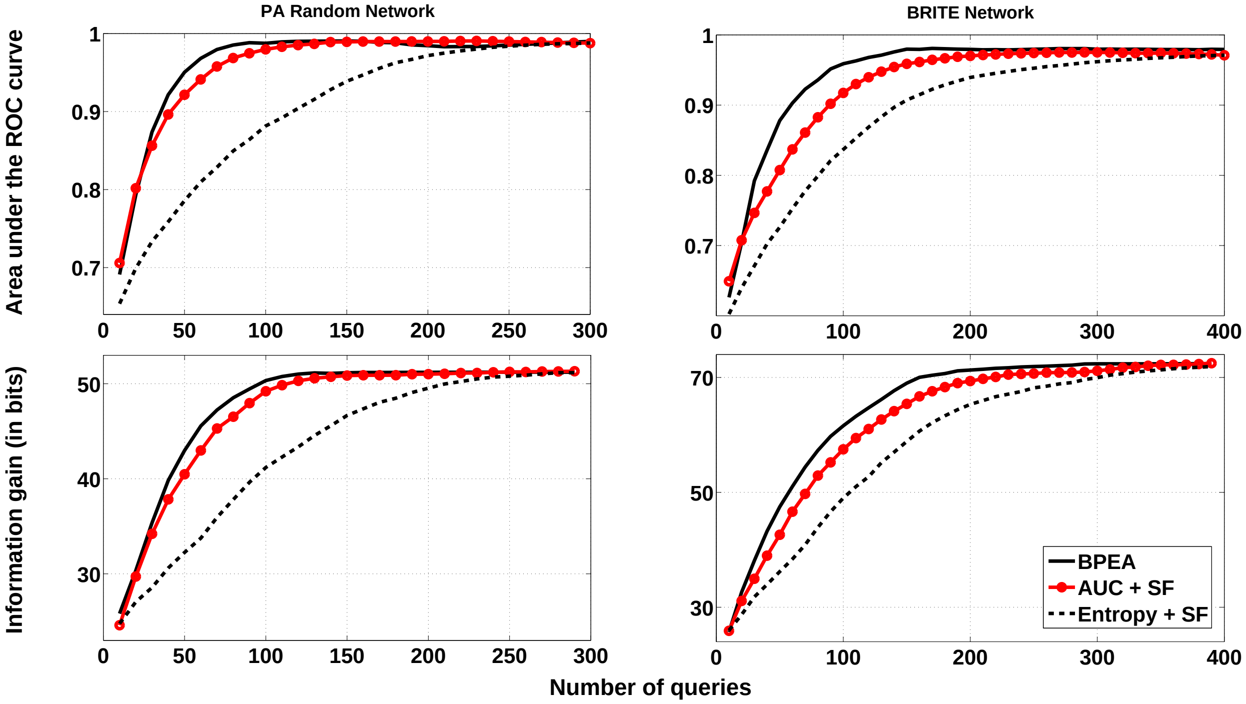
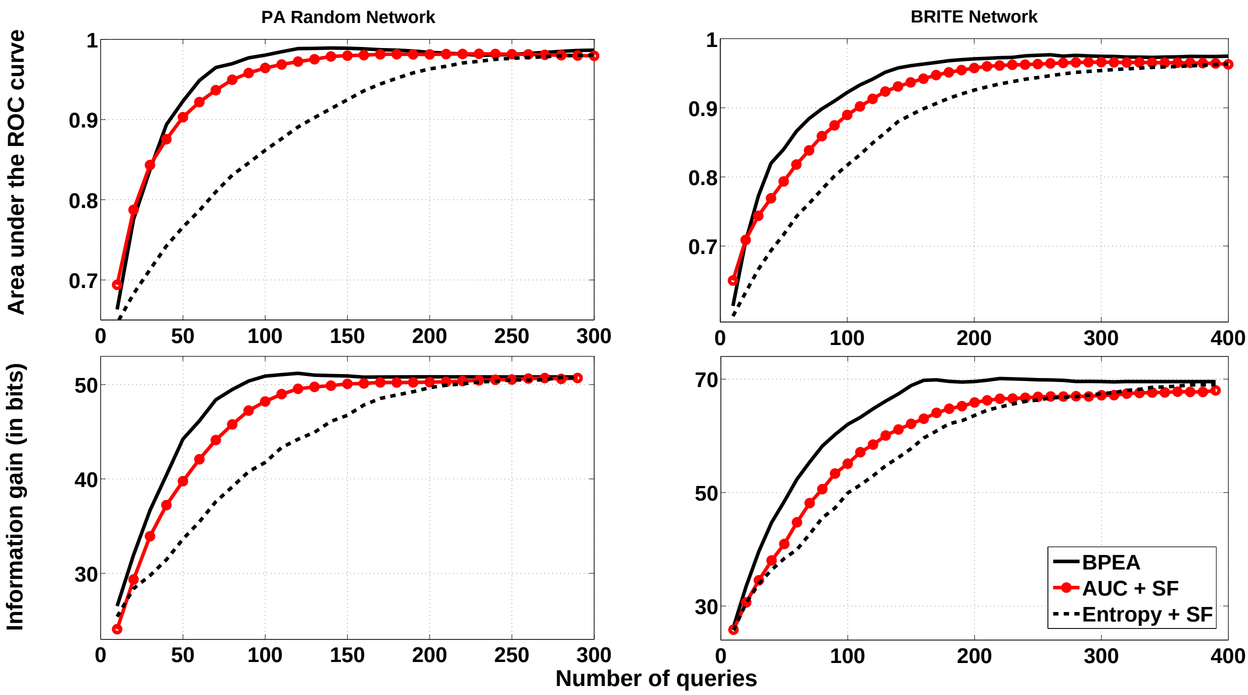
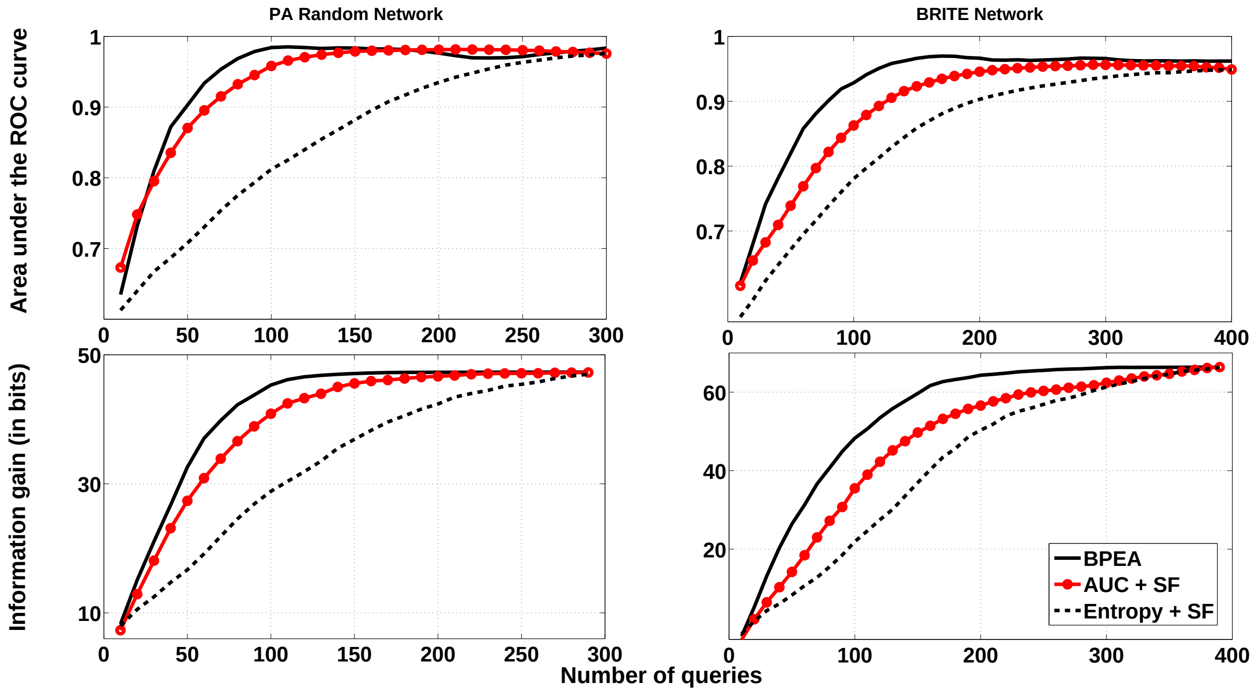


(a)

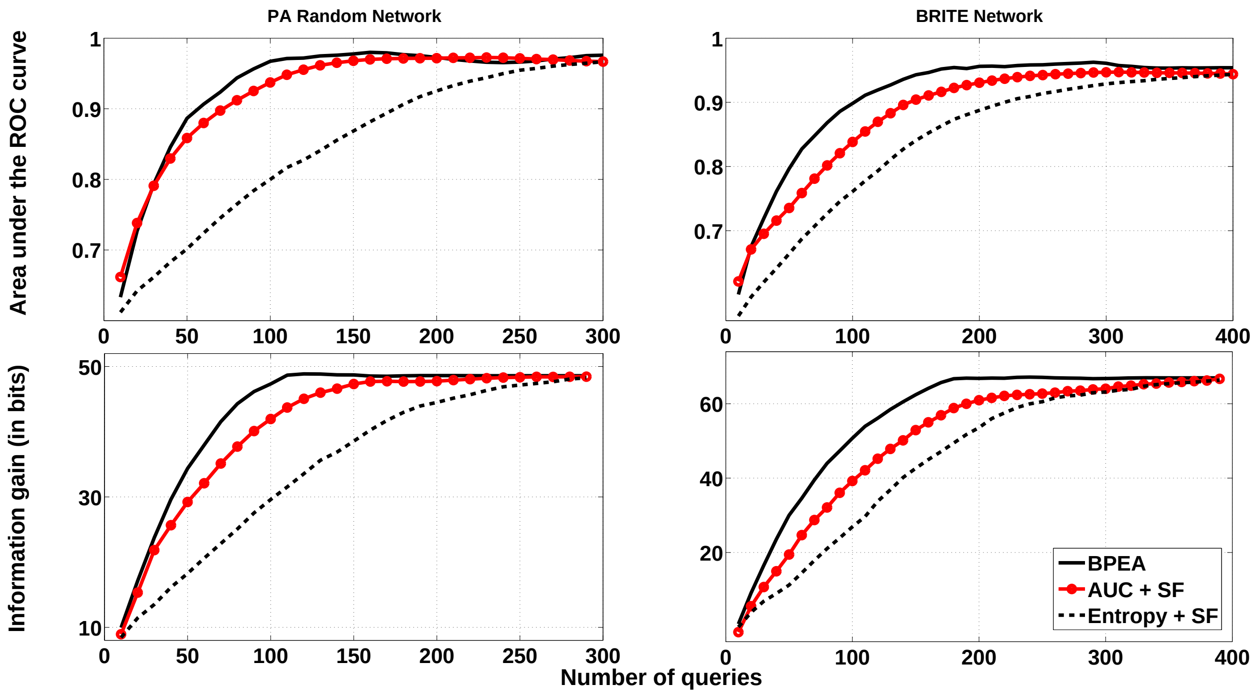


(b)

Fig. 1. The plots in the first column correspond to a dataset generated using the PA model, and the second column corresponds to a BRITE network. The figure in the top corresponds to $(\alpha, \rho_i, \rho_l) = (0.03, 0.05, 0.05)$, and the figure in the bottom corresponds to $(\alpha, \rho_i, \rho_l) = (0.03, 0.1, 0.1)$.



(a)



(b)

Fig. 2. The plots in the first column correspond to a dataset generated using the PA model, and the second column corresponds to a BRITE network. The figure in the top corresponds to $(\alpha, \rho_i, \rho_l) = (0.05, 0.05, 0.05)$, and the figure in the bottom corresponds to $(\alpha, \rho_i, \rho_l) = (0.05, 0.1, 0.1)$.

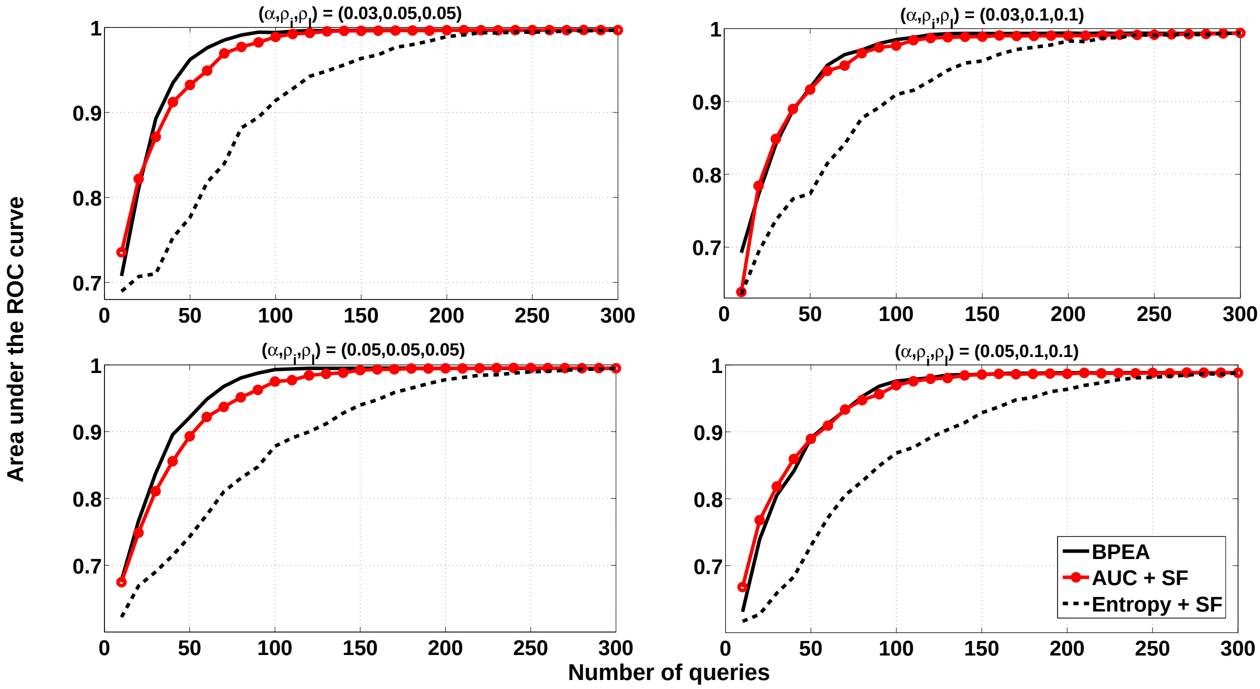


Fig. 3. The plots in this figure correspond to a dataset generated using PA model. The AUC is computed by ranking the objects using posterior probabilities obtained from Belief Propagation (rather than single fault approximation).

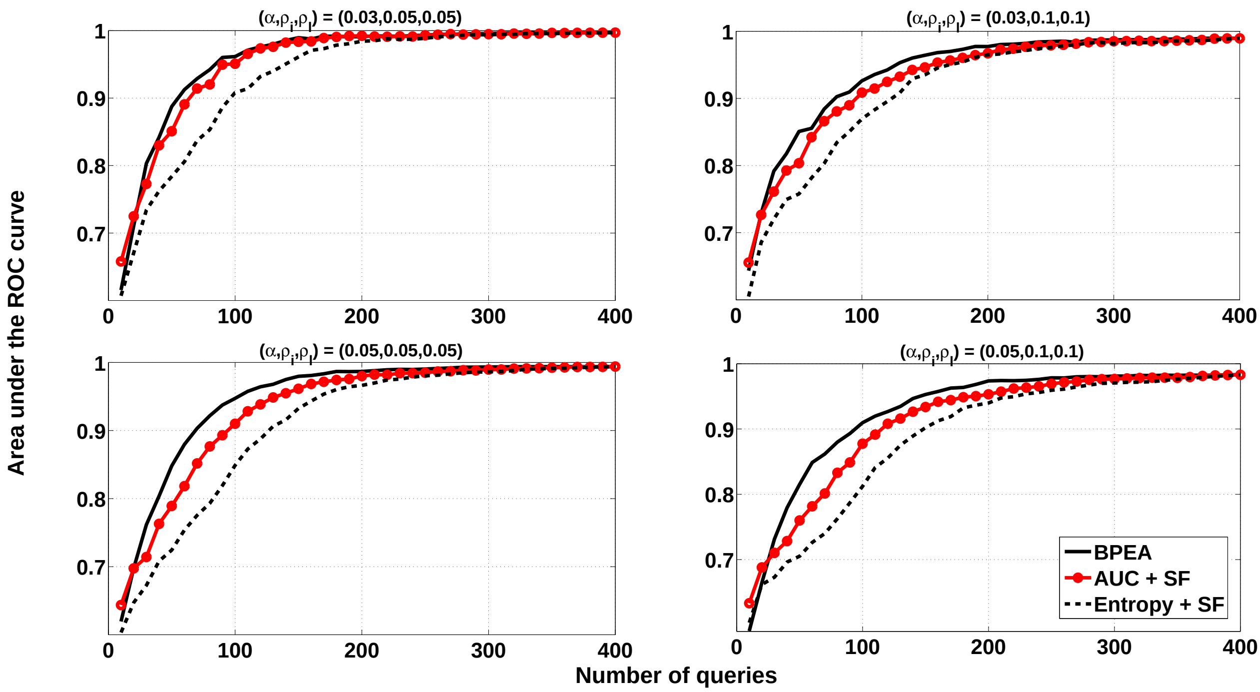


Fig. 4. The plots in this figure correspond to a dataset generated using BRITE. The AUC is computed by ranking the objects using posterior probabilities obtained from Belief Propagation (rather than single fault approximation).

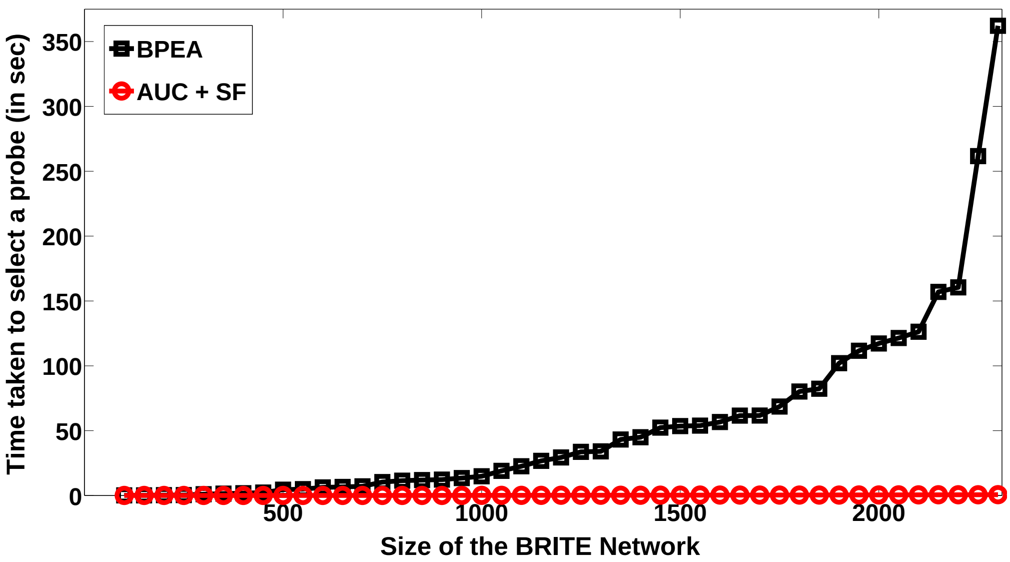


Fig. 5. This plot compares the time complexity of selecting a query using BPEA and AUC+SF.