# APPLICATION: RAYLEIGH FADING CHANNEL

Goal| detect a sinusoid at a known frequency $f_0$, $0 < f_0 < 1/2$.

$$\left.\begin{aligned} H_0 &: x(n) = w(n) \\ H_1 &: x(n) = A\cos(2\pi f_0 n + \phi) + w(n) \end{aligned}\right\} n = 0, 1, \ldots, N-1$$

where $w(n) \overset{iid}{\sim} N(0, \sigma^2)$.

Assumptions|

1: Time varying channel: $A$ and $\phi$ are not fixed, but change with time due to moving transmitter/receiver or changing environmental conditions.

2. Multipath arrivals: Received signal is superposition of several versions of transmitted signal.

Examples| Wireless comm, sonar

We have developed two general strategies for detection with unknown parameters, the Bayes Factor and GLRT.

Given our assumptions the Bayesian approach makes more sense.

Observe

$$A \cos(2\pi f_0 n + \phi) = a \cos 2\pi f_0 n + b \sin 2\pi f_0 n$$

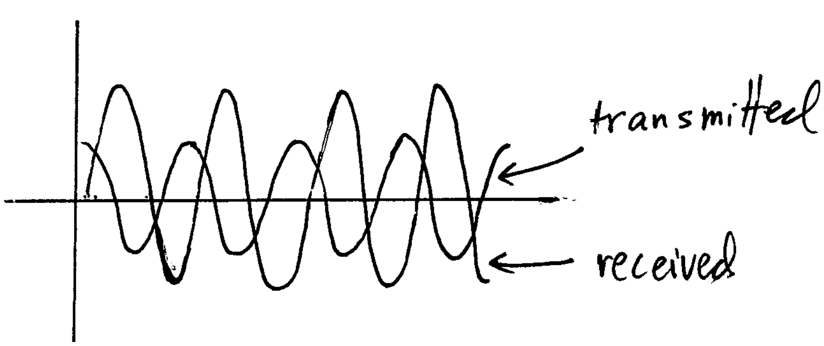
where

$$a = A \cos \phi, \qquad b = A \sin \phi$$

Let's specify the following prior:

$$\underline{\theta} = \begin{bmatrix} a \\ b \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{bmatrix} \right)$$

due to many multipath arrivals and appeal to central limit theorem




same frequency
different phase
and amplitude

The phrase "Rayleigh fading" comes from the fact that

$$A = \sqrt{a^2 + b^2}$$

has a Rayleigh distribution.

$$f(A) = \begin{cases} \frac{A}{\tau^2} \exp\left(\frac{-A^2}{2\tau^2}\right), & A > 0 \\ 0 & A < 0 \end{cases}$$

Also note

$$\phi = \arctan\left(-\frac{b}{a}\right) \sim \text{Uniform}[0, 2\pi]$$

Bayes Factor

$$\frac{f(\underline{x} \mid H_1)}{f(\underline{x} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

where

$$f(\underline{x} \mid H_1) = \int f(\underline{x} \mid H_1, \underline{\theta}) f(\underline{\theta} \mid H_1) \, d\underline{\theta}$$

Exercise (a) Determine $f(\underline{x} | H_1)$ (b) Use the MIL and a suitable approximation to simplify the detector (c) Interpret the detector (d) Choose threshold to ensure $P_F = \alpha$.

$$\text{MIL: } (A + BCD)^{-1} = A^{-1} - A^{-1}B\left(DA^{-1}B + C^{-1}\right)^{-1}DA$$

Solution (a) Writing $\underline{\theta} = \begin{bmatrix} a \\ b \end{bmatrix}$ and

$$H = \begin{bmatrix} 1 & 0 \\ \cos(2\pi f_0) & \sin(2\pi f_0) \\ \vdots & \vdots \\ \cos(2\pi f_0 (N-1)) & \sin(2\pi f_0 (N-1)) \end{bmatrix}$$

our model for $H_1$ is

$$H_1 : \underline{X} = H\underline{\theta} + \underline{W}$$

where

$$\underbrace{\underline{\theta} \sim N(\underline{0}, \tau^2 I_{2\times 2}), \quad \underline{W} \sim N(\underline{0}, \sigma^2 I_{N\times N})}_{\text{independent}}$$

Therefore $$\underline{X} = \begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ \underline{W} \end{bmatrix}$$

$$\sim N(\underline{0}, \sigma^2 I_{N\times N} + \tau^2 H H^T)$$

(b) <u>log LRT</u>:

$$-\frac{1}{2}\left[\underline{x}^T(\sigma^2 I + \tau^2 H H^T)^{-1}\underline{x} - \underline{x}^T(\sigma^2 I)^{-1}\underline{x}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \log \eta$$

Let's apply the <u>matrix inversion lemma</u>:

$$(\sigma^2 I + \tau^2 H H^T)^{-1} = \frac{1}{\sigma^2}\underline{I} -$$

$$\frac{1}{\sigma^2} I H \left(H^T \frac{1}{\sigma^2} I H + \frac{1}{\tau^2} I_{2\times 2}\right)^{-1} \underline{H}^T \cdot \left(\frac{1}{\sigma^2} I\right)$$

$$\text{Now } H^T H \approx \begin{bmatrix} N/2 & 0 \\ 0 & N/2 \end{bmatrix}$$

$\boxed{\text{exact when } f_0 = m/N \text{ approximate for large } N\text{, otherwise}}$

$$\text{So } \frac{1}{\sigma^2} H^T H + \frac{1}{\tau^2} I_{2\times 2} = \begin{pmatrix} \frac{N}{2\sigma^2} + \frac{1}{\tau^2} & 0 \\ 0 & \frac{N}{2\sigma^2} + \frac{1}{\tau^2} \end{pmatrix}$$

$$\Rightarrow (\quad)^{-1} = \frac{1}{\frac{N}{2\sigma^2} + \frac{1}{\tau^2}} I_{2\times 2}$$

$\Longrightarrow$: Log LRT $=$

$$\frac{1}{2}\underline{x}^T\left(\frac{1}{\sigma^4}\cdot\frac{1}{\frac{N}{2\sigma^2}+\frac{1}{\tau^2}}HH^T\right)\underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \log \eta$$

or

$$\underline{x}^T(HH^T)\underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

Now $\underline{x}^T(HH^T)\underline{x} = (H^T\underline{x})^T(H^T\underline{x})$

$$= \| H^T\underline{x}\|^2$$

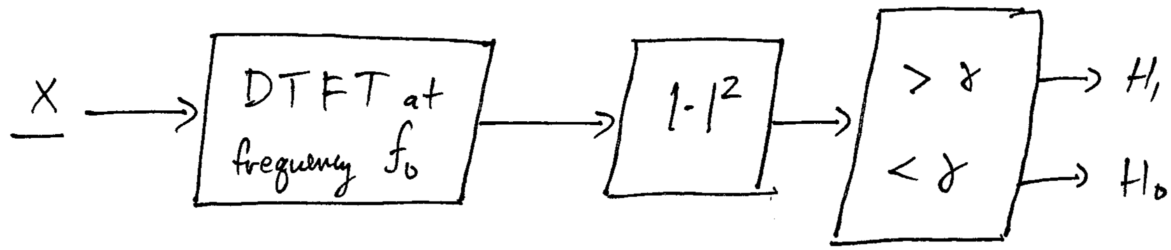
$$= \left\| \begin{bmatrix} \sum x(n)\cos(2\pi f_0 n) \\ \sum x(n)\sin(2\pi f_0 n) \end{bmatrix} \right\|^2$$

$$= \left(\sum x(n)\cos(2\pi n f_0)\right)^2 + \left(\sum x(n)\sin(2\pi n f_0)\right)^2$$

$$= \left|\sum x(n)e^{-i2\pi f_0 n}\right|^2$$

$$\equiv \Gamma(\underline{x})$$

(c) Interpretation: Optimal detector given by




Fast implementation possible with FFT.

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(d) How should we set $\gamma$ so that $P_F = \alpha$?

Observe $\sum x(n) \cos(2\pi f_0 n) \sim N\left(0, \frac{N\sigma^2}{2}\right)$

under $H_0$.

Similarly, $\sum x(n) \sin(2\pi f_0 n) \sim N\left(0, \frac{N\sigma^2}{2}\right)$

under $H_0$.

$$\Rightarrow \quad \frac{2}{N\sigma^2} \Gamma(\underline{x}) \sim \chi_2^2$$

$$\Rightarrow \quad P_F = \Pr\{\Gamma(\underline{x}) > \gamma\} = \Pr\left\{\frac{2}{N\sigma^2}\Gamma(\underline{x}) > \frac{2}{N\sigma^2}\gamma\right\}$$

$$= Q_{\chi_2^2}\left(\frac{2}{N\sigma^2}\gamma\right) \Longrightarrow \gamma = \frac{\sigma^2 N}{2} Q_{\chi_2^2}^{-1}(\alpha)$$

## Summary

- To detect a sinusoid with unknown phase/amplitude, just look at the magnitude (squared) of the frequency component.
- Test statistic and threshold independent of prior variance $\tau^2$.