CFAR DETECTION

Signal Detection in Unknown Noise Level

When the noise variance or covariance are not known, the signal detection problem becomes significantly more difficult.

- Decision rules can be derived using the techiques we have been discussing, such as the GLRT
- Selecting a meaningful threshold, however, can be very difficult

Why? Both hypotheses depend on the noise! Therefore, we can't compute performance probabilities P_F and P_D.

How can we handle such situations?

Answer: Derive decision rules and thresholds that don't depend on P_F!

Unknown Noise Variance

Suppose we have the signal detection problem

$$H_0: \quad x(n) = w(n) \qquad\qquad n = 0, \ldots, N-1$$

$$H_1: \quad x(n) = s(n) + w(n)$$

Where $s(n)$ is a known signal waveform with a fixed amplitude

$$\underline{w} \sim N(\underline{0}, \sigma^2 R)$$

where R is a known covariance structure, with $\text{trace}(R) = N$. That is R is normalized.

σ^2 is unknown noise power

*Knowing R is equivalent to assuming a known correlation function for the noise, up to a constant which is the overall noise power σ^2

Log Likelihood Ratio

$$\underline{s}^T R^{-1} \underline{x} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\sigma^2 \log \eta + \tfrac{1}{2}\underline{s}^T R^{-1} \underline{s}}_{\gamma}$$

test statistic does not depend on σ^2

<u>However</u>, both hypotheses do involve σ^2 and consequently P_D and P_F are functions of σ^2

$\Rightarrow$ we can't determine a γ to guarantee a desired P_F

What can we do?

Perhaps the GLRT will provide a test statistic more amenable to analysis.

<u>Exercise</u> Find the GLRT. Does the test statistic depend on σ^2 under H_0?

GLRT

$$\tilde{\Lambda}(\underline{x}) = \frac{\max_{\sigma^2} f(\underline{x} \mid H_1, \sigma^2)}{\max_{\sigma^2} f(\underline{x} \mid H_0, \sigma^2)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$f(\underline{x} \mid H_1, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2} \, |R|^{1/2}} \cdot \exp\left\{ -\frac{1}{2\sigma^2} (\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s}) \right\}$$

$$f(\underline{x} \mid H_0, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2} \, |R|^{1/2}} \exp\left\{ -\frac{1}{2\sigma^2} \underline{x}^T R^{-1} \underline{x} \right\}$$

$$\hat{\sigma}^2_{ML} \mid H_1 = \frac{(\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})}{N}$$

$$\hat{\sigma}^2_{ML} \mid H_0 = \frac{\underline{x}^T R^{-1} \underline{x}}{N}$$

$\Longrightarrow$

$$\tilde{\Lambda}(\underline{x}) = \frac{\dfrac{1}{(2\pi \, \hat{\sigma}^2_{ML} \mid H_1)^{N/2}}}{\dfrac{1}{(2\pi \, \hat{\sigma}^2_{ML} \mid H_0)^{N/2}}} = \left(\frac{\underline{x}^T R^{-1} \underline{x}}{(\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})} \right)^{\frac{N}{2}}$$

Derivation of ML noise estimate:

$$f(\underline{x} \mid H_1, \sigma^2) = (2\pi\sigma^2)^{-\frac{N}{2}} |R|^{-\frac{1}{2}} \exp\left\{ \frac{-1}{2\sigma^2} \cdot Q \right\}$$

$$Q = (\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})$$

$$\Rightarrow \log f(\underline{x} \mid H_1, \sigma^2) = -\frac{N}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} Q$$

+ constant (not depending on σ^2)

$$\Rightarrow \frac{\partial}{\partial \sigma^2} \log f(\underline{x} \mid H_1, \sigma^2) = \frac{-N}{2\sigma^2} + \frac{Q}{2(\sigma^2)^2}$$

Set to 0 $\Rightarrow \dfrac{N}{2\sigma^2} = \dfrac{Q}{2(\sigma^2)^2}$

$$\Rightarrow \hat{\sigma}^2_{ML} \mid H_1 = \frac{Q}{N} = \frac{(\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})}{N}$$

GLRT:

$$\underbrace{\left(\frac{\underline{x}^T R^{-1}\underline{x}}{(\underline{x}-\underline{s})^T R^{-1}(\underline{x}-\underline{s})}\right)^{\frac{N}{2}}}_{\substack{\text{test statistic} \\ \tilde{\Lambda}(\underline{x})}} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\eta}_{\text{threshold}}$$

To set the threshold to achieve a desired P_F we need to consider the distribution of the test statistic under H_0.

Since both hypotheses depend on the unknown σ^2, we might suppose that $\tilde{\Lambda}(\underline{x})$ does as well.

But what if it didn't? What if $\tilde{\Lambda}(\underline{x})$ didn't depend on σ^2?

Definition | CFAR

If the distribution of the test statistic under H_0 is independent of the unknown parameter (noise variance), then the detector is a <u>constant false-alarm rate</u> (CFAR) detector.

That is, no matter what the unknown parameter is, for a given threshold level, P_F is constant.

Let's see if the GLRT has the CFAR property in this case.

Recall GLRT $\Rightarrow$

$$\tilde{\lambda}(x) = \left(\frac{\underline{x}^T R^{-1} \underline{x}}{(\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})} \right)^{\frac{N}{2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Now since $a^b, b>0$ is a monotonic transformation, we have the following equivalent test:

$$\frac{\underline{x}^T R^{-1} \underline{x}}{(\underline{x}-\underline{s})^T R^{-1} (\underline{x}-\underline{s})} \underset{H_0}{\overset{H_1}{\gtrless}} \eta^{2/N}$$

Under H_0, $\underline{x} = \underline{w} = \sigma \tilde{\underline{w}}$

where $\tilde{\underline{w}} \sim N(\underline{0}, R)$

and the test statistic is written

$$\frac{\sigma^2 \tilde{\underline{w}}^T R^{-1} \tilde{\underline{w}}}{(\sigma \tilde{\underline{w}} - \underline{s})^T R^{-1} (\sigma \tilde{\underline{w}} - \underline{s})}$$

In order for the test statistic to be invariant to the unknown noise variance, σ must cancel in the numerator and denominator.

$$\frac{\sigma^2 \underline{\tilde{W}}^T R^{-1} \underline{\tilde{W}}}{(\sigma \underline{\tilde{W}} - \underline{s})^T R^{-1} (\sigma \underline{\tilde{W}} - \underline{s})}$$

test statistic

Unfortunately, σ can't be eliminated and hence P_F will depend on σ.

$\Rightarrow$ GLRT is <u>not</u> CFAR

Remarkably, the situation is dramatically different if we assume that the signal amplitude is also unknown.

Consider the following scenario.

$$H_0 : \underline{x} = \underline{w}$$

$$H_1 : \underline{x} = A\underline{s} + \underline{w}$$

A unknown

$\underline{w} \sim N(\underline{0}, \sigma^2 R)$

trace$(R) = N$, R known

σ^2 unknown

Under H_0 we can express the observation as

$$\underline{x} = \sigma \underline{\tilde{w}}, \quad \underline{\tilde{w}} \sim N(\underline{0}, R)$$

Under H_1

$$\underline{x} = A\underline{s} + \underline{\sigma} \tilde{w}, \quad \tilde{w} \sim N(\underline{0}, R)$$

$$= \sigma(A'\underline{s} + \tilde{w}),$$ where again A' is just an unknown amplitude $A' = \frac{A}{\sigma}$

In both cases we can view σ as a scaling factor applied to the entire observation.

So, in the case where both the noise power and signal amplitude are unknown, we can view uncertainty in the noise variance as an arbitrary scaling of the data.

This interpretation will enable a GLRT with the CFAR property.

To form a GLRT for this case we also must estimate the unknown signal amplitude in hypothesis H_1.

$$f(\underline{X} \mid H_1, A, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2} |R|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2\sigma^2}(\underline{x}-A\underline{s})^T R^{-1}(\underline{x}-A\underline{s})\right\}$$

So under H_1 we must jointly estimate A and σ^2.

$$(\hat{A}_{ML}, \widehat{\sigma^2_{ML}}) = \arg\max_{A,\sigma^2} f(\underline{x} \mid H_1, A, \sigma^2)$$

Maximizing $f(\underline{x} \mid H_1, A, \sigma^2)$ is equivalent to maximizing $\log f(\underline{x} \mid H_1, A, \sigma^2)$.

$$\frac{\partial}{\partial A} \log f(\underline{x} \mid H_1, A, \sigma^2) = \frac{1}{\sigma^2} \underline{s}^T R^{-1} (\underline{x} - A\underline{s})$$

Setting this to zero we find

$$\hat{A}_{ML} = \frac{\underline{s}^T R^{-1} \underline{x}}{\underline{s}^T R^{-1} \underline{s}} \quad , \text{ independent of } \sigma^2$$

The joint maximum occurs at $\frac{\partial^2}{\partial \sigma^2 \partial A} \log f(\underline{x} \mid H_1, A, \sigma^2) = 0$

or

$$\frac{\partial}{\partial \sigma^2} \log f(\underline{x} \mid H_1, \sigma^2; \hat{A}_{ML}) = 0$$

And so

$$\hat{\sigma}^2_{ML} \mid H_1 = (\underline{x} - \hat{A}_{ML}\underline{s})^T R^{-1} (\underline{x} - \hat{A}_{ML}\underline{s})/N$$

$$= \left(\underline{x} - \frac{\underline{s}^T R^{-1}\underline{x}}{\underline{s}^T R^{-1}\underline{s}}\underline{s}\right)^T R^{-1} \left(\underline{x} - \frac{\underline{s}^T R^{-1}\underline{x}}{\underline{s}^T R^{-1}\underline{s}}\underline{s}\right)/N$$

Cumbining this with

$$\hat{\sigma}^2_{ML} \mid H_0 = \underline{x}^T R^{-1}\underline{x}/N \quad (\text{as before})$$

we get a GLRT (in simplified form)

$$\frac{\underline{x}^T R^{-1}\underline{x}}{\left(\underline{x} - \frac{\underline{s}^T R^{-1}\underline{x}}{\underline{s}^T R^{-1}\underline{s}}\underline{s}\right)^T R^{-1}\left(\underline{x} - \frac{\underline{s}^T R^{-1}\underline{x}}{\underline{s}^T R^{-1}\underline{s}}\underline{s}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

Under H_0, the test statistic can be written as

$$\frac{\sigma^2 \underline{\tilde{w}}^T R^{-1}\underline{\tilde{w}}}{\left(\sigma\underline{\tilde{w}} - \sigma\frac{\underline{s}^T R^{-1}\underline{\tilde{w}}}{\underline{s}^T R^{-1}\underline{s}}\underline{s}\right)^T R^{-1}\left(\sigma\underline{\tilde{w}} - \sigma\frac{\underline{s}^T R^{-1}\underline{\tilde{w}}}{\underline{s}^T R^{-1}\underline{s}}\underline{s}\right)}$$

and the σ^2 factor on top and bottom cancel!

Therefore, the probability density of the test statistic does not depend on σ^2.

$\Rightarrow$ A threshold can be chosen to insure a specified P_F for any value of σ^2.

The GLRT has the CFAR property!

To actually set the threshold γ we need to relate γ to P_F which requires the test statistic's distribution under H_0.

Let's look at the test statistic more carefully, and to keep things simple set $R = I$ (white noise).

Test statistic: $(R = I)$

$$\frac{\underline{x}^T \underline{x}}{\left(\underline{x} - \frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underline{s}\right)^T \left(\underline{x} - \frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underline{s}\right)} \equiv t(\underline{x})$$

Note

$$\frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} \underline{s} = \underline{s} \frac{\underline{s}^T \underline{x}}{\underline{s}^T \underline{s}} = \underbrace{\frac{\underline{s}\,\underline{s}^T}{\underline{s}^T \underline{s}}}_{\text{does this look familiar?}} \cdot \underline{x}$$

The test statistic is written as

$$t(\underline{x}) = \frac{\underline{x}^T \underline{x}}{(\underline{x} - P_s \underline{x})^T (\underline{x} - P_s \underline{x})}$$

Look at the denominator:

$$(\underline{x} - P_s \underline{x})^T (\underline{x} - P_s \underline{x}) =$$

(a)

$$=$$

We now have

$$t(\underline{x}) = \frac{\underline{x}^T \underline{x}}{\underline{x}^T \underline{x} - \underline{x}^T P_s \underline{x}}$$

$$= \frac{\underline{x}^T (I - P_s) \underline{x} + \underline{x}^T P_s \underline{x}}{\underline{x}^T (I - P_s) \underline{x}}$$

$$= 1 + \frac{\underline{x}^T P_s \underline{x}}{\underline{x}^T (I - P_s) \underline{x}}$$

This gives us a modified test

$$t'(\underline{x}) = \frac{\underline{x}^T P_s \underline{x}}{\underline{x}^T (I - P_s) \underline{x}} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma - 1 \equiv \gamma'$$

Ok. So what? Well let's try to determine the distribution of the numerator and denominator of $t'(\underline{x})$.

Proposition If P is a rank r projection matrix and $\underline{X} \sim N(\underline{0}, \sigma^2 I)$, then $\frac{\underline{X}^T P \underline{X}}{\sigma^2} \sim \chi^2_r$

Proof:

$$P = \mathcal{U} \cdot \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \mathcal{U}^T \quad \text{(from 551)}$$

$$\Rightarrow \quad P = \sum_{i=1}^{r} \underline{u}_i \underline{u}_i^T$$

$$\Rightarrow \quad \underline{X}^T P \underline{X} = \sum_{i=1}^{r} \underline{X}^T \underline{u}_i \underline{u}_i^T \underline{X} = \sum_{i=1}^{r} (\underline{u}_i^T \underline{X})^2$$

Now $\underline{u}_i^T \underline{X} \sim N(0, \sigma^2)$ since $\underline{u}_i^T \underline{u}_i = 1$.

In addition, if $i \neq j$, $\underline{u}_i^T X$ and $\underline{u}_j^T \underline{X}$ are uncorrelated (and hence independent) because

(b) $$E\left[(\underline{u}_i^T \underline{X})(\underline{u}_j^T \underline{X})\right] =$$

Def | If $U \sim \chi^2_p$, $V \sim \chi^2_q$ are independent, and $Z = \dfrac{U/p}{V/q}$, we say Z has an F-distribution with p, q degrees of freedom.

Recall our test statistic $t'(\underline{x}) = \dfrac{\underline{x}^T P \underline{x}}{\underline{x}^T (I-P) \underline{x}} = \dfrac{\underline{x}^T P \underline{x} / \sigma^2}{\underline{x}^T (I-P) \underline{x} / \sigma^2}$

Clearly the numerator and denominator are chi-squared RVs.
Are they independent? Writing

$$I - P = \sum_{i=r+1}^{N} \underline{u}_i \underline{u}_i^T$$

we can argue that $\underline{u}_i^T \underline{x}$, $i \le r$, and $\underline{u}_j^T \underline{x}$, $j \ge r+1$, are independent (as before), and therefore the two chi-squared RVs are independent.

In our case, $r = 1$ $\left(P = P_s = \dfrac{\underline{s}\,\underline{s}^T}{\underline{s}^T \underline{s}} \right)$.

Therefore, under H_0

$$(N-1)\, t'(\underline{x}) \sim F_{1, N-1}$$

$$\Longrightarrow \gamma' = \frac{1}{N-1} Q^{-1}_{F_{1,N-1}}(\alpha)$$

ensures $P_F = \alpha$ regardless of σ^2.

Summary:

CFAR: test statistic's distribution under H_0 is <u>independent</u> of unknown parameters $\Rightarrow$ constant false alarm rate

Most tests are <u>not</u> CFAR.

In special cases, like the unknown noise variance and unknown signal amplitude scenario, the GLRT is CFAR and F-distributed.

This allows us to design a detector and set a threshold to achieve a desired P_F even though σ^2 is unknown.

Key

a. $(\underline{x} - P_s\underline{x})^T(\underline{x} - P_s\underline{x})$

$$= \underline{x}^T\underline{x} - \underline{x}^T P_s \underline{x} - \underline{x}^T P_s^T \underline{x} + \underline{x}^T P_s^T P_s \underline{x}$$

$$= \underline{x}^T\underline{x} - \underline{x}^T P_s \underline{x}$$

[$P = P^T$ and $P = P^2$ for projections]

$$= \underline{x}^T(I - P_s)\underline{x}$$

b. $E[(\underline{u}_i^T \underline{x})(\underline{u}_j^T \underline{x})]$

$$= E[\underline{u}_i^T \underline{x} \cdot \underline{x}^T \underline{u}_j]$$

$$= \underline{u}_i^T \cdot \sigma^2 I \cdot \underline{u}_j$$

$$= 0$$

since $\underline{u}_i^T \underline{u}_j = \delta_{ij}$