

# DETECTION THEORY

## Hypothesis Testing

In a hypothesis testing problem, we are given data  $\underline{x} = [x_1, \dots, x_N]^T$ , and we must decide which of two or more hypotheses best fits the data.

### Example

Suppose  $x_n \stackrel{\text{iid}}{\sim} \mathcal{N}(\theta, 1)$

but we are not sure about  $\theta$ .

we suspect either  $\theta = -1$  or  $\theta = 1$ :

$$H_0: \theta = -1$$

$$H_1: \theta = 1$$

## Binary vs. M-ary Test

In general, there can be  $M \geq 2$  hypotheses,  $H_1, \dots, H_M$ .

### Example

Suppose we want to devise a speech recognition system to recognize the digits 0, 1, ..., 9:

$$H_0: \underline{x} \sim \text{"zero"}$$

$$H_1: \underline{x} \sim \text{"one"}$$

⋮

$$H_9: \underline{x} \sim \text{"nine"}$$

If  $M=2$ , we have a binary testing problem.

### Example

$$H_0: \underline{x} \sim \text{"no"}$$

$$H_1: \underline{x} \sim \text{"yes"}$$

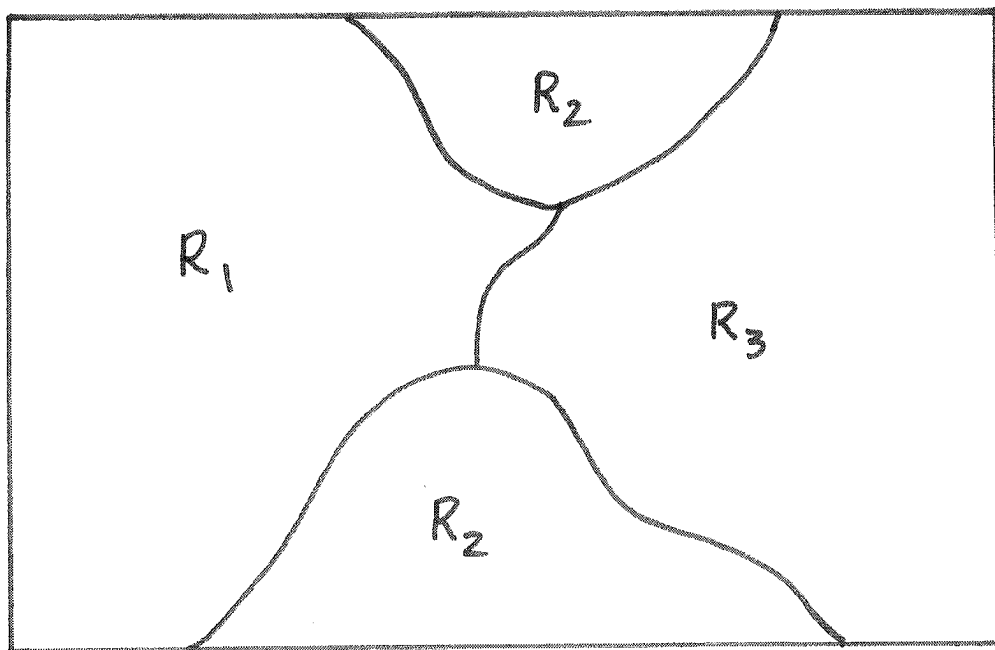
In hypothesis testing the goal is to construct a decision rule / detector / test, i.e. a mapping

$$h: \mathbb{R}^N \rightarrow \{ \mathcal{H}_1, \dots, \mathcal{H}_M \}$$

assigning an observation  $\underline{x}$  to a hypothesis.

A decision rule partitions the input space into decision regions

$$R_k = \{ \underline{x} \in \mathbb{R}^N : h(\underline{x}) = \mathcal{H}_k \}.$$



$$N=2, M=3$$

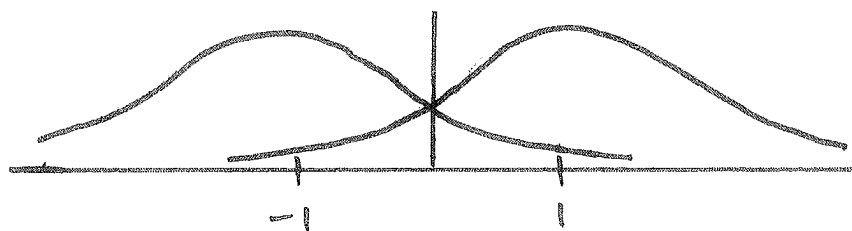
Ex | Suppose  $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ , where

$$x_n \stackrel{\text{iid}}{\sim} \mathcal{N}(\theta, 1)$$

and consider the testing problem

$$\mathcal{H}_0: \theta = -1$$

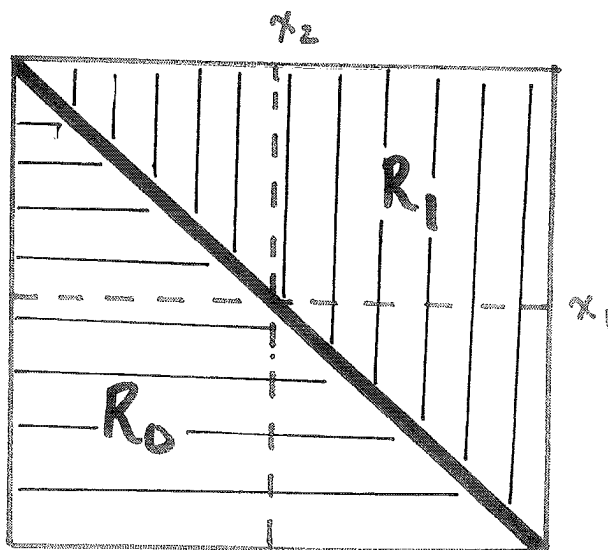
$$\mathcal{H}_1: \theta = 1$$



A reasonable test is

$$h(\underline{x}) = \begin{cases} \mathcal{H}_0 & \text{if } \bar{x} \leq 0 \\ \mathcal{H}_1 & \text{if } \bar{x} > 0 \end{cases}$$

where  $\bar{x} = \frac{1}{2}(x_1 + x_2)$



## Simple vs. Composite Hypotheses

If a hypothesis  $\mathcal{H}$  specifies a unique distribution for  $\underline{x}$ , we say  $\mathcal{H}$  is simple.

If  $\mathcal{H}$  specifies a class of possible distributions for  $\underline{x}$ , we say  $\mathcal{H}$  is composite.

### Example

Suppose we want to detect a sinusoid

$$s_n = \cos(2\pi f_0 n + \phi), n=0,1,\dots,N-1$$

where  $f_0$  is known but  $\phi$  is unknown.

Let  $w_n \sim \mathcal{N}(0, \sigma^2)$ ,  $\sigma^2$  known.

$$\mathcal{H}_0: \underline{x} = \underline{s}$$

$$\mathcal{H}_1: \underline{x} = \underline{s} + \underline{w}$$

Here  $\mathcal{H}_0$  is simple and

$\mathcal{H}_1$  is composite (since  $\phi$  unknown)

# Terminology

Null hypothesis: An interesting event  
did not happen

## One-Sided Test

$$H_0: \theta \leq \theta_0 \quad \theta \in \mathbb{R}$$

$$H_1: \theta > \theta_0$$

## Two-Sided Test

$$H_0: \underline{\theta} = \underline{\theta}_0$$

$$H_1: \underline{\theta} \neq \underline{\theta}_0$$

$$\underline{\theta} \in \mathbb{R}^N$$

"hypothesis  $H_1$  lies on both sides of  $H_0$ "

## Thresholding Rule

$$T(\underline{x}) \begin{cases} > \gamma \\ < \gamma \end{cases} \begin{matrix} H_1 \\ H_0 \end{matrix}$$

$T(\underline{x})$  is a scalar statistic.

## Detection Theory

Detection theory is another name for hypothesis testing in the context of signal processing problems.

In the next few weeks, we will study the basic theory of hypothesis testing, and apply it to various signal detection problems.