APPLICATION: CHANNEL ESTIMATION

Kalman Filtering for Time-Varying Channel Estimation

Multipath Communications Channel:

This effect is the result of many propagation paths, each of which delays and attenuates the input signal.

Additionally, the channel is time-varying due to movement of the source, receiver, and/or scatters. Therefore, the channel is acting like a linear time-varying filter.
Due to the time-varying nature of the channel, a sinusoidal input does not produce a pure sinusoid at the output:

\[ \text{input} \rightarrow h(t) \rightarrow \text{output} \]

Instead, the output is a narrowband process. Assuming the channel is relatively slowly varying (compare to the frequency of the input) we can view the input sinusoid as being amplitude modulated by the time-varying channel. This effect is referred to as fading and such channels are called fading multipath channels.
If we sample the output of the channel, then a very good model is the low-pass tapped delay line model:

\[ y(n) = \sum_{k=0}^{p-1} h_n(k) v(n-k) \]

input  \rightarrow output  \leftarrow impulse response depends on time \( n \)

This is simply an FIR filter with time-varying coefficients. In practice we wouldn't observe this perfect output, but rather a noise-corrupted version of it:

\[ x(n) = \sum_{k=0}^{p-1} h_n(k) v(n-k) + w(n) \]

where \( w(n) \) is observation noise.
The goal of channel estimation is to determine the linear time-varying filter $h_n(k)$ based on the input $v(n)$ and measured output $x(n)$. Is this possible?

Assume $v(n) = 0$ for $n < 0$. Then

$$X(0) = h_0(0)v(0) + h_0(1)v(-1) + w(0)$$
$$= h_0(0)v(0) + w(0)$$

$$X(1) = h_1(0)v(1) + h_1(1)v(0) + w(1)$$

$$X(2) = h_2(0)v(2) + h_2(1)v(1) + w(2)$$

$$\vdots$$

For each $n \geq 1$ we have two new parameters we must estimate!

Even in the absence of measurement noise we have more unknowns than equations and we can't determine the filter.
What can we do?

Well, suppose that the filter weights are not changing too rapidly from sample to sample. This is known as a **slow-fading channel model**.

Probabilistically, we can view the slowly varying channel as a vector-valued Gauss-Markov process:

\[
\mathbf{h}_n = A \mathbf{h}_{n-1} + \mathbf{u}_n
\]

where \( \mathbf{h}_n = [h_n(0), \ldots, h_n(p-1)]^T \), \( A \) is a \( p \times p \) matrix designed to reflect the correlation expected between filter weights at different time samples, and \( \mathbf{u}_n \) is a white Gaussian noise vector process with covariance \( \mathbf{Q} \).

That is,

\[
\ldots, \mathbf{u}_{n-2}, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots \text{ are iid vectors}
\]

and \( \mathbf{u}_n \sim \mathcal{N}(0, \mathbf{Q}) \)
A standard simplifying assumption is to assume that $A$ and $Q$ are diagonal $\Rightarrow$ filter weights are uncorrelated with each other.

This is called an uncorrelated scattering model.

The measurement/observation model in vector form is

$$x_n = v_n^T b_n + w_n$$

where $v_n = [v(n), v(n-1), \ldots, v(n-p+1)]^T$.

With this notation and our Gauss-Markov model for the time-varying filter, we can now devise a Kalman filter to estimate and track the channel.
In the case we have the state equation:

\[ \mathbf{h}_{n+1} = A\mathbf{h}_n + \mathbf{u}_n, \quad n \geq 0 \]

(with \( \mathbf{h}_n \) in place of \( \mathbf{s}_n \) now)

Furthermore, assume that

\[ \mathbf{h}_0 \sim N(0, \mathbf{R}_0) \]

with \( \mathbf{R}_0 \) also diagonal.

**Measurement equation:**

\[ \mathbf{x}_n = \mathbf{V}_n^T \mathbf{h}_n + \mathbf{w}_n \]

Note that \( \mathbf{V}_n \) is known, but time-varying. In our earlier discussion the \( \mathbf{H} \) matrix of the observation model was constant. The Kalman filter still is applicable here, we just replace \( \mathbf{H} \) with \( \mathbf{V}_n^T \).
Kalman Filter

\( \Sigma_n = H_n, \quad H = V_n^T, \quad B = I \)

\[
\begin{align*}
\hat{x}_{n|n-1} &= A \hat{x}_{n-1|n-1} \\
M_{n|n-1} &= A M_{n-1|n-1} A^T + Q \\
K_n &= \frac{M_{n|n-1} - V_n}{V_n^T M_{n|n-1} V_n + \sigma_w^2} \\
\hat{x}_{n|n} &= \hat{x}_{n|n-1} + K_n (x_n - V_n^T \hat{x}_{n|n-1}) \\
M_{n|n} &= (I - K_n V_n) M_{n|n-1}
\end{align*}
\]
Example \((p = 2)\)

**Slow-Fading Model Parameters**

\[
A = \begin{bmatrix}
0.999 & 0 \\
0 & 0.999
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
0.0001 & 0 \\
0 & 0.0001
\end{bmatrix}
\]

To reflect little or no knowledge about the initial state of the channel:

\[h_0 \sim N(0, \begin{bmatrix} 1 & 0 \\
0 & 1 & 0
\end{bmatrix})\]

**Channel model:**

\[X(n) = h_n(0) V(n) + h_n(1) V(n-1) + W(n)\]

- \(V(n) = \) input to channel, known square wave
- \(h_n(k) = \) time-varying channel model (linear time-varying FIR filter)
- \(W(n) = \) white Gaussian observation noise
noise-free channel output $y_n = \frac{v_n^T}{\lambda_n} h_n$

observation channel output $x$

channel $h$

↑ filter changes drastically (e.g., car drives out of a tunnel)