EIGENDECOMPOSITIONS & THE SPECTRAL THEOREM

The Spectral Theorem

<u>Definition</u> If $\mathcal{U} \in \mathbb{C}^{N \times N}$ is such that

$$\mathcal{U}^H \mathcal{U} = \mathcal{U}\mathcal{U}^H = I_{N \times N}$$

then $\mathcal{U}$ is said to be <u>unitary</u>.

If $\mathcal{U} \in \mathbb{R}^{N \times N}$ is such that

$$\mathcal{U}^T \mathcal{U} = \mathcal{U}\mathcal{U}^T = I_{N \times N}$$

then $\mathcal{U}$ is said to be <u>orthogonal</u>

[slightly confusing since the columns of $\mathcal{U}$ are in fact orthonormal]

Intuitively, such matrices are distance preserving since

$$\begin{aligned} \|\mathcal{U}\underline{x} - \mathcal{U}\underline{y}\|^2 &= (\mathcal{U}\underline{x} - \mathcal{U}\underline{y})^H (\mathcal{U}\underline{x} - \mathcal{U}\underline{y}) \\ &= (\underline{x} - \underline{y})^H \mathcal{U}^H \mathcal{U} (\underline{x} - \underline{y}) \\ &= \|\underline{x} - \underline{y}\|^2 \end{aligned}$$

Unitary and orthogonal matrices effect a change of coordinate system.

Theorem | (Spectral Theorem)

If $A \in \mathbb{C}^{N \times N}$ is Hermitian, then there exist a unitary matrix $\mathcal{U}$ and a <u>real</u> diagonal matrix Λ such that

$$A = \mathcal{U} \Lambda \mathcal{U}^H.$$

If $A \in \mathbb{R}^{N \times N}$ is symmetric, the same result holds where now $\mathcal{U}$ is orthogonal.

Proof | See Moon and Stirling, Mathematical Methods and Algorithms for Signal Processing.

Eigenvalues and eigenvectors

Suppose A is Hermitian/symmetric. Write

$$A = \mathcal{U} \Lambda \mathcal{U}^H$$

according to the spectral theorem. Let

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_N \end{bmatrix}$$

Since $A\mathcal{U} = \mathcal{U}\Lambda$, we conclude that the λ_i are eigenvalues with $\underline{u}_i$ the associated eigenvector:

$$A \underline{u}_i = \lambda_i \underline{u}_i \,, \qquad i = 1, \dots, N.$$

The spectral theorem also gives rise to following spectral decomposition of A:

$$A = \sum_{i=1}^{N} \lambda_i \underline{u}_i \underline{u}_i^H$$

Positive (semi) definite matrices

Let A be a Hermitian / symmetric $N \times N$ matrix.

We say A is <u>positive definite</u> (PD) if

$$\underline{x} \neq 0 \implies \underline{x}^H A \underline{x} > 0.$$

We say A is <u>positive semi-definite</u> (PSD) if

$$\forall \underline{x} \quad \underline{x}^H A \underline{x} \geq 0.$$

[PSD is also called nonnegative definite]

Exercise Show that A is PD (PSD) iff the eigenvalues of A are positive (nonnegative).

Solution | By the spectral theorem, $\underline{u}_1, \ldots, \underline{u}_N$ is an orthonormal collection (a basis, in fact) such that

$$A \underline{u}_i = \lambda_i \underline{u}_i .$$

For each $i = 1, \ldots, N$ we have

$$\begin{aligned} \lambda_i &= \lambda_i \cdot \underline{u}_i^H \underline{u}_i \\ &= \underline{u}_i^H \cdot \lambda_i \underline{u}_i \\ &= \underline{u}_i^H A \underline{u}_i \quad \begin{cases} > 0 & \text{if } A \text{ is PD} \\ \geq 0 & \text{if } A \text{ is PSD.} \end{cases} \end{aligned}$$

Conversely, suppose $\lambda_i > 0$ (≥ 0) $\forall i$.

Then for $\underline{x} \neq 0$ we have

$$\begin{aligned} \underline{x}^H A \underline{x} &= \sum_{i=1}^{n} \lambda_i \underline{x}^H \underline{u}_i \underline{u}_i^H \underline{x} \\ &= \sum_{i=1}^{n} \lambda_i \| \underline{u}_i^H \underline{x} \|^2 \end{aligned}$$

$$\begin{cases} > 0 & \text{if } \lambda_i > 0 \quad \forall i \\ \geq 0 & \text{if } \lambda_i \geq 0 \quad \forall i \end{cases}$$