LINEAR PREDICTION

Linear prediction is an application of LMMSE estimation theory that is widely used in speech processing, spectral estimation, and elsewhere.

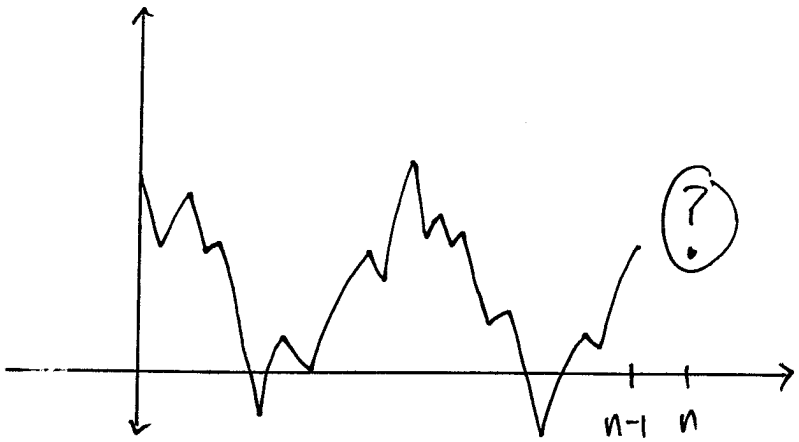
Let $\{x[n]\}$ be a zero-mean, WSS random process with ACF

$$r_{xx}[k] = E\{x[n]\, x[n+k]\}.$$

We observe

$$\underline{x} = [x[n-1] \cdots x[n-p]]^T$$

and our task is to predict the next value $x[n]$.



Let's view $x[n]$ as an unknown parameter to be estimated:

$$\theta = x[n]$$

A linear estimator for $x[n]$ has the form

$$\hat{\theta} = \hat{x}[n] = \sum_{k=1}^{p} h_p[k]\, x[n-k].$$

We know that the optimal predictor coefficients must satisfy the Wiener-Hopf equations

$$R_{xx}\, \underline{h}_p = R_{x\theta}$$

where $\underline{h}_p = [h_p[1] \cdots h_p[p]]^T$.

To simplify notation, denote

$$r_k = r_{xx}[k].$$

Then

$$R_{xx} = E\{\underline{x}\,\underline{x}^T\} = \begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_{p-1} \\ r_1 & r_0 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & r_0 & & \vdots \\ \vdots & \vdots & & \ddots & \\ r_{p-1} & r_{p-2} & \cdots & & r_0 \end{bmatrix} =: R_p$$

Toeplitz ↗

and

$$R_{x\theta} = E\left\{ \underline{x} \cdot x[n] \right\} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix} =: \underline{r}_p$$

Then the optimal linear predictor satisfies

$$R_p \cdot \underline{h}_p = \underline{r}_p$$

$$\Updownarrow$$

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_{p-1} \\ r_1 & r_0 & \cdots & r_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p-1} & r_{p-2} & \cdots & r_0 \end{bmatrix} \begin{bmatrix} h_p[1] \\ h_p[2] \\ \vdots \\ h_p[p] \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix}$$

Mean-Squared Prediction Error

$$E\left\{ \left(x[n] - \hat{x}[n] \right)^2 \right\}$$

$$=$$

(a)

Base of recursion

The WH equations for $p = 1$ are

$$[r_0] \cdot [h_1[1]] = [r_1]$$

$$\Rightarrow \quad h_1[1] = \frac{r_1}{r_0} .$$

This will initialize the recursive algorithm

General recursion

We wish to express

$$\underline{h}_p = \begin{bmatrix} \underline{h}_{p-1} \\ 0 \end{bmatrix} + \begin{bmatrix} \underline{d}_{p-1} \\ k_p \end{bmatrix}$$

The goal is to find $\underline{d}_{p-1}$, k_p.

Our approach is to exploit the recursive structure of the WH equations.

According to the WH equations

$$R_p \cdot \underline{h}_p = \underline{r}_p .$$

$$\Updownarrow$$

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_{p-1} \\ r_1 & r_0 & \cdots & r_{p-2} \\ \vdots & \vdots & \ddots & \\ r_{p-1} & r_{p-2} & & r_0 \end{bmatrix} \begin{bmatrix} h_p(1) \\ h_p(2) \\ \vdots \\ h_p(p) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix}$$

Now observe

$$R_p = \begin{bmatrix} R_{p-1} & \tilde{\underline{r}}_{p-1} \\ \tilde{\underline{r}}_{p-1}^T & r_0 \end{bmatrix} , \quad \underline{r}_p = \begin{bmatrix} \underline{r}_{p-1} \\ r_p \end{bmatrix}$$

where $\tilde{\underline{r}}_p :=$ "$\underline{r}_p$ upside-down". Plugging in the recursive formula for $\underline{h}_p$ we get

$$\begin{bmatrix} R_{p-1} & \tilde{\underline{r}}_{p-1} \\ \tilde{\underline{r}}_{p-1}^T & r_0 \end{bmatrix} \cdot \left\{ \begin{bmatrix} \underline{h}_{p-1} \\ 0 \end{bmatrix} + \begin{bmatrix} \underline{d}_{p-1} \\ k_p \end{bmatrix} \right\} = \begin{bmatrix} \underline{r}_{p-1} \\ r_p \end{bmatrix}$$

This gives us a system of equations

$$R_{p-1}\,\underline{h}_{p-1} + R_{p-1}\,\underline{d}_{p-1} + \tilde{\underline{r}}_{p-1}\,k_p = \underline{r}_{p-1} \qquad (1)$$

$$\tilde{\underline{r}}_{p-1}^{T}\cdot\underline{h}_{p-1} + \tilde{\underline{r}}_{p-1}^{T}\cdot\underline{d}_{p-1} + r_0\,k_p = r_p \qquad (2)$$

Recall: we're assuming $\underline{h}_{p-1}$ known and we need to solve for $\underline{d}_{p-1}$, k_p.

How can we simplify the first equation?

$$R_{p-1}\,\underline{h}_{p-1} = \underline{r}_{p-1} \quad \Longrightarrow \quad \underline{d}_{p-1} = -k_p\, R_{p-1}^{-1}\,\tilde{\underline{r}}_{p-1}$$

Exercise | Show that $R_{p-1}^{-1}\,\tilde{\underline{r}}_{p-1} = \tilde{\underline{h}}_{p-1}$, or equivalently, $R_{p-1}\,\tilde{\underline{h}}_{p-1} = \tilde{\underline{r}}_{p-1}$. For concreteness, you may wish to consider $p = 3$ or 4.

Solution | Suppose

$$A \cdot \underline{b} = \underline{c}$$

where A is $m \times n$, $\underline{b}$ is $n \times 1$, $\underline{c}$ is $m \times 1$.

When is $A\tilde{\underline{b}} = \tilde{\underline{c}}$?

If $A = (a_{ij})$, then define

$$\tilde{A} = (\tilde{a}_{ij})$$

where $\tilde{a}_{ij} = a_{m-i,\, n-j}$. Then

$$A\underline{b} = \underline{c} \iff \sum_{j=1}^{n} a_{ij} b_j = c_i \quad \forall i$$

$$\iff \sum_{j=1}^{n} a_{i,n-j} b_{n-j} = c_i \quad \forall i$$

$$\iff \sum_{j=1}^{n} a_{m-i,n-j} b_{n-j} = c_{m-i} \quad \forall i$$

$$\iff \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{b}_j = \tilde{c}_i$$

$$\iff \tilde{A} \cdot \tilde{\underline{b}} = \tilde{\underline{c}}$$

So we need $A = \tilde{A}$. This is true when A is symmetric and Toeplitz, as is the case for R_{p-1}.

Conclusion: $\boxed{\underline{d}_{p-1} = -k_p \, \underline{\tilde{h}}_{p-1}}$

Thus far we have shown

$$\underline{h}_p = \begin{bmatrix} \underline{h}_{p-1} \\ 0 \end{bmatrix} + \begin{bmatrix} \underline{d}_{p-1} \\ k_p \end{bmatrix} = \begin{bmatrix} \underline{h}_{p-1} \\ 0 \end{bmatrix} + k_p \begin{bmatrix} -\underline{h}_{p-1} \\ 1 \end{bmatrix}.$$

It remains to find k_p.

Recall equation (2):

$$\underline{\tilde{r}}_{p-1}^T \, \underline{h}_{p-1} + \underline{\tilde{r}}_{p-1}^T \, \underline{d}_{p-1} + r_0 k_p = r_p.$$

Plugging in $\underline{d}_{p-1} = -k_p \underline{\tilde{h}}_{p-1}$ we get

$$\underline{\tilde{r}}_{p-1}^T \, \underline{h}_{p-1} + k_p \left(-\underline{\tilde{r}}_{p-1}^T \, \underline{\tilde{h}}_{p-1} + r_0 \right) = r_p$$

or

$$\boxed{k_p = \frac{r_p - \underline{\tilde{r}}_{p-1}^T \, \underline{h}_{p-1}}{-\underline{r}_{p-1}^T \, \underline{h}_{p-1} + r_0}}$$

The two boxed formulas define the general Levinson-Durbin recursion.

Extensions

The L-D algorithm may be extended to other settings including

1. ℓ-step prediction: given $x[n-1], \ldots, x[n-p]$, predict $x[n+\ell]$
2. multiple predictions: given $x[n-1], \ldots, x[n-p]$, predict $x[n], \ldots, x[n+1]$.

Application: Linear Predictive Coding

Given: A signal $x[0], x[1], \ldots, x[N]$

Task: Compress signal (store with as few bits as possible while still providing an accurate representation.

Idea: Store the first value as is, and encode the prediction errors of the remaining values:

$$e[n] = x[n] - \hat{x}[n]$$

where

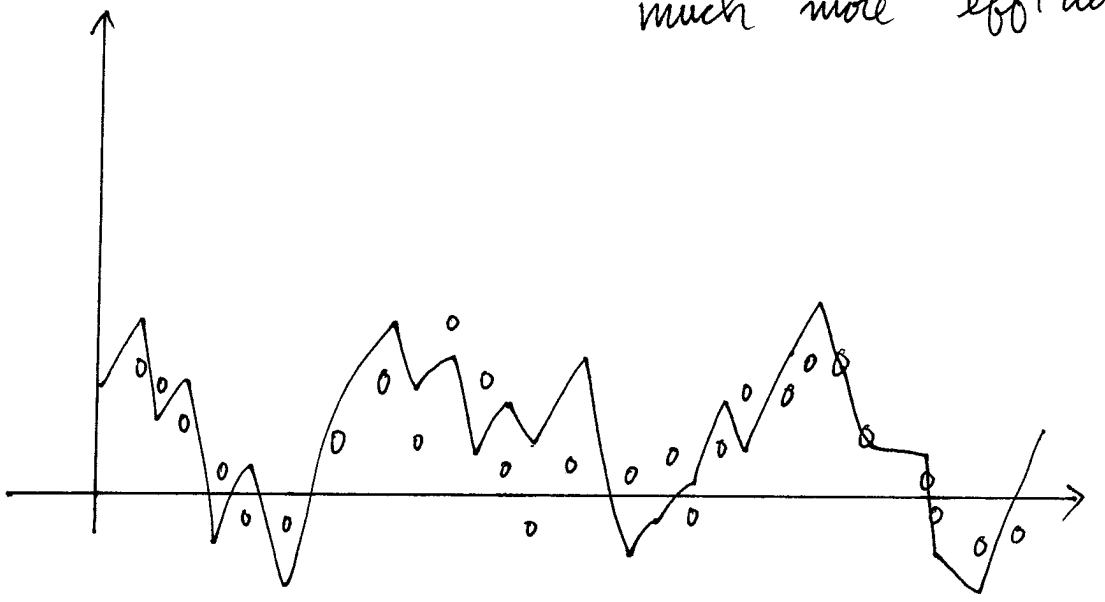
$$\hat{x}[n] = \sum_{k=1}^{n} h_n[k]\, x[n-k]$$

In other words, there are two equivalent representations

$$x[0],\ x[1],\ \ldots,\ x[N]$$

$$x[0],\ e[1],\ \ldots,\ e[N]$$

but the prediction errors have much smaller variance and can therefore be encoded much more efficiently.



LPC is a key ingredient in modern speech processing applications such as compression and synthesis. Since speech is not a stationary process, speech signals are encoded in short blocks and signal characteristics are updated for each block (otherwise the prediction errors would get really big).

Summary

- Linear prediction: application of LMMSE theory
- WSS process $\Rightarrow$ Toeplitz auto-covariance matrix
- Toeplitz ACV matrix $\Rightarrow$ Levinson-Durbin algorithm for fast matrix inversion and predictor coefficient updates.
- Important application: LPC, widely used in speech modeling.

Key a.

$$E\{(x[n]-\hat{x}[n])^2\}$$

$$= E\{(x[n]-\underline{h}_p^T\underline{x})(x[n]-\underline{h}_p^T\underline{x})\}$$

$$= E\{(x[n]-\underline{h}_p^T\underline{x})\cdot x[n]\}$$

$$= E\{x[n]^2\} - \underline{r}_p^T R_p^{-1} E\{\underline{x}\,x[n]\}$$

$$= r_0 - \underline{r}_p^T R_p^{-1}\underline{r}_p$$

by orthogonality principle