Array Processing

Suppose we want to determine the direction of propagation, or bearing, of a signal.

**Directional sensor**

- expensive
- one direction at a time

**Array of omnidirectional sensors**

- cheap
- multiple directions/sources at a time
- requires signal processing
Assumptions on the signal source:

- **near-field** → spherical waves at array
- **far-field** → plane waves at array

 prácticamente planar
We will study the following scenario

- plane waves (far-field sources)
- linear array of equally spaced array
- sinusoidal signals
- white noise at sensors

The Picture

\[ \theta \]

\[ d \]

\[ 0 \quad 1 \quad \ldots \quad N-1 \]

\[ N \] sensors

\[ d = \text{intersensor distance} \]

\[ \text{origin} = \text{centroid of sensors (not a sensor location if } N \text{ is even)} \]

\[ c = \text{speed of plane wave} \]
Exercise 1 Suppose the signal is $s(t)$ at the origin. Express the signal at sensor number $n$ in terms of $d, c, n$, and $\theta$.

Solution

\[
time = \frac{\text{distance}}{\text{speed}} = \frac{d}{c} \left( n - \frac{N-1}{2} \right) \cos \theta
\]

\[
s^{(n)}(t) = s \left( t + \frac{d}{c} \left[ n - \frac{N-1}{2} \right] \cos \theta \right)
\]
Now assume \( s(t) \) is a complex sinusoid:

\[
s(t) = A \exp \left\{ i \left( 2\pi f_c t + \phi_c \right) \right\}
\]

where

\[
f_c = \text{carrier frequency}
\]

\[
\phi_c = \text{carrier phase}
\]

By the previous exercise, at the \( n^{th} \) sensor,

\[
s^{(n)}(t) = A \exp \left\{ i \left( 2\pi f_c t + 2\pi \left( \frac{f_c}{c} \right) \left[ n - \frac{N-1}{2} \right] \cos \theta + \phi_c \right) \right\}
\]

\[
= A \exp \left\{ i \left( 2\pi f_c t + 2\pi \frac{c}{\lambda} \left[ n - \frac{N-1}{2} \right] \cos \theta + \phi_c \right) \right\}
\]

\[\uparrow \quad \text{since } c = \frac{\lambda}{f_c}\]

\underline{The Big Idea}

View \( t \) as fixed and \( n \) as a variable. What kind of "spatial" signal do we observe?
\[ s^{(d)}[n] = A \cdot \exp \left\{ i \left( 2\pi f_n + \phi \right) \right\} \]

where

\[ f = \frac{d}{\lambda} \cos \theta \]

\[ \phi = 2\pi f_c t + 2\pi \frac{d}{\lambda} \cdot \frac{N-1}{2} \cdot \cos \theta + \phi_c \]

**Conclusion**: The spatial signal is again a complex sinusoid.

Assuming complex WGN \( z[n] \) corrupts the sensor measurement at sensor \( n \), we observe

\[ x[n] = A \exp \left\{ i \left( 2\pi f_n + \phi \right) \right\} + z[n] \]

\[ \Rightarrow \] we can use \( x[0], x[1], \ldots, x[N-1] \) as our data to compute a frequency estimate \( \hat{f} \) using the techniques we just finished studying

\[ \Rightarrow \hat{\theta} = \cos^{-1} \left[ \frac{A}{d} \hat{f} \right] \]
Multiple Signals

A major advantage of arrays of omnidirectional sensors over directional sensors is the ability to estimate the bearing of multiple signals at the same time.

Suppose there are $p$ sinusoids, each with the same carrier frequency $f_c$.

Then our spatial signal is

$$x[n] = \sum_{j=1}^{p} A_j \exp\{i(2\pi f_j n + \phi_j)\} + z[n]$$

$$\Rightarrow \hat{f}_j, \ldots, \hat{f}_p \text{ using sinusoidal frequency estimation}$$

$$\Rightarrow \hat{\theta}_j = \cos^{-1}\left[\frac{1}{\Delta} \hat{f}_j\right]$$

**Question:** Why do we need to assume each sinusoid has the same carrier frequency?
Answer: Recall \( \lambda = \frac{c}{f_c} \). If the carrier-frequencies are different, we don’t know what \( \lambda \) to use when relating \( \hat{f}_j \) to \( \hat{\theta}_j \).

High Resolution Frequency Estimation

We need it for 3 reasons:

- angles / bearings may be very close
- array size may be constrained by space or cost (small \( N \))
- \( f_j = \frac{d}{\lambda} \cos \theta_j \in \left[-\frac{d}{\lambda}, \frac{d}{\lambda}\right] \), so if \( \frac{d}{\lambda} \) is small, the frequencies are very bunched together.
  (if \( \lambda \) is large, physical constraints may prevent \( d \) from being large enough)
Comments

- If $\Phi \frac{d}{\lambda} > \frac{1}{2}$, aliasing can occur.

  In particular, $\theta \in [0, \pi)$

  $\Rightarrow f \in \left[ -\frac{d}{\lambda}, \frac{d}{\lambda} \right]

  $\text{which is larger than } [-\frac{1}{2}, \frac{1}{2}]$

- Hence, $\frac{d}{\lambda} = \frac{1}{2}$ is ideal.

- A linear array cannot distinguish one side of the array from the other:

  $$\cos \theta = \cos (-\theta)$$

  $\Rightarrow$ need multiple arrays in such cases
Array Processing

... is a broad area

→ nonlinear arrays
→ nonuniform sensor spacing
→ location estimation
→ target tracking
→ 3D (we discussed 2D)

Spectral estimation bears on these problems too!

References:

D. Johnson and D. Dudgeon, Array Signal Processing, Prentice Hall, 1993
