**SIGNAL SPACES**

We will view signals as points in certain mathematical spaces. The spaces have common structure, so it will be useful to study them in the abstract.

**Metric Spaces**

A *metric space* is a set $X$ together with a *metric (distance function)* $d: X \times X \rightarrow \mathbb{R}$ such that for all $x, y, z \in X$:

- **M1** $d(x, y) = d(y, x)$
- **M2** $d(x, y) \geq 0$
- **M3** $d(x, y) = 0 \iff x = y$
- **M4** $d(x, z) \leq d(x, y) + d(y, z) \quad \text{triangle inequality}$

**Example** $\ell_p$ space

$X = \mathbb{R}^n$

$$d_p(x, y) = \begin{cases} \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}, & 1 \leq p < \infty \\ \max_{i=1,\ldots,n} |x_i - y_i|, & p = \infty \end{cases}$$
Each dp is different and thus defines a different metric space. \( p = 2 \Leftrightarrow \) Euclidean space.

The \( \Delta \) ineq. follows from Minkowski's ineq. for \( p < \infty \).

Picture for \( n = 2, \ p = 2 \):

\[
\begin{align*}
A & \leq B + C \\
\text{Example} & : \quad X = C^d[a, b] \\
\end{align*}
\]

\[
d_{\infty}(x, y) := \max_{t \in [a, b]} |x(t) - y(t)|
\]
Let $K$ be a collection of scalars, $K = \mathbb{R}$ or $\mathbb{C}$.

Let $V$ be a set equipped with two operations:

1. Vector addition: $x + y$
2. Scalar multiplication: $a \cdot x$

We say $V$ is a vector space over $K$ if,

VS1  (a) Closure of addition:
\[ \forall x, y \in V, \quad x + y \in V \]

(b) Existence of additive identity:
\[ \exists 0 \in V \text{ s.t. } \forall x \in V, \quad x + 0 = 0 + x = x. \]

(c) Existence of additive inverse:
\[ \forall x \in V, \quad \exists y \in V \text{ s.t. } x + y = 0. \]

Notation: $y = -x$

(d) Associativity of addition:
\[ \forall x, y, z \in V, \quad (x + y) + z = x + (y + z) \]

(e) Commutativity of addition:
\[ \forall x, y \in V, \quad x + y = y + x. \]
VS2 Properties of scalar multiplication: For all \( x, y \in V, \ a, b \in K \):

1. \( a \cdot x \in V \)
2. \( a (b \cdot x) = (ab) \cdot x \)
3. \( (a + b) \cdot x = a \cdot x + b \cdot x \)
4. \( a (x + y) = a \cdot x + a \cdot y \)
5. \( 1 \cdot x = x \)

Example: \( \mathbb{R}^n \) over \( \mathbb{R} \):

\[
x + y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}
\]

\[
a \cdot x = a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix}
\]

Other examples:

\( \mathbb{C}^n / \mathbb{C} \)
\( \mathbb{C}^n / \mathbb{R} \)
\( \text{not } \mathbb{R}^n / \mathbb{C} \) (scalar mult. not closed)
Example 1 \[ V = C[a, b], \quad K = \mathbb{R} \]

\[ f + g \mapsto h, \quad h(t) := f(t) + g(t) \]

\[ a \cdot f \mapsto h, \quad h(t) := a \cdot f(t) \]

The elements of a vector space are called vectors. Thus, functions can be vectors. This is a key concept!

**Terminology** \( V \subseteq \mathbb{R} \) = linear space = linear vector space

**Normed Vector Spaces**

Let \( V \) be a vector space over \( K \), \( K = \mathbb{R} \) or \( \mathbb{C} \).

A norm is a function \( \| \cdot \| : V \rightarrow \mathbb{R} \) such that

\[ N1 \quad \| x \| \geq 0 \quad \forall x \in V \]

\[ N2 \quad \| x \| = 0 \iff x = 0 \]

\[ N3 \quad \| a \cdot x \| = |a| \cdot \| x \| \quad \forall x \in V, \quad a \in K \]

\[ N4 \quad \| x + y \| \leq \| x \| + \| y \| \quad \forall x, y \in V \]  \( \text{triangle inequality} \)
A vector space together with a norm is called a **normed vector space**, or **normed linear space**.

**Example** | $l_p$ norm on $V = \mathbb{R}^n$ over $\mathbb{R}$

$$||x||_p := \begin{cases} \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}, & 1 \leq p < \infty \\ \max_{i=1,...,n} |x_i|, & p = \infty \end{cases}$$

Different $l_p$ norms induce different geometries. Consider the unit ball in $\mathbb{R}^2$, $\{x : ||x||_p = 1\}$

- $p = 1$
- $p = 2$
- $p = 4$
- $p = \infty$

**Note** | A normed V.S. is a metric space, with induced metric

$$d(x,y) = ||x-y||.$$ 

**Δ ineq:**

$$||x-z|| = ||x-y+y-z|| \leq ||x-y|| + ||y-z||$$

$$d(x,y) \leq d(x,y) + d(y,z)$$
Inner Product Spaces

Let $V$ be a vector space over $K$, $K = \mathbb{R}$ or $\mathbb{C}$.

An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow K$ such that for all $x, y, z \in V$, $a \in K$,

IP1 $\langle x, y \rangle = \langle y, x \rangle$

IP2 $\langle a \cdot x, y \rangle = a \cdot \langle x, y \rangle$

IP3 $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

IP4 $\langle x, x \rangle > 0$ with equality if and only if $x = 0$.

A vector space together with an inner product is called an inner product space.

Example: Complex Euclidean space, $V = \mathbb{C}^n$, $K = \mathbb{C}$

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \bar{y}_i$$

Example: $V = \{ f : [0, \infty) \rightarrow \mathbb{C} \}$, $K = \mathbb{C}$,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt$$
An IPS is a NVS with induced norm

\[ \| x \| = \sqrt{\langle x, x \rangle}. \]

Proof of the Δ inequality relies on the Cauchy-Schwarz inequality, which states that for any inner product and any \( x, y \in V \),

\[ |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}. \]

IP spaces have many more important properties that we will study in detail later.
Completeness

Let \((X,d)\) be a metric space.

A sequence \(x_1, x_2, \ldots\) in \(X\) converges if there exists \(x \in X\) such that, \(\forall \varepsilon > 0\), \(\exists N\) such that \(n > N \Rightarrow d(x_n, x) < \varepsilon\).

Example \(\quad X = \mathbb{R}, \quad d(x,y) = |x-y|, \quad x_n = \frac{1}{n} \rightarrow 0 \) (choose \(N = \frac{1}{\varepsilon}\)).

A sequence \(x_1, x_2, \ldots\) is called a Cauchy sequence if \(\forall \varepsilon > 0\), \(\exists N\) such that \(m, n > N \Rightarrow d(x_m, x_n) < \varepsilon\).

It can be shown that every convergent sequence is Cauchy. The converse, however, is false.

Example \(\quad\) Consider \(X = C[0,1]\), the real valued continuous functions on \([0,1]\), and \(d(f,g) = \left( \int_0^1 f(t)g(t) \, dt \right)^{1/2}\).

Consider the sequence \(f_n(t) = \begin{cases} n^2, & 0 \leq t \leq \frac{1}{n} \\ 1, & \frac{1}{n} < t \leq 1 \end{cases}\).

Then the limit function \(f(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \end{cases}\) is discontinuous, hence no limit in \(X\) exists. Yet \(\{f_n\}\) is Cauchy.
Example 1. \( \mathbb{Q} \) = \{ fractional numbers \}, \( d(x, y) = |x - y| \).

3, 3.1, 3.14, 3.141, 3.1415, ... \( \rightarrow \pi \not\in \mathbb{Q} \),
yet sequence is Cauchy.

A complete metric space is one for which every Cauchy sequence converges.

A complete normed VS is called a Banach space.

Example 1. \( p \geq 1 \)

\[ L^p(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{C} \mid \| f \|_p < \infty \} \]

where \( \| f \|_p = \begin{cases} \left( \int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p}, & 1 \leq p < \infty \\ \sup_{t \in \mathbb{R}} |f(t)|, & p = \infty \end{cases} \)

is complete.
A complete IPS is called a Hilbert space. Hilbert spaces will be extremely important in this course.

**Example 1** \( l^2 \) space

\[
V = \{ x \in \mathbb{C}^n : \sum |x_i|^2 < \infty \}
\]

\[
\langle x, y \rangle = \sum x_i \overline{y_i}
\]

is a Hilbert space, even when \( n = \infty \).

**Example 2** \( L^2(\mathbb{R}) \)

\[
V = \{ f: \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} f(t)^2 \, dt < \infty \}
\]

\[
\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt
\]

is a Hilbert space.
CT FOURIER TRANSFORM

Fourier Transform in \( L^1(\mathbb{R}) \)

Let \( f: \mathbb{R} \rightarrow \mathbb{C} \). The continuous time Fourier transform of \( f \) is a function \( \hat{f}: \mathbb{R} \rightarrow \mathbb{C} \) defined by

\[
\hat{f}(\omega) := \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt
\]

**Example**

If \( f(t) = 1_{[-a,a]}(t) := \begin{cases} 1 & \text{if } -a \leq t \leq a \\ 0 & \text{else} \end{cases} \)

then

\[
\hat{f}(\omega) = \int_{-a}^{a} e^{i\omega t} \, dt = \frac{e^{i\omega a} - e^{-i\omega a}}{i\omega} = \frac{2\sin \omega a}{\omega}
\]

using the formula \( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \).

When is \( \hat{f} \) well-defined, i.e. when does the integral converge?

If \( f \in L^1(\mathbb{R}) \), then

\[
|\hat{f}(\omega)| = \left| \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \right| \leq \int_{-\infty}^{\infty} |f(t)| e^{-i\omega t} |dt|
\]

\[
= \int_{-\infty}^{\infty} |f(t)| \, dt < \infty
\]

Furthermore, if \( f \in L^1(\mathbb{R}) \), then \( \hat{f}(\omega) \) is continuous (exercise).
\[ \int |g| < \infty \Rightarrow \int g \text{ exists} \]

Just like:

\[ \sum |a_n| < \infty \]

\[ \Rightarrow \sum a_n \text{ exists} \]
Inverse Fourier Transform

When can we recover \( f \) from \( \hat{f} \)?

**Theorem**  \( \hat{f} \in L'(\mathbb{R}) \) and \( f \in L'(\mathbb{R}) \), then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwt} \, dw$$

**Why do we need \( \hat{f} \in L'(\mathbb{R}) \)?**

**Proof**  See Mallat Ch 2.

**Interpretation:** \( f \) = superposition of sinusoids, \( \hat{f}(w) \) = frequency content of \( f \) at \( w \).

**Conclusion:**  \( f(t) \xrightarrow{\text{CTFT}} \hat{f}(w) \xrightarrow{\text{CTFT}} 2\pi f(-t) \)

Therefore we often speak of FT pairs

Unfortunately, for many signals of interest, either \( f \notin L'(\mathbb{R}) \) or \( \hat{f} \notin L'(\mathbb{R}) \). In most signal processing books/courses, it is simply assumed that the forward and inverse formulas hold, with proof. While a comprehensive treatment of Fourier transforms is beyond the scope of the course, we will see how to extend the FT from \( L'(\mathbb{R}) \) to \( L^2(\mathbb{R}) \), the space of finite energy signals.
The following result has broad implications in the context of LTI systems, which we will discuss later.

**Theorem**  
If \( f, h \in L'(\mathbb{R}) \), and \( g = \hat{h} * f \), where

\[
g(t) = (h * f)(t) = \int_{-\infty}^{\infty} h(u) f(t-u) \, du,
\]

then \( g \in L'(\mathbb{R}) \) and \( \hat{g}(w) = \hat{h}(w) \hat{f}(w) \).

**Proof**

\[
\hat{g}(w) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(u) f(t-u) \, du \right) e^{-iwt} \, dt
\]

\[= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(u) f(t-u) \, du \right) e^{-iwt} \, dt
\]

\[= \int_{-\infty}^{\infty} h(u) e^{-iwt} \, du \cdot \left( \int_{-\infty}^{\infty} f(t) e^{-iwt} \, dt \right)
\]

\[= \hat{h}(w) \cdot \hat{f}(w).
\]
Implications

1. Frequency-domain filtering

\[ f(t) \xrightarrow{\text{CTFT}} \hat{f}(\omega) \]

\[ \hat{f}(\omega) \xrightarrow{\hat{h}(\omega)} \hat{h}(\omega) \hat{f}(\omega) \]

\[ \mathcal{L}\{f(t)^2\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(\omega) \hat{f}(\omega) e^{i\omega t} d\omega \]

\[ \hat{h}(\omega) \text{ is called the transfer function of } \mathcal{L}. \]
2. **Eigenfunctions**

\[ L \{ e^{i \omega t} \} = \int_{-\infty}^{\infty} h(u) e^{i \omega (t-u)} \, du \]

\[ = e^{i \omega t} \int_{-\infty}^{\infty} h(u) e^{-i \omega u} \, du \]

\[ = \hat{h}(\omega) e^{i \omega t} \]

\[ \text{eigenvalue} = \text{factor by which frequency } \omega \text{ is amplified/attenuated.} \]

3. **Causality**: A system is causal if \( L \{ f \} \) at time \( t \) does not depend on \( f(u), u > t \).

Since

\[ L \{ f(t) \} = \int_{-\infty}^{\infty} h(u) f(t-u) \, du, \]

an LTI system is causal provided \( h(u) = 0 \) for \( u < 0 \).

4. **Stability**: A system is bound if \( L \{ f \} \) is bounded if \( f \in L^1(\mathbb{R}) \) whenever \( f \) is bounded. A necessary and sufficient condition is

\[ \int |h(u)| \, du < \infty \]

i.e., \( h \in L'^1(\mathbb{R}) \). (see Mallat)
Consider the function
\[ f(t) = \frac{\sin t}{t} = \text{sinc} (t) \]

Unfortunately, \( f(t) \notin L^1(\mathbb{R}) \). However,
\[ \int_{-\infty}^{\infty} |f(t)|^2 \, dt \leq 2 \left( 1 + \int_{1}^{\infty} \frac{1}{t^2} \, dt \right) < \infty \]
thus \( f \in L^2(\mathbb{R}) \).

**Theorem** If \( f \) and \( h \) belong to \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \),
then
\[ \int f(t) \overline{h(t)} \, dt = \frac{1}{2\pi} \int \hat{f}(w) \overline{\hat{h}(w)} \, dw \quad \text{(Parseval)} \]

Therefore, if \( f = h \),
\[ \int |f(t)|^2 \, dt = \frac{1}{2\pi} \int |\hat{f}(w)|^2 \, dw \quad \text{(Plancherel)} \]

**Proof** Set \( g = f^* h^* \) where \( h^*(t) = \overline{h(-t)} \).

Then
\[ \int f(t) \overline{h(t)} \, dt = \int f(t) h^*(-t) \, dt = g(0) \]
\[ = \frac{1}{2\pi} \int \hat{g}(w) \, dw = \frac{1}{2\pi} \int \hat{f}(w) \cdot \overline{\hat{h}(w)} \, dw \]
\[ = \frac{1}{2\pi} \int \hat{f}(w) \cdot \overline{\hat{h}(w)} \, dw \]
Next time
swap the names,
since the second is
more often called Perseus
(is Mallat mistaken?)
Now suppose $f \in L^2(\mathbb{R})$ but $f \notin L'(\mathbb{R})$.

Fact: $L'(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, which means

$\exists$ a sequence $f_n, f_{n+1}, ... \in L'(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\lim_{n \to \infty} \| f_n - f \|_2 = 0.$$  

Since $f_n \in L'(\mathbb{R})$, $\hat{f}_n$ is well-defined.

By Plancherel, for any $m,n$,

$$\| \hat{f}_n - \hat{f}_m \|_2 = \sqrt{2\pi} \| f_n - f_m \|_2.$$  

Since $\{ f_n \}$ converges, it is a Cauchy sequence. Therefore $\{ \hat{f}_n \}$ is a Cauchy sequence. Since $L^2(\mathbb{R})$ is complete, $\exists$ a function, call it $\hat{f}$, such that

$$\lim_{n \to \infty} \| \hat{f}_n - \hat{f} \|_2 = 0.$$  

We define $\hat{f}$ to be the CTFT of $f$.

It can be shown that $\hat{f}$ satisfies the convolution, Plancherel, and Parseval formulas, as well as the other basic properties of the CTFT.
Uniqueness?
Properties? (panel, convolution, etc.)
Example

\[ f(t) = \frac{\sin \frac{\pi t}{5}}{\pi t}, \quad 5 > 0 \]

\[ f_n(t) = \begin{cases} f(t), & 1 + 1 \leq n \\ 0, & \text{else} \end{cases} \]

\[ \Rightarrow \hat{f}(\omega) = 1_{[-5, 5]}(\omega) \]

Diracs

The Dirac \( \delta_{\frac{\pi}{5}}(t) \) is defined by the property

\[ \int f(t) \delta_{\frac{\pi}{5}}(t - \tau) \, dt = f(\tau). \]

Thus we define

\[ \hat{\delta}_{\frac{\pi}{5}}(\omega) = e^{-i\omega \frac{\pi}{5}} \]

Aside

By analogy with the inversion formula, it is common to define the CTFT of \( e^{i\frac{\pi}{5}t} \) to be

\[ 2\pi \delta_{\frac{\pi}{5}}(\omega) \]
The interpretation of $S_\omega(t)$ is as a pure spike at $w = \omega$. Since $e^{i\omega t}$ is a pure (complex) sinusoid with frequency $\omega$, this is consistent with the interp. of $\hat{f}(w)$ as the frequency content at $w$.

Note however that $e^{i\omega t}$ is in neither $L^1(\mathbb{R})$ nor $L^2(\mathbb{R})$, nor is it bounded! Fortunately, we won't formally require to compute its Fourier transform.

**Impulse Trains**

In sampling theory it is convenient to use impulse trains, or Dirac combs,

$$c_T(t) = \sum_{n=-\infty}^{\infty} S(t - nT).$$

By linearity of the CTFT,

$$\hat{c}_T(w) = \sum_{n=-\infty}^{\infty} e^{-i\omega nT}$$

**Theorem** (Poisson formula)

$$\hat{c}_T(w) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$
The equality in this theorem means that for any function $\hat{\phi}(\omega)$,

$$
\int_{-\infty}^{\infty} \hat{\phi}(\omega) \cdot \left( \sum_{n=-\infty}^{\infty} e^{-i\omega nT} \right) d\omega = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \hat{\phi} \left( \frac{2\pi k}{T} \right).
$$

**Proof**: Mallet, Ch. 2.
A CT signal is a function $f : \mathbb{R} \rightarrow \mathbb{C}$

A CT system is a function $L$ mapping CT signals to CT signals, e.g.,

$$L\left\{ f(t) \right\} = 2f(t) + f(t-1) + t^2$$

In many SP applications, systems are also linear:

$$L\left\{ af(t) + bg(t) \right\} = aL\left\{ f(t) \right\} + bL\left\{ g(t) \right\}$$

and time invariant:

if $L\left\{ f(t) \right\} = g(t)$

then $L\left\{ f(t-\tau) \right\} = g(t-\tau)$

**Impulse Response**

Suppose $L$ is LTI, and define the impulse response of $L$

$$h(t) = L\left\{ s(t) \right\}.$$
We will show that \( h(t) \) characterized \( L \) completely. If \( f(t) \) is any CT signal, then

\[
f(t) = \int_{-\infty}^{\infty} f(u) \delta(t-u) \, du
\]

Then

\[
L \{ f(t) \} = \int_{-\infty}^{\infty} f(u) L \{ \delta(t-u) \} \, du \quad \text{(by linearity)}
\]

\[
= \int_{-\infty}^{\infty} f(u) h(t-u) \, du \quad \text{(by time-invariance)}
\]

\[
= \int_{-\infty}^{\infty} h(u) f(t-u) \, du
\]

\[
= (h \ast f)(t)
\]

\* By viewing the integral as the limiting value of a sequence of \( \text{"Riemann sums,"} \sum_{i} f(u_{i}) L \{ \delta(t-u_{i}) \} \Delta u_{i} \).\n
Technically, an additional continuity assumption of \( L \) is needed to justify passing the limit through \( L \).
When can we recover $f(t)$ from discrete time samples $f(nT),\ n \in \mathbb{Z}$?

$T > 0$ is called the sampling interval.

Define

$$f_d(t) = \sum_{n=-\infty}^{\infty} f(nT) \delta(t-nT).$$

How is $f_d$ related to $f$?

**Proposition**

$$\hat{f}_d(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}(\omega - \frac{2\pi n}{T}).$$
Proof

\[ f_d(t) = f(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = f(t) \quad \text{impulse train} = c_T(t) \]

\[ \Rightarrow \hat{f}_d(\omega) = \frac{1}{2\pi} \hat{f}(\omega) \ast \hat{c}_T(\omega) \]

\[ = \frac{1}{2\pi} \hat{f}(\omega) \ast \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}(\omega - \frac{2\pi k}{T}) \]

Now suppose \( \hat{f}(\omega) = 0 \) for \( |\omega| > \frac{\pi}{T} \).

Then we can recover \( \hat{f}(\omega) \) with an ideal low pass filter \( \hat{f}_c(\omega) \)

\[ = T \cdot \mathbb{1}_{-\frac{\pi}{T}, \frac{\pi}{T}}(\omega) = \begin{cases} T, & |\omega| \leq \frac{\pi}{T} \\ 0, & \text{else} \end{cases} \]
In the time domain we have

\[ f(t) = f_d(t) * h_T(t) \]

\[ = \sum_{k=-\infty}^{\infty} f(nT) \cdot h_T(t-nT) \]

where

\[ h_T(t) = \frac{\sinh (\pi t / T)}{\pi t / T} \]

Thus we have proved:

**Theorem 1 (Whittaker sampling theorem).**

If \( | \hat{f}(\omega) | = 0 \) for \( |\omega| > \frac{\pi}{T} \), then

\[ f(t) = \sum_{k=-\infty}^{\infty} f(nT) \cdot h_T(t-nT) \]

\[ \text{The Nyquist Rate} \]

Suppose a signal has maximum frequency \( \omega_0 \). At what rate does it need to be sampled for perfect reconstruction?

\[ \omega_0 \leq \frac{\pi}{T} \iff \frac{1}{T} \geq \frac{\omega_0}{\pi} \]

↑ Sampling rate
It is also common to express frequency in terms of cycles per second,

\[ 2\pi f = \omega \]

\[ \uparrow \]

\[ \text{cycles/sec} \]

\[ \text{radians/sec} \]

\[ \text{cycles/sec (Hz)} \]

Then we require

\[ 2\pi f_0 \leq \frac{\pi}{T} \iff \frac{1}{T} \geq 2f_0 \]

which says we need to sample at least twice the maximum frequency (in Hz). Then \( 2f_0 \) is called the Nyquist rate.

**Aliasing**

What happens if we sample below the Nyquist rate?

? \( \not= f(t) \)!

\[ \uparrow \]

\[ \not= \hat{f}(w) \]

\[ \frac{\pi}{T} \]

\[ -\frac{\pi}{T} \]

\[ \frac{\pi}{T} \]
When it is not possible to sample at the Nyquist rate, an anti-aliasing filter may be applied before sampling.

\[ f(t) \rightarrow h(t) \rightarrow \tilde{f}(t) \]

\[ \tilde{h}(\omega) = 1_{[-\omega_0, \omega_0]}(\omega), \]

where \( \omega_0 = \frac{\pi}{T} \).

Example \[ CDs \]

Human audio range: 20 kHz
CD sampling rate: 44.1 kHz

Why the surplus? In practice, ideal lowpass filter is impossible. Allows for a transition band.
DT FOURIER TRANSFORM

DT Signals

A DT (discrete-time) signal is a sequence

... \( f[-2], f[-1], f[0], f[1], f[2], \ldots \)

or \( \{ f[n] \}_{n \in \mathbb{Z}} \) for short.

Just like \( L^p \) spaces for CT signals, we have \( L^p \) spaces for DT signals:

\[ L^p(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{C} \mid \| f \|_p < \infty \} \]

with norm

\[ \| f \|_p := \left( \sum_{n \in \mathbb{Z}} |f[n]|^p \right)^{1/p}, \quad 1 \leq p < \infty \]

\[ \| f \|_\infty = \sup_{n \in \mathbb{Z}} |f[n]|, \quad p = \infty \]

DTFT

The DTFT of a DT signal is

\[ \hat{f}(\omega) := \sum_{n \in \mathbb{Z}} f[n] e^{-j\omega n} \]

Note that \( \hat{f} : \mathbb{R} \to \mathbb{C} \). If \( f \in L^1(\mathbb{Z}) \), then

\[ \sum_{n \in \mathbb{Z}} |f[n]| e^{-j\omega n} = \sum_{n} |f[n]| < \infty, \text{ and therefore } \hat{f}(\omega) \text{ converges.} \]
Since $e^{-i(\omega + 2\pi k)n} = e^{-i\omega n}$ for all $k$, $\hat{f}(\omega)$ is periodic with period $2\pi$.

Therefore we view $\hat{f}$ as a function on $[-\pi, \pi]$.

This leads us to define the spaces

$$L^p([a,b]) := \{ f: [a,b] \to \mathbb{C} \mid \int_a^b |f(t)|^p \, dt < \infty \}$$

when $p = 2$ we have the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, dt, \quad 1 \leq p < \infty$$

Fact: If $1 \leq p < q < \infty$, then $L^q([a,b]) \subseteq L^p([a,b])$

This is not true for $L^p(\mathbb{R})$.

**Inversion Formula**

**Theorem** If $\hat{f}(\omega) \in L^2([-\pi, \pi])$, then

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\omega n} \, d\omega = \langle \hat{f}(\omega), e^{i\omega n} \rangle$$

See Mallat, Ch 3.

Interpretation: $e^{i\omega n} = \text{discrete complex sinusoid with freq } \omega$, $\hat{f}(\omega) = \text{amount of that freq. component in } f[n]$.
Example 1: If \( \hat{f}(\omega) = 1_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\omega) \), \( 0 < \frac{\pi}{2} < \pi \), then

\[
\hat{f}[n] = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\omega n} \, d\omega
\]

\[
= \frac{\sin \frac{n}{2}}{\pi n}
\]

\[
\hat{f}[n] \quad \mapsto \quad f(n)
\]

Properties:

The DTFT also has a Parseval formula:

\[
\|f\|_2^2 = \sum |f[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 \, d\omega = \|\hat{f}\|_2^2
\]
Note that if we define the CT signal
\[ f(t) = \sum_{n=-\infty}^{\infty} f[n] s(t-n) \]
then
\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} f[n] s(t-n) \right) e^{-j\omega t} \, dt \]

\[ = \sum_{n=-\infty}^{\infty} f[n] e^{-j\omega n} \]

\[ = \text{DTFT of } \{f[n]\} \]

Therefore, the DTFT inherits all the usual properties of the CTFT.

**Theorem**  If \( f, h \in L^1(\mathbb{R}) \), then \( g = f \ast h \in L^1(\mathbb{R}) \),

where
\[ g[n] := \sum_{k=-\infty}^{\infty} f[k] h[n-k], \]

and
\[ \hat{g}(\omega) = \hat{f}(\omega) \cdot \hat{h}(\omega). \]
Define the \( \delta \) function (impulse):

\[
\delta[n] = \begin{cases} 
1 & n=0 \\
0 & n \neq 0
\end{cases}
\]

Suppose \( L \) is a DT, LTI system:

- **DT System**: map \( f: \mathbb{Z} \rightarrow \mathbb{C} \) to itself.

- **Linear, Time-Invariant**: same as CT, but with \( t \) restricted to \( \mathbb{Z} \).

The **impulse response** of \( L \) is

\[
h[n] = L\{\delta[n]\}
\]

If \( f[n] \) is any DT signal, then

\[
f[n] = \sum_{k=-\infty}^{\infty} f[k] \delta[n-k]
\]

\[
\Rightarrow L\{f[n]\} = \sum f[k] L\{\delta[n-k]\} = \sum f[k] h[n-k] = \mathcal{H} \{f[n]\}
\]
Therefore,

\[ L \{ f[n] \} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{H}(\omega) \hat{F}(\omega) e^{i\omega n} d\omega \]

\[ \uparrow \]

frequency domain filter

Causal system: \( h[n] = 0 \) for \( n < 0 \)

Stable system: \( h \in L^1(\mathbb{Z}) \), \( \Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \)

The design of discrete-time filters is taught in 4/51.
SAMPLING RATE CONVERSION

In many applications, especially audio, it is desirable to change the sampling rate. We could reconstruct an analog signal and resample, but in practice the approximate interpolation introduces errors. We would like to operate entirely in the DT domain.

\[
\begin{align*}
  x[n] & \quad \Downarrow M \quad \rightarrow \quad y[n], \quad M \geq 1 \quad \text{an integer} \\
  y[n] &= x[M\cdot n]
\end{align*}
\]

Is this system linear? time invariant?
How is $\hat{y}(\omega)$ related to $\hat{x}(\omega)$?

Consider the case $M=2$.

Define $r[m] = \frac{1}{2} \left( 1 + (-1)^m \right) = \begin{cases} 1 & \text{m even} \\ 0 & \text{m odd} \end{cases}$

Then

$$\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[2n] e^{-i\omega n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot r[m] e^{i(\frac{\omega}{2}) \cdot m}$$

$$= \frac{1}{2} \sum_{m=-\infty}^{\infty} x[m] e^{-i(\frac{\omega}{2}) m} + \frac{1}{2} \sum x[m] (-1)^m e^{-i(\frac{\omega}{2} - \pi) m}$$

$$= \frac{1}{2} \hat{x}(\frac{\omega}{2}) + \frac{1}{2} \sum x[m] e^{i(\frac{\omega}{2} - \pi) m}$$

$$= \frac{1}{2} \hat{x}(\frac{\omega}{2}) + \frac{1}{2} \hat{x}(\frac{\omega}{2} - \pi)$$

---

dilated spectrum
dilated, shifted spectrum
Consider now general \( M \). How can the previous argument be generalized?

Introduce
\[
\nu[m] = \begin{cases} 
1 & \text{if } m = Mn \\
0 & \text{else}
\end{cases} = \frac{1}{M} \sum_{k=0}^{M-1} e^{-i \frac{2\pi m k}{M}}
\]

Then
\[
\hat{y}(\omega) = \sum_n \chi[Mn] \cdot e^{-i \omega n}
= \sum_m \nu[m] \nu[m] e^{-i \frac{\omega}{M} m} \quad [m=Mn]
= \frac{1}{M} \sum_{k=0}^{M-1} \sum_m \nu[m] e^{-i \left( \frac{\omega}{M} - \frac{2\pi k}{M} \right) m}
= \frac{1}{M} \sum_{k=0}^{M-1} \hat{x} \left( \frac{\omega}{M} - \frac{2\pi k}{M} \right)
= \frac{1}{M} \sum \hat{x} \left( \frac{\omega - 2\pi k}{M} \right)
\]
$\hat{X}(\omega)$

$\hat{R}\left(\frac{\omega}{3}\right)$

$\hat{X}\left(\omega - \frac{2\pi}{3}\right)$

$\hat{X}\left(\omega - \frac{4\pi}{3}\right)$

$\hat{Y}(\omega)$
Suppose \( \hat{x}(\omega) = 0 \) for \( |\omega| > \frac{\pi}{m} \)

\[ M = 3 \]

\[ \Rightarrow \quad \text{on} \quad [-\pi, \pi], \]

\[ \hat{y}(\omega) = \frac{1}{M} \hat{x}\left(\frac{\omega}{m}\right) \]

When \( x \) is bandlimited to \( \left[ -\frac{\pi}{m}, \frac{\pi}{m} \right] \), how can we recover \( x \) from \( y \)? We need a way to compress the spectrum of \( y \).
Upsampling

\[ x[n] \rightarrow \uparrow M \rightarrow y[n] \]

\[ y[n] = \begin{cases} x[n/M] & \text{if } n \text{ is a multiple of } M \\ 0 & \text{else} \end{cases} \]

\[ \hat{y}(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n} \]

\[ = \sum_{m=-\infty}^{\infty} x[m] e^{-i\omega m M} \]

\[ = \hat{x}(\omega M) \]
So how can we recover \( x \) from a downsampled version?

Assume \( \hat{x}(w) = 0 \) for \( |w| > \frac{\pi}{M} \).

What is this saying in the time domain?
Consider the following system

\[ x[n] \rightarrow \uparrow M \rightarrow \hat{\mathcal{H}}_M(\omega) \rightarrow y[n] \]

where

\[ \hat{\mathcal{H}}_M(\omega) := M \cdot 1_{[-\frac{\pi}{M}, \frac{\pi}{M}]}(\omega). \]

The impulse response of \( \hat{\mathcal{H}}_M(\omega) \) is

\[ h_M[n] = \frac{\sin(\pi n/M)}{\pi n/M} \]

Therefore

\[ y[n] = \left( \sum_{k=-\infty}^{\infty} x[k] \ast s[n-kM] \right) \ast h_M[n] \]

\[ = \sum_{k=-\infty}^{\infty} x[k] \cdot h_M[n-kM] \]
Therefore, this stage interpolates the upsampled signal to fill in the zero values. Note that at multiples of $M$, the interpolation is exact, i.e. $y[n] = x[\frac{n}{M}]$, because

$$h[0] = \begin{cases} 1 \\ \sum_{k=-\infty}^{\infty} \end{cases}$$

$$h[n] = 0, \quad n = \pm M, \pm 2M, \pm 3M, \ldots$$

Now we can analyze the entire system:

**DT Sampling Theorem**

Suppose $\hat{x}(w)$ is bandlimited to $[-\frac{\pi}{M}, \frac{\pi}{M}]$.

Then

$$x[n] = \sum_{k=-\infty}^{\infty} x[Mk] \frac{\sin[\pi(n-kM)/M]}{\pi(n-kM)/M}$$

That is, $x[n]$ can be recovered after $M$-fold downsampling.

\[ x \rightarrow \downarrow M \rightarrow \uparrow M \rightarrow \hat{h}_M(w) \rightarrow x \]
Aliasing

In practice, we may not know that a signal is bandlimited. If $\hat{x}(\omega) \neq 0$ for $\frac{\pi}{m} < |\omega| < \pi$, then downsampling will result in aliasing (high frequencies get folded down to low frequencies).

Therefore, as in CT sampling, we will pre-filter with an anti-aliasing filter with passband $[-\frac{\pi}{m}, \frac{\pi}{m}]$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram}
\end{figure}
Changing the Sampling Rate by a Rational Factor

Our general goal is to change the sampling rate by a factor \( L/M \).

\[ \chi_c(t) \rightarrow \text{anti-aliasing filter if } T > T' \quad \chi_c(t) \]

\( L < M \)

\[ \chi[n] \rightarrow \text{DT processing} \rightarrow y[n] \]

\[ \text{sample with interval } T' = T \cdot M \]

We would like to do our processing in the discrete domain to avoid the imprecision associated with the 2-3 analog stages.

\[ T' < T \]

What does the picture look like for \( T' > T \)?
If \( T' > T \), need to
LPF to avoid aliasing

\[
M = 1
\]

\[
\kappa_c(t) = \sum_{k=-\infty}^{\infty} \kappa[k] \frac{\sin \left[ \frac{\pi}{T} (t - kT) / T \right]}{\pi \left( \frac{t - kT}{T} \right)}
\]

(from Whittaker–Shannon interpolation formula)

\( T' < T \), so no need to apply anti-aliasing filter

\[
\Rightarrow y_c(t) = \kappa_c(t)
\]

\[
\Rightarrow y[n] = y_c(nT/L)
\]

\[
= \sum_{k=-\infty}^{\infty} \kappa[k] \frac{\sin \left[ \frac{\pi}{L} (nT/L - kT) / (nT/L) \right]}{\pi \left( \frac{nT/L - kT}{nT/L} \right)}
\]

\[
= \sum_{k=-\infty}^{\infty} \kappa[k] \cdot \frac{\sin \left[ \frac{\pi}{L} (n - kL) / L \right]}{\pi \left( \frac{n - kL}{L} \right)}
\]
This is precisely the output of the system

\[ x[n] \xrightarrow{\uparrow L} \hat{h}_L(\omega) \rightarrow y[n] \]

where

\[ \hat{h}_L(\omega) = L \cdot 1_{[-\frac{\pi}{L}, \frac{\pi}{L}]}(\omega) \]

\[ L = 1 \]

Here the anti-aliasing filter is necessary

\[ x_c(t) \xrightarrow{\frac{-\pi}{T}} \rightarrow y_c(t) \rightarrow \]

Sample at 

\[ T' = nT_m \]

\[ y[n] = y_c(nT_m) \]
On the homework you will show that this process is equivalent to

\[ x[n] \rightarrow 1_{[-\frac{\pi}{M}, \frac{\pi}{M}]}(\omega) \rightarrow lM \rightarrow y[n] \]

which is a downsampler preceded by a discrete anti-aliasing filter.

**General Case**

**Decimator:**

\[ \rightarrow 1_{[-\frac{\pi}{M}, \frac{\pi}{M}]}(\omega) \rightarrow lM \rightarrow \rightarrow D_m \rightarrow \]

**Interpolator:**

\[ \rightarrow \uparrow L \rightarrow l \cdot 1_{[-\frac{\pi}{L}, \frac{\pi}{L}]}(\omega) \rightarrow \rightarrow I_L \rightarrow \]
Therefore

\[ T \rightarrow D_m \rightarrow T' = TM \rightarrow I_L \rightarrow T'' = T \frac{M}{L} \]

or

\[ T \rightarrow I_L \rightarrow T' = \frac{T}{L} \rightarrow D_m \rightarrow T'' = T \frac{M}{L} \]

change the sampling rate by a factor of \( \frac{M}{L} \).

Which order makes more sense? Does it depend on \( L, M \)?

Consider the first:

- \( D_m \rightarrow I_L \rightarrow T'' \)
Thus, regardless of whether $L > M$ or $L < M$, we retain more of the signal by applying the interpolator before the decimator.

In short, interpolation shrinks the spectrum, thereby reducing the impact of the decimator’s anti-aliasing filter.

As a bonus, we can combine the interpolation and anti-aliasing filters.

where

$$\tilde{\xi} = \min\left(\frac{\pi}{M}, \frac{\pi}{L}\right)$$
The complete process now looks like this:

```plaintext
if $L < M$

$x_c(t)$

$\frac{1}{\pi} \left[ \frac{1}{M} \right] \rightarrow y_c(t)$

reconstruct with interval $T$

$x[n] \rightarrow T\rightarrow P_m \rightarrow y[n]$

sample with interval $T = \frac{T}{L}$
```

On the homework you will formally show that both paths from $x[n]$ to $y[n]$ are the same. The rules are obviously the same, but in fact the signals are identical. E.g., could

![Practical Considerations](image)

Suppose we wish to play a sound using a card. We need to change the tone from 44100 kHz to 81 for some...
The conversion factor is
\[
\frac{44100}{8192} = \frac{11025}{2048} = \frac{m}{L}
\]

Do you see any problems with this system?

- Tiny passband \(\Rightarrow\) huge filter length
- Only 1 out of every 2048 interpolator outputs is nonzero
- Only 1 out of every 11025 decimator inputs is not discarded

Solution: change the rate in stages.

Polyphase filters can be used to change rates very efficiently
could show
example, e.g. from
Proclus + Munkian.
See also Ron's notes.
LINEAR ALGEBRA

Linear algebra is the study of vector space (linear spaces) and linear transformations.

Linear Combinations

Let $V$ be a vector space with scalars $K = \mathbb{R}$ or $\mathbb{C}$.

We say $v \in V$ is a linear combination of $v_1, \ldots, v_n \in V$ if $\exists a_1, \ldots, a_n \in K$ s.t. $v = \sum_{i=1}^{n} a_i v_i$.

Note that $n$ is finite.

Examples

(a) 

(b) $f(t) = 2t^3 - 3t^2 + 9t - 4$

$$f_i(t) = t^i, \quad i = 0, 1, 2, 3$$

$v_2$
Consider writing a handout that gives more precise statements of Lin Alg. and proofs about basis and span of basis in terms of vectors, set, span set, etc.
Linear Independence

Let \( U \subseteq V \). We say \( U \) is \( \text{linearly independent} \) if, for any \( n \), and any \( v_1, \ldots, v_n \in U \),
\[
\sum a_i v_i = 0 \implies a_i = 0, \quad i = 1, \ldots, n.
\]

If, on the other hand, \( \exists x_1, \ldots, x_n \in U \) and \( a_1, \ldots, a_n \in K \), not all 0, s.t.
\[
\sum a_i v_i = 0,
\]
we say \( U \) is \( \text{linearly dependent} \).

Examples
(a) \( V = \mathbb{R}^3 \), \( K = \mathbb{R} \), \( U = \left\{ \begin{bmatrix} u_{11} \\ u_{22} \\ u_{32} \end{bmatrix}, \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix}, \begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} \right\} \)

LI \( \iff \) non-coplanar

(b) \( f_1(t) = \cos(t) \), \( f_2(t) = \sin(t) \), \( f_3(t) = \cos(t + 2) \)

\[
f_3(t) = \cos(2) \cdot \cos(t) - \sin(2) \sin(t)
= a_1 f_1(t) + a_2 f_2(t)
\]
\( \implies \) LD
Span

Let \( U \subseteq V \). The span of \( U \) is the set of all LC's of vectors in \( U \), i.e.

\[
\text{span}(U) = \langle U \rangle := \left\{ \sum_{i=1}^{n} a_i v_i \mid v_1, \ldots, v_n \in U, a_1, \ldots, a_n \in \mathbb{K} \right\}
\]

Example

(a) \( U = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\} \)

\( \text{span}(U) = \text{x-y plane} \)

(b) \( U = \left\{ 1, t - 2, 2t^2 + 3t - 4, 3t^2 - 1 \right\} \)

\( \text{span}(U) = \left\{ \text{polynomials in } t \text{ of degree } \leq 2 \right\} \)

Basis

A (Hamel) basis of \( V \) is a subset \( U \) s.t. \( U \) is LI and \( V = \langle U \rangle \).
Example 1 (a) \( V = \mathbb{R}^n, \quad K = \mathbb{R} \)

\[ u_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T \]

\[ i \text{th position} \]

(b) \( V = \{ \text{all polynomials w/ complex coefficients} \}
\text{ of degree} \leq 5 \}, \quad K = \mathbb{C} \)

\[ U = \{ 1, t, t^2, t^3, t^4, t^5 \} \]

why are these LI?

fundamental thm of algebra: if \( \sum_{i=0}^{5} a_i t^i = p(t) \)

and some \( a_i \neq 0 \), then \( p(t) \) has 5 roots.
Include TFE

(i) $U$ is a basis
(ii) every $v$ has unique...
(iii) $U$ is a maximal LI set
(iv) $U$ is a minimal spanning set

Don't prove, provide reference.
Theorem: $U$ is a basis for $V$ $\iff$ every $v \in V$ has a unique representation (up to order)

$$v = \sum_{i=1}^{n} a_i u_i$$

where $u_i \in U$, $a_i \in K$, $a_i \neq 0$.

Proof:
- Existence: Since $V = \text{span}(U)$, and by definition of span,
- Uniqueness: Assume $\exists v \in V$ s.t.

$$v = \sum_{i=1}^{m} a_i u_i = \sum_{j=1}^{n} b_j v_j$$

where $u_i, v_j \in V$, $a_i, b_j \in K$, $a_i, b_i \neq 0$.

Then

$$a_1 u_1 + \ldots + a_m u_m + (-b_1) v_1 + \ldots + (-b_n) v_n = 0$$

Since $U$ is LI, this implies that (perhaps after reordering the $u_i$'s),

$$u_i = v_i \text{ and } a_i = b_i \quad \forall i$$

$\Rightarrow$ representation is unique.
**Theorem** I \[ \text{If } U_1 \text{ and } U_2 \text{ are bases for } V, \]
\[ \text{then } |U_1| = |U_2|. \]

Therefore, we can define the **dimension** of a vector space to be the cardinality of any basis, denoted \( \dim(V) \).

**Fact 1** Let \( V \) be a vector space

(a) If \( U_L \subseteq V \) is L.I., and \( U_S \subseteq V \) spans \( V \), then \( |U_L| \leq |U_S| \).

(b) If \( U \subseteq V \) is L.I., then \( U \) can be extended to a basis.

(c) If \( U \subseteq V \) spans \( V \), then \( U \) contains a basis.

(d) If \( \dim(V) = n < \infty \), then \( U \) is L.I., and \( |U| = n \), then \( U \) is a basis.

(e) If \( \dim(V) = n < \infty \), span \((U) = V\), and \( |U| = n \), then \( U \) is a basis.
Example \[ V = \mathbb{C}^N, \quad K = \mathbb{C} \]

standard basis \implies \text{dim}(V) = N.

Consider the "DFT basis," \( U = \{ u_0, \ldots, u_{N-1} \} \) where

\[
U_k = \begin{bmatrix}
1 \\
e^{\frac{i 2 \pi \cdot k}{N}} \\
\vdots \\
e^{\frac{i 2 \pi \cdot k \cdot (N-1)}{N}}
\end{bmatrix} = \begin{bmatrix}
U_k[0] \\
\vdots \\
U_k[N-1]
\end{bmatrix}
\]

We have seen that for any \( f = [f[0] \ldots f[N-1]]^T \in \mathbb{C}^N \),

\[
f[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{f}[k] e^{\frac{i 2 \pi \cdot kn}{N}}, \quad n = 0, \ldots, N-1.
\]

Expressing all of these scalar equations in a single vector equation we have

\[
f = \sum_{k=0}^{N} a_k \ u_k,
\]

where \( a_k = \frac{1}{2\pi} \hat{f}[k] \).

Thus, \( \text{span}(U) = V \). Since \( |U| = N \), we deduce that \( U \) is a basis.
A subspace is a set $S \subseteq V$ that is closed under both vector addition and scalar multiplication.

Note that a subspace is a vector space in its own right. In particular, it has a dimension.

**Example 1**  \[ V = \mathbb{R}^2, \ K = \mathbb{R} \]

(a) \[ S = \langle u \rangle \]

(b) In general, if $U \subseteq V$ is any set, and $S = \text{span}(U)$, then $S$ is a subspace.
(c) Band-limited signals

\[ V = \mathcal{L}^2(\mathbb{R}), \quad K = \mathbb{C} \]

\[ S = \left\{ f \in \mathcal{L}^2(\mathbb{R}) \mid \hat{f}(w) = 0 \text{ for } |w| > \omega_0 \right\} \]

If \( f_1, f_2 \in S \), \( a_1, a_2 \in \mathbb{C} \), and \( f = a_1 f_1 + a_2 f_2 \),

then

\[ \hat{f}(w) = a_1 \hat{f}_1(w) + a_2 \hat{f}_2(w) = 0 \]

for \( |w| > \omega_0 \).

---

Direct Sum

Let \( S, T \subseteq V \) be subspaces. We say \( V \) is the (inner) direct sum of \( S \) and \( T \), written

\[ V = S \oplus T \]

if \( \forall v \in V \), there is unique \( s \in S, t \in T \), s.t.

\[ v = s + t. \]
**Fact 1** \( V = S \oplus T \) iff

(i) \( \forall v \in V, \exists s \in S, t \in T \) s.t. \( v = s + t \)

(ii) \( S \cap T = \{0\} \)

\( T \) trivial subspace

If \( V = S \oplus T \), we say \( S \) and \( T \) are **complements**.

**Example**

\( V = \mathbb{R}^{n \times n} : = \{ n \times n \text{ matrices w/ real entries} \} \)

\( S = \{ A \in V \mid A^T = A \} \) (symmetric)

\( T = \{ A \in V \mid A^T = -A \} \) (skew-symmetric)

Are \( S + T \) subspace?

Let \( A \in \mathbb{R}^{n \times n} \). We can write

\[
A = \frac{1}{2} \left( A + A^T \right) + \frac{1}{2} \left( A - A^T \right)
\]

\[
= \frac{1}{2} \left( \begin{array}{cc}
S & S \\
T & T
\end{array} \right)
\]

This establishes (i).

Now suppose \( A \in S \cap T \). Then \( A^T = A \) and \( A^T = -A \)

\( \Rightarrow \) \( A = 0 \). This establishes (ii).

Thus, \( V = S \oplus T \).
If \( V = S \oplus T \), and 
\( \mathcal{U}_S, \mathcal{U}_T \) are bases for 
\( S \oplus T \), then 
\( \mathcal{U}_S \cup \mathcal{U}_T \) is a basis 
for \( V \). Conclude 
\[ \dim (V) = \dim (S) + \dim (T) \]
LINEAR TRANSFORMATIONS

Let $V, W$ be vector spaces with the same set of scalars $K = \mathbb{R}$ or $\mathbb{C}$.

A function $L : V \mapsto W$ is a linear transformation if, $\forall u, v \in V$, and $\forall a, b \in K$,

$$L(au + bv) = aL(u) + bL(v)$$

Why do we assume $V, W$ have same scalars?

Examples

(a) $V = W = L^2(\mathbb{R})$, $K = \mathbb{C}$

For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$:

$$L \{ af + bg \} (w) = \int_{-\infty}^{\infty} (af(t) + bg(t))e^{-iwt}dt$$

$$= a \int_{-\infty}^{\infty} f(t)e^{-iwt}dt + b \int_{-\infty}^{\infty} g(t)e^{-iwt}dt$$

$$= aL \{ f \} (w) + bL \{ g \} (w)$$

For $f, g \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$, use density argument.
(b) \( V = \mathbb{R}^2 \), \( W = \mathbb{R} \), \( K = \mathbb{R} \\
V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rightarrow w = 2v_1 - 3v_2 \\
Why \ is \ this \ linear?\)

(c) More generally, suppose \( V = K^m \), \( W = K \)

Consider the transformation

\[
L : \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \rightarrow c_1 v_1 + \ldots + c_m v_m = \sum_{i=1}^m c_i v_i
\]

where \( c_1, \ldots, c_m \in K \) are fixed. Then

\[
L (av + bw) = L \left( \begin{bmatrix} av_1 + bw_1 \\ \vdots \\ av_m + bw_m \end{bmatrix} \right)
\]

\[
= \sum c_i (av_i + bw_i)
\]

\[
= a \sum c_i v_i + b \sum c_i v_i
\]

\[
= a L (v) + b L (w)
\]

What if we added a constant term? Quadratic?
(d) \( V = K^m, \ W = K^m \)

\[
L : \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \rightarrow \begin{bmatrix} c_{11} v_1 + \cdots + c_{1m} v_m \\ \vdots \\ c_{m1} v_1 + \cdots + c_{mm} v_m \end{bmatrix}
\]

Why is this linear? Apply previous case to each row.

**Matrices**

The last example can be represented as a matrix:

\[
\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mm} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}
\]

So every matrix transformation is linear. When \( L \) maps one Euclidean space to another, the converse is true.

**Theorem**

If \( L : K^m \rightarrow K^n \) is linear, then \( L \) has a matrix representation.
Proof 1. Let \( \{e_1, \ldots, e_m\} \) be the standard basis of \( K^m \).

Let \( \nu \in K^m \) have the (unique) expansion

\[
\nu = \nu_1 e_1 + \cdots + \nu_m e_m
\]

\[
= \begin{bmatrix}
\nu_1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \cdots + \begin{bmatrix}
0 \\
\vdots \\
0 \\
\nu_m
\end{bmatrix} = \begin{bmatrix}
\nu_1 \\
\vdots \\
\nu_m
\end{bmatrix}
\]

Since \( L \) is linear,

\[
L(\nu) = \sum_{j=1}^{m} \nu_j L(e_j) = \nu
\]

This is a vector equation in \( K^n \).

Let's write it component-wise. Denote

\[
L(e_j) = \begin{bmatrix}
c_{1j} \\
\vdots \\
c_{nj}
\end{bmatrix}, \quad j = 1, \ldots, m
\]

Then

\[
L(\nu) = \begin{bmatrix}
w_1 \\
\vdots \\
w_n
\end{bmatrix} = \begin{bmatrix}
c_{11} & \cdots & c_{1m} \\
\vdots & \ddots & \vdots \\
c_{n1} & \cdots & c_{nm}
\end{bmatrix} \begin{bmatrix}
\nu_1 \\
\vdots \\
\nu_m
\end{bmatrix}
\]
Inverses

Let \( L : V \rightarrow W \) be linear. We say \( L \) is

• **injective** / **1-to-1** if
  \[ u \neq v \Rightarrow L(u) \neq L(v) \]

• **surjective** / **onto** if
  \[ \forall w \in W, \exists v \in V \text{ s.t. } L(v) = w. \]

• **bijective** if
  \( L \) is one-to-one and onto,

If \( L \) is bijective it has an **inverse**

\[ L^{-1} : W \rightarrow V \]

which is defined to be a function satisfying

\[ L^{-1}(L(v)) = v \quad \forall v \in V \quad \text{"left inverse"} \]

\[ L(L^{-1}(w)) = w \quad \forall w \in V \quad \text{"right inverse"} \]

In other words,

\[ L^{-1} \circ L = I_V \quad \text{identity transformation} \]

\[ L \circ L^{-1} = I_W \]
Examples

(a) CTFT on $L^2(\mathbb{R})$

$$L^{-1} \{ \hat{f} \} (t) = \frac{1}{2\pi} \int \hat{f}(\omega) e^{i\omega t} d\omega$$

for $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

(b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I$$

The problem is that $L$ is not injective

$$L \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$v_3$ could be anything, output doesn't change.
(c) \[
L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad L^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{if } ad-bc \neq 0.
\]

**Isomorphisms**

If \( L: V \rightarrow W \) is a bijection, we also call \( L \) an isomorphism, and say that \( V \) and \( W \) are isomorphic, denoted \( V \cong W \).

**Theorem**

If \( L: V \rightarrow W \) is an isomorphism, and \( B \) is a basis for \( V \), then \( L(B) = \{ L(b) \mid b \in B \} \) is a basis for \( V \).

**Proof:** Must show (i) \( L(B) \) is LI (ii) \( \text{span}(L(B)) = W \).

(i) Suppose \( \sum_{i=1}^{n} q_i L(b_i) = 0 \) in \( W \).

Apply \( L^{-1} \) to both sides. Since \( L^{-1} \) is linear (homework)

\[0 = L^{-1}(0) = \sum q_i b_i \text{ in } V \]

\[\Rightarrow q_i = 0 \text{ for all } b_i, \text{ since } B \text{ is LI.}\]

(ii) Let \( w \in W \), and let \( v \in V \) s.t. \( L(v) = w \).

Since \( V = \text{span}(B) \), we can write \( v = \sum q_i b_i \).

Then \( w = \sum q_i L(b_i) \).

**Corollary**

If \( V \cong W \), then \( \text{dim}(V) = \text{dim}(W) \).

**Corollary**

Only square matrices are invertible.
Subspaces Associated with a LT \( L: V \rightarrow W \), linear

- **range/image**

  \[
  R(L) = \{ w \in W \mid w = L(\mathbf{v}) \text{ for some } \mathbf{v} \in V \} \]

- **nullspace/kernel**

  \[
  N(L) = \{ \mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0} \} \]

  Why are these subspaces?

**Example**

\[
L = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}
\]

**range:**

\[
\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 6v_2 \\ v_1 + 3v_2 \end{bmatrix} = (v_1 + 3v_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

\( \Rightarrow R(L) = \text{span} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \)

**nullspace:**

\[
\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1 + 3v_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0
\]

\( \Rightarrow v_1 = -3v_2 \Rightarrow N(L) = \text{span} \left( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) \)

\[\begin{array}{c}
\text{Range} \quad R(L) \\
\text{Nullspace} \quad N(L)
\end{array}\]
If \( L \) is a matrix, \( L = [e_1, e_2, \ldots, e_m] \), then 
\[
Lv = v_1 \cdot e_1 + \cdots + v_m \cdot e_m .
\]
Therefore,
\[
R(L) = \text{span}(e_1, \ldots, e_m) = \text{colspan}(L).
\]
which is also called the column span of \( L \).

Therefore

- \( N(L) = \{ \mathbf{0} \} \iff e_1, \ldots, e_m \) are LI

- Given \( w \in \mathbb{R}^n \), the equation
  
  \[
  Lw = w
  \]
  has a solution \( \iff w \in \text{colspan}(L) \).
More generally, for any LT \( L : V \to W \),

- \( L \) is injective \( \iff \) \( N(L) = \{0\} \).
- \( L \) is surjective \( \iff \) \( R(L) = W \).

Let's prove the first statement.

\[ \rightarrow \) Suppose \( L \) is injective. We know

\[ L(0) = 0, \quad L(0) = 0. \] So if \( L(v) = 0 \),

then \( v = 0 \). Thus \( N(L) = \{0\} \).

\[ \leftarrow \) Suppose \( N(L) = \{0\} \). If \( L(u) = L(v) \),

then \( 0 = L(u) - L(v) = L(u - v) \)

\( \Rightarrow u - v \in N(L) \Rightarrow u - v = 0 \Rightarrow u = v \).
Rank Plus Nullity Theorem

If \( L : V \rightarrow W \) is linear, we define

\[
\text{rank} \ (L) = \dim \ (R(L)) \quad \text{null} \ (L) = \dim \ (N(L)).
\]

Theorem: For any \( LT \ L : V \rightarrow W \), then

and therefore

\[
\text{rank} \ (L) + \text{null} \ (L) = \dim \ (V).
\]

Example:

\[
L = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 1 & 7 \end{bmatrix}
\]

\[
\dim \ (V) = \quad \text{rank} \ (L) = \quad \text{null} \ (L) =
\]

Proof: Let \( N(L)^c \) be a complement of \( N(L) \), so that

\[
V = N(L) \oplus N(L)^c.
\]

Then \( \dim \ (V) = \text{rank} \ (L) + \dim \ (N(L)^c) \).

We will show \( N(L)^c \cong R(L) \).
Let $L^c : N(L)^c \rightarrow R(L)$ be the restriction of $L$ to $N(L)^c$. We will show it is an isomorphism.

(i) Injective: If $L^c(v) = 0$ for some $v \in N(L)^c$, then $v \in N(L) \iff v \in N(L) \cap N(L)^c = \{0\}$
$\Rightarrow v = 0$.

(ii) Surjective: Let $w \in R(L)$. Then $\exists v \in V$ s.t. $L(v) = w$. Since $V = N(L) \oplus N(L)^c$,
$\exists s \in N(L)$, $t \in N(L)^c$ s.t. $v = s + t$.
Then $w = L(v) = L(s + t) = L(s) + L(t) = L(t)$.

Corollary: Let $L : K^n \rightarrow K^n$, $K = \mathbb{R}$ or $\mathbb{C}$.

Then

$L$ is invertible $\iff$ rank $(L) = n$
$\iff$ null $(L) = 0$
$\iff$ the columns of $L$ are LI
$\iff$ " " " " " span $K^n$

Do you know any other equivalent conditions?

- $\det(L) \neq 0$
- $\forall w \in \mathbb{K}^n$, $Lv = w$ has a unique solution
- Gaussian elimination $\Rightarrow$ identity
ORTHOGONALITY

Recall, an \textit{inner product space} is a vector space $V$ equipped with an \textit{inner product}, which is a function $\langle \cdot , \cdot \rangle : V \times V \to K$ such that $\forall x, y, z \in V, a \in K$

IP1 $\langle x, y \rangle = \overline{\langle y, x \rangle}$ \hspace{1cm} \text{conj- symm.}

IP2 $\langle ax, y \rangle = a \langle x, y \rangle$ \hspace{1cm} \text{linearity in first variable}

IP3 $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

IP4 $\langle x, x \rangle > 0$ with equality iff $x = 0$ \hspace{1cm} \text{pos. def.}

\textbf{Theorem} \hspace{.5cm} (Cauchy-Schwarz inequality)

Let $V$ be an IPS. For any $x, y \in V$,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

where $\|x\| := \sqrt{\langle x, x \rangle}$, with equality iff $x, y$ are L.D.

This implies that for all $x, y$

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} \leq 1.$$

Recall that for $x, y \in \mathbb{R}^2$ with $\langle x, y \rangle = x_1y_1 + x_2y_2$,

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos \theta.$$
Therefore we can define the angle between two vectors in Euclidean space by defining

\[ \theta = \cos^{-1}\left( \frac{\langle x, y \rangle}{\|x\| \|y\|} \right) \]

which makes sense by the Cauchy-Schwarz inequality.

Thus we can define the notion of the angle between non-geometric quantities like polynomials.

We won’t be too concerned about general angles, but we will focus on the case \( \theta = \pm \frac{\pi}{2} \), i.e. \( \langle x, y \rangle = 0 \).

If \( \langle x, y \rangle = 0 \) we say \( x \) and \( y \) are orthogonal, and write \( x \perp y \).

\[ \text{Orthogonal / orthonormal sets/bases} \]

Let \( U \subseteq V \). We say \( U \) is an

- **orthogonal set** if \( \langle u, v \rangle = 0 \) for all \( u \neq v \) in \( U \).
- **orthonormal set** if \( U \) is an orthogonal set and \( \|u\| = 1 \) for all \( u \in U \).
orthogonal basis if $U$ is a basis and an orthog. set
orthonormal basis " " " " basis " " orthonormal set.

Example

$V = \mathbb{R}^2$, $\langle \cdot, \cdot \rangle = \text{dot product}$

$\left( -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$

$\left( \frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right)$

Note

If $U$ is orthogonal, then

$\{ \frac{u}{\|u\|} \mid u \in U \}$ is orthonormal.

Fact

If $U$ is an orthogonal set, then $U$ is L.I.

(homework).

Fact

(For the"green" Theorem)

If $u \perp v$, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$
Example \text{ DFT vectors.} \quad V = \mathbb{C}^N, \quad k = \mathbb{C}.

\[ U = \{ u_0, \ldots, u_{N-1} \}, \quad u_k = \begin{bmatrix} 1 \\ e^{2\pi i \frac{k}{N}} \\ \vdots \\ e^{2\pi i \frac{(N-1)k}{N}} \end{bmatrix} \]

\[ <u_k, u_l> = \sum_{j=0}^{N-1} e^{i2\pi \frac{k_j}{N}} e^{-i2\pi \frac{l_j}{N}} \]

\[ = \sum_{j=0}^{N-1} e^{i2\pi \delta \frac{(k-l)}{N}} = \sum_{\delta=0}^{N-1} q^\delta, \quad q = e^{\frac{2\pi i}{N}(k-l)} \]

\[ = \begin{cases} N & \text{if } k = l \\ \frac{1-q^N}{1-q} = 0 & \text{if } k \neq l \end{cases} \]

\[ \Rightarrow U \text{ is orthonormal} \]
\[ \Rightarrow U \text{ is LTI} \]
\[ \Rightarrow U \text{ is a basis.} \]

Also, \[ \tilde{U} = \{ \tilde{u}_0, \ldots, \tilde{u}_{N-1} \}, \quad \tilde{u}_k = \frac{1}{\sqrt{N}} u_k, \]
is an orthonormal basis.
**Fourier Expansions**

If \( \mathcal{U} \) is an ONB for \( V \),

**Theorem** Suppose \( \dim(V) = n < \infty \) and \( \mathcal{U} = \{u_1, \ldots, u_n\} \) is an ONB for \( V \). Then \( \forall \mathbf{v} \in V \),

\[
\begin{align*}
(i) \quad \mathbf{v} &= \sum_{i=1}^{n} \langle \mathbf{v}, u_i \rangle u_i \\
(ii) \quad \|\mathbf{v}\|^2 &= \sum_{i=1}^{n} |\langle \mathbf{v}, u_i \rangle|^2
\end{align*}
\]

**Proof** (i) Since \( \mathcal{U} \) is a basis, \( \exists a_1, \ldots, a_n \in \mathbb{K} \) s.t. \( \mathbf{v} = \sum a_i u_i \). Then

\[
\langle \mathbf{v}, u_j \rangle = \langle \sum a_i u_i, u_j \rangle = \sum a_i \langle u_i, u_j \rangle = a_j.
\]

(ii) \( \|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i,j} \langle \mathbf{v}, u_i \rangle \langle \mathbf{v}, u_j \rangle = \sum_{i,j} \langle \mathbf{v}, u_i \rangle \langle \mathbf{v}, u_j \rangle \langle u_j, u_i \rangle \langle u_i, u_j \rangle = \sum_{i,j} |\langle \mathbf{v}, u_i \rangle|^2 \]

or use Pythagorean Thm.
Example 1. \( \hat{u} = \text{normalized DFT basis} = \{\hat{u}_0, \ldots, \hat{u}_{N-1}\} \).

\( V = \mathbb{C}^N, \ K = \mathbb{C} \)

Then

\[
\langle v, \hat{u}_k \rangle = \sum_{n=0}^{N-1} v[n] \cdot \frac{1}{\sqrt{N}} e^{-i 2\pi \frac{nk}{N}}
\]

\[= \text{Normalized DFT coeff.} \]

Therefore, we conclude the reconstruction formula

\[ v = \sum_{k} \langle v, \hat{u}_k \rangle \hat{u}_k \]

Parseval

\[ \|v\|_2^2 = \sum_{k} |\langle v, \hat{u}_k \rangle|^2 \]

To recover the original formulas, we just rescale the above formulas.
PROJECTIONS

Let $V$ be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let $S$ be a subspace. If $V = S \oplus T$ for some $T$, we say that $T$ is an algebraic complement of $S$. In general, every $S$ has many complements.

Given $S, T$ s.t. $V = S \oplus T$, we define the projection onto $S$ (relative to $T$) to be the function

$$\Pi_S : V \to V$$

given by $\Pi_S (v) = s$, where $v = s + t$, $s \in S$, $t \in T$.  

Properties:

- $\Pi_s^2 = \Pi_s$
- $\Pi_s + \Pi_T = I_V$
- $\Pi_s \cdot \Pi_T = 0$
- $R(L) = S$
- $N(L) = T$

It would be nice if there was a unique complement we could associate to $S$.

Orthogonal Complements: Now assume $V$ is an IPS.

For any subspace $S$, we define its orthogonal complement

\[ S^\perp := \{ v \in V \mid v \perp s \quad \forall s \in S \} \]

Examples:

(i) $V = \mathbb{R}^2$

\[
\begin{array}{c}
\begin{array}{c}
S \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
S \\
\end{array}
\end{array}
\]
(ii) \( S^\perp = \mathbb{R}_3\text{-axis} \)

(iii) \( V = L^2(\mathbb{R}), \quad \langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt \)

\( S = \{ \text{odd functions} \}, \quad T = \{ \text{even functions} \} \)

Suppose \( f \in S, g \in T \). Then

\[
\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt
\]

\[
= \int_{-\infty}^{0} f(t) \overline{g(t)} \, dt + \int_{0}^{\infty} f(t) \overline{g(t)} \, dt
\]

\[
= 0
\]

So \( T \subseteq S^\perp \). Is \( T = S^\perp \)?

We know \( V = S \oplus T \). Let \( v \in S^\perp \), and write

\( v = s + t \). Then \( 0 = \langle v, s \rangle = \langle s + t, s \rangle \)

\[
= \langle s, s \rangle \implies s = 0 \implies v \in T.
\]

Thus \( S^\perp = T \).
Properties of $S^+$:

- $S^+$ is a subspace: if $a, b \in K$, and $s \in S$
  \[ \langle u, s \rangle = 0 \quad \forall s \in S \]
  \[ \langle v, s \rangle = 0 \quad \forall s \in S \]
  then $\langle au + bv, s \rangle = a \langle u, s \rangle + b \langle v, s \rangle = 0 \quad \forall s \in S$.

- $S \cap S^+ = \{0\}$
  If $s \in S \cap S^+$, then $\langle s, s \rangle = 0$
  \[ \Rightarrow s = 0. \]

- If $U$ is a basis for $S$, and $v \perp u \in U$, then $v \in S^+$.

**Orthogonal Projections**

If $V = S \oplus S^+$, we can define the orthogonal projection onto $S$ to be the projection onto $S$ relative to $S^+$.

Is it always the case that $V = S \oplus S^+$?
That is, is the orthog. comp. always an alg. comp?
Example \( V = \mathbb{R}^n, \quad K = \mathbb{R} \)

\[ S = \{ v \in V \mid v_i = 0 \text{ for all but finitely many } i \} \]

Note \( e_i \in S \) where \( e_i \) has a 1 in the \( i \)-th position and 0 elsewhere.

If \( v \in S^+ \), then \( v + e_i \forall i \)

\[ \Rightarrow \langle v, e_i \rangle = v_i = 0 \Rightarrow v = 0 \]

So \( S^+ = \{0\} \Rightarrow V \neq S \oplus S^+ \).

Theorem \textit{(Projection Theorem, version 1)}

If \( V \) is an IPS, \( S \subset V \) a subspace, and \( \dim(S) < \infty \), then \( V = S \oplus S^+ \).

Proof: Let \( U = \{u_1, \ldots, u_n\} \) be an ONB for \( S \).

Let \( v \in V \), and set \( s = \sum_{i=1}^{n} \langle v, u_i \rangle u_i \in S \)

and \( t = v - s \). Now for any \( i \),

\[ \langle t, u_i \rangle = \langle v - s, u_i \rangle = \langle v, u_i \rangle - \langle v, u_i \rangle = 0 \]

\[ \Rightarrow t \in S^+ \]. This shows \( S, S^+ \) span \( V \).

Since \( S \cap S^+ = \{0\} \), we conclude \( V = S \oplus S^+ \).
Where did we use the assumption \( \dim(S) = n < \infty \)?

When we claimed \( S := \sum_{i=1}^{n} \langle v, u_i \rangle u_i \in S \), which it may not be if \( n = \infty \). Consider the previous example.

**Nearest Point Property**

**Theorem** If \( V = S \oplus S^\perp \), then \( \forall v \in V \),

\[ \Pi_S(v) \text{ is the unique solution of } \]

\[
\min_{s' \in S} \| v - s' \|
\]

**Proof**: Let \( v \in V \). Write \( v = s + t \), where \( s \in S \), \( t \in S^\perp \), so that \( \Pi_S(v) = s \). Then for any \( s' \in S \),

\[
\| v - s' \|^2 = \| s - s' + t \|^2
\]

\[= \| s - s' \|^2 + \| t \|^2 \]

which is minimized when \( s' = s \) only.

**Orthogonality Principle**

\( v = \Pi_S v + s \quad \forall s \in S \). If we take \( s \in \{s_1, s_2, \ldots \} = \text{basis for } S \), we can often set up a system of equations to solve for \( \Pi_S v \).
From now on, when we mention projection, we mean a thin proj. unless otherwise indicated.
It turns out that many important problems in signal processing can be understood using projections.

- **Fitting**: project onto $S$ to estimate a noisy signal known to belong to $S$.

- **Approximation**: find the best fit to a signal using a signal from $S$. 
PROJECTION MATRICES AND LEAST SQUARES

Let's now focus on the case $V = \mathbb{K}^n$, and
\[ \langle x, y \rangle = y^H x \] (standard dot product)

Let $S$ be an arbitrary subspace, spanned by linearly independent vectors $a_1, \ldots, a_p \in \mathbb{K}^n$. Denote
\[
A = \begin{bmatrix} a_1 & \cdots & a_p \end{bmatrix} \quad (N \times p)
\]

**Theorem**

\[
\Pi_S = A \cdot (A^H A)^{-1} A^H
\]

**Proof:** $A^H A$ is invertible since rank $(A) = p$ (Homework)

Let $b_1, \ldots, b_{n-p}$ span $S^\perp$ and set $B = [b_1 \ldots b_{n-p}].$

For any $v \in V$, $v = s + t$, $s \in S$, $t \in S^\perp$,
\[
= A \theta + B \phi, \quad \theta \in \mathbb{K}^p, \quad \phi \in \mathbb{K}^{n-p}.
\]

Then
\[
A (A^H A)^{-1} A^H v = A (A^H A)^{-1} A^H (A \theta + B \phi)
\]
\[
= A (A^H A)^{-1} A^H A \theta + A (A^H A)^{-1} A^H B \phi
\]
\[
= A \theta = s
\]

\[ \square \]
If \( \alpha_1, \ldots, \alpha_p \) are orthogonal, then
\[
\Pi_S = A \cdot A^H
\]
and we obtain
\[
\Pi_S (\alpha) = A \cdot A^H \alpha = A \cdot \begin{bmatrix}
\langle \alpha, \alpha_1 \rangle \\
\vdots \\
\langle \alpha, \alpha_p \rangle
\end{bmatrix} = \sum_{i=1}^{p} \langle \alpha, \alpha_i \rangle \alpha_i
\]
which is a Fourier expansion.

We saw this previously in the proof of the projection. Thus,

Note: \( A^H A = \begin{bmatrix}
\langle \alpha_1, \alpha_1 \rangle & \cdots & \langle \alpha_1, \alpha_p \rangle \\
\vdots & \ddots & \vdots \\
\langle \alpha_p, \alpha_1 \rangle & \cdots & \langle \alpha_p, \alpha_p \rangle
\end{bmatrix} \) is called the Grammian. We saw on the homework that \( A^H A \) is invertible when \( \text{rank}(A) = p \).
The problem

$$\min_{\Theta} \| \nu - A \Theta \|^2$$

is called a least squares problem.

From the nearest point property, and the previous theorem, we deduce that the least squares solution is unique

$$(A^H A)^{-1} A^H \nu.$$

The matrix

$$A^+ := (A^H A)^{-1} A^H$$

is called the pseudoinverse of $A$. 
Linear Regression

Observe \((x_i, y_i), i = 1, \ldots, n\)

Suppose \(y_i = ax_i + b + e_i\). How can we estimate \(a, b\)? Let's minimize the sum of squared errors:

\[
(\hat{a}, \hat{b}) = \arg\min_{(a, b)} \sum_{i=1}^{n} \left[ y_i - (ax_i + b) \right]^2
\]

\[
= \| e \|^2
\]

Now

\[
\hat{e} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
\]

\[
= y - A \hat{\theta}
\]

So

\[
\hat{\theta} = \arg\min_{\theta} \| y - A \theta \|^2 = (A^T A)^{-1} A^T y
\]
Denoising Sinusoidal Signals

Suppose we observe a signal in noise

\[ x[n] = s[n] + \varepsilon[n], \quad n = 0, \ldots, N-1. \]

where

\[ s[n] = C \cdot \cos\left[2\pi fn + \phi\right] \]

\( C \in \mathbb{C}, \) unknown

\( \phi \in [-\pi, \pi], \) unknown

\( f \in \left[-\frac{1}{2}, \frac{1}{2}\right], \) known

How can we estimate \( s[n] \)?
Denote \( \mathbf{x} = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \), \( \mathbf{s} = \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} \), \( \mathbf{e} = \begin{bmatrix} e[0] \\ \vdots \\ e[N-1] \end{bmatrix} \)

so that \( \mathbf{x} = \mathbf{s} + \mathbf{e} \).

Notice

\[
\cos(2\pi f_n + \phi) = \frac{e^{i(2\pi f_n + \phi)} - i(2\pi f_n + \phi)}{2}
\]

\[
= \frac{1}{2} e^{i\phi} e^{i2\pi f_n} + \frac{1}{2} e^{-i\phi} e^{-i2\pi f_n}
\]

Therefore

\[
\mathbf{s} = \begin{bmatrix} 0 \\ e^{i2\pi f} \\ \vdots \\ e^{i2\pi f (N-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-i2\pi f} \\ \vdots \\ e^{-i2\pi f (N-1)} \end{bmatrix}
\]

\[
= c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2
\]

\( \mathbf{e} \) spans \( \{ \mathbf{a}_1, \mathbf{a}_2 \} \) = \( \mathbf{s} \).
Furthermore, we can recover $C$, $\phi$ from $c_1, c_2$:

$$C = \sqrt{4c_1 c_2}$$

$$\phi = \frac{1}{2} \text{arg} \left( \frac{c_1}{c_2} \right)$$

So signals like $s[n]$ coincide exactly span $\xi_1, \xi_2, \xi_3$.

So let's estimate $\hat{s}$ by projecting onto $S$

$$\hat{s} = \Pi_S x = A \left( A^HA \right)^{-1} A^H x$$

Will this recover $s$ exactly? No, because in general, $\Pi_S e \neq 0$, so $\hat{s} = s + \Pi_S e \neq s$
Least Squares FIR Filter Design

Consider an LTI system with impulse response $h[n]$ having length $N$:

$$x[n] \rightarrow \underbrace{h[n]} \rightarrow y[n]$$

$$y[n] = \sum_{k=0}^{N-1} h[k] x[n-k]$$

Suppose we wish to design $\{h[n]\}$ to have a certain frequency response. In particular, suppose we wish $\hat{h}(\omega)$ to match some desired response $\hat{h}_d(\omega)$ at a finite set of chosen frequencies $w_0, \ldots, w_{L-1} \in [-\pi, \pi]$, $L > N$.

How can we minimize

$$\sum_{k=0}^{L-1} \left| \hat{h}(w_k) - \hat{h}_d(w_k) \right|^2$$
Let's write
\[ \hat{h}(\omega) = \sum_{n=0}^{N-1} h[n] e^{-i\omega n} \]
\[ = \langle \frac{h}{\varphi}, \varphi(\omega) \rangle \]

where
\[ \frac{h}{\varphi} = \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{bmatrix} \]
\[ \varphi(\omega) = \begin{bmatrix} 1 \\ e^{i\omega} \\ \vdots \\ e^{i\omega (N-1)} \end{bmatrix} \]

Then
\[ \sum_{k=0}^{N-1} |\hat{h}(\omega_k) - \hat{d}(\omega_k)|^2 \]
\[ = \left\| \begin{bmatrix} \hat{d}(\omega_0) \\ \vdots \\ \hat{d}(\omega_{N-1}) \end{bmatrix} - \begin{bmatrix} \varphi(\omega_0)^H \\ \vdots \\ \varphi(\omega_{N-1})^H \end{bmatrix} \begin{bmatrix} h[0] \\ \vdots \\ h[N-1] \end{bmatrix} \right\|^2 \]
\[ = \left\| \hat{d} - A \cdot \hat{h} \right\|^2 \]
Therefore the LS optimal filter coefficients are given by

\[ \hat{h}^* = A^+ \hat{b}_d = (A^H A)^{-1} A^H \hat{b}_d \]

What happens if \( w_k = \frac{2\pi k}{L} \)? Then

\[
A = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & e^{-i2\pi \frac{1}{L}} & \cdots & e^{-i2\pi \frac{N-1}{L}} \\
& \vdots & \ddots & \vdots \\
1 & e^{-i2\pi \frac{N-1}{L}} & \cdots & e^{-i2\pi \frac{(L-1)(N-1)}{L}} \\
\end{bmatrix}
\]

\((L \times N)\)

\[
= \begin{bmatrix}
\hat{u}_0 & \hat{u}_1 & \cdots & \hat{u}_{N-1}
\end{bmatrix}
\]

\(\hat{u}_n\) DFT basis vectors for length-\(L\) DFT

\[
A^H A = L \cdot I_{N \times N}
\]

\[
\hat{b}^* = \frac{1}{L} A^H \hat{b}_d = \frac{1}{L} \sum_{k=0}^{L-1} \hat{b}_d \left( \frac{2\pi k}{L} \right) \hat{u}_k \left( \frac{2\pi k}{L} \right) = L
\]

\(\hat{b}^*\) Fast implementation w/ FFT.
Rethink this.
Not worth much
class time, if
any.
In general, \( h[n] \in \mathbb{C} \) even if \( h[0] \in \mathbb{R} \).

What if we wanted \( h[n] \in \mathbb{R} \)?

If we enforce the constraint \( h[n] = h[N-n], \ n \geq 1 \), then \( A \) is real \( \Rightarrow \) \( B^* \) is real.
Consider the set $V$ of all complex-valued, zero mean random variables.

Aside: A complex-valued random variable has the form $x = x_R + ix_I$ where $x_R, x_I$ are jointly distributed real RVs. Then $E[x] := E[x_R] + iE[x_I]$, $\text{Var}(x) := E[|x - E[x]|^2]$. Then $V$ is a vector space over $\mathbb{C}$.

Now consider the inner product

$$\langle x, y \rangle = E[xy^*]$$

Is this a valid inner product?

The induced norm is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{E[|x|^2]} = \sqrt{\text{Var}(x)}$$

By Cauchy-Schwartz, $-1 \leq \rho_{xy} \leq 1$, where

$$\rho_{xy} := \frac{E[xy]}{\sqrt{E[x^2]E[y^2]}}$$

is the correlation coefficient.
\[ = \text{Var} (y) \\
1 \text{Var} (x) \]
Note that in this norm, \( X = Y \iff E[|x - y|^2] = 0 \)

**Linear Prediction**

Consider a random process \( \{x[n]\}_{n=-\infty}^{\infty} \),

which is zero-mean and wide-sense stationary, i.e.

\[
E[x[n]] = 0 \quad \forall n
\]

\[
r_{xx}[k] := \langle x[n], x[n-k] \rangle
\]

\[
\text{auto-correlation function (ACF)}
\]

\[
= E[x[n]x[n-k]]
\]

is independent of \( n \).  

Suppose we observe \( x[n-1], \ldots, x[n-p] \)
and we wish to predict \( x[n] \).

Unless the \( x[n] \) are uncorrelated, we expect to be able do better than random guessing.
Let's look for a linear estimator

\[ \hat{x}[n] = \sum_{k=1}^{p} h_p[k] x[n-k] \]

In other words, we want to determine \( h_p = \begin{bmatrix} h_p[1] \\ \vdots \\ h_p[p] \end{bmatrix} \)
to have the smallest prediction error,

\[ e[n] = x[n] - \hat{x}[n] \]

When we say "small," we mean in the sense of small

\[ \| e[n] \|^2 = E \left[ \left| x[n] - \hat{x}[n] \right|^2 \right] \]

Define the subspace

\[ S = \text{span} \{ x[n-1], \ldots, x[n-p] \} \]

Then we seek the point \( \hat{x}[n] \) in \( S \) that is closest to \( x[n] \). Since \( S \) is finite dimensional, we know \( V = S \oplus S^\perp \). Therefore \( \hat{x}[n] \) is the projection of \( x[n] \) onto \( S \), and

\[ e[n] = x[n] - \hat{x}[n] \perp S \quad \forall s \in S. \]

\( \perp \) or orthogonality principle
Solution via Orthogonality Principle

By the orthogonality principle we know

\[ x[n] - \hat{x}[n] \perp x[n-k], \]

\[ k = 1, \ldots, p. \] That is

\[ 0 = \langle x[n] - \hat{x}[n], x[n-k] \rangle \]

\[ = \langle x[n] - \sum_{l=1}^{p} h_p[l] x[n-l], x[n-k] \rangle \]

\[ = \langle x[n], x[n-k] \rangle - \sum_{l=1}^{p} h_p[l] \langle x[n-l], x[n-k] \rangle \]

\[ = r_{xx}[k] - \sum_{l=1}^{p} h_p[l] \cdot r_{xx}[k-l] \]

This holds for each \( k \). As a matrix equation

\[
\begin{bmatrix}
    r_{xx}[0] & r_{xx}[-1] & \cdots & r_{xx}[p-1] \\
    r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[p-2] \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{xx}[p-1] & r_{xx}[p-2] & \cdots & r_{xx}[0]
\end{bmatrix}
\begin{bmatrix}
    h_p[1] \\
    h_p[2] \\
    \vdots \\
    h_p[p]
\end{bmatrix}
= 
\begin{bmatrix}
    r_{xx}[1] \\
    r_{xx}[2] \\
    \vdots \\
    r_{xx}[p]
\end{bmatrix}
\]

or

\[ R_p \cdot h_p = r_p \]
Therefore the optimal prediction filter is the solution of a linear system of equations.

This solution can be arrived at by other means, e.g. gradients, but none are as simple and elegant as the orthogonality principle.

**Algorithms**

Since $R_p$ is Toeplitz, the system can be solved efficiently.

In addition, it is possible to update $h_{p+1}$ from $h_p$, once $x[n]$ is observed, using the Levinson-Durbin algorithm.
FOURIER SERIES IN HILBERT SPACE

Goal of this lecture and the next: Generalize Fourier series and projections to infinite dimensional subspaces.

We have seen that if $V$ is a vector space with an ONB $\Omega = \{e_1, \ldots, e_n\}$, then

$$v = \sum_{k=1}^{\infty} \langle v, e_k \rangle e_k$$

and

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2$$

for all $v \in V$.

What if a finite ONB does not exist?

**Orthonormal Sequences**

An orthonormal sequence is an orthonormal set, $\{e_1, e_2, \ldots, e_3\}$, indexed by $\mathbb{N}$. 
Examples

- Standard basis in

\[ l^2 = \{ (x_1, x_2, \ldots) \in \mathbb{C}^\infty \mid \sum_{i=1}^\infty |x_i|^2 < \infty \} \]

\[ \langle x, y \rangle = \sum_{i=1}^\infty x_i \overline{y_i} \]

- \( \left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n \in \mathbb{Z}} \) in \( L^2(-\pi, \pi) \)

\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, dt \]

Theorem (Bessel's inequality)

If \( \{e_i\}_i \) is an orthonormal sequence, then

\[ \sum_{i=1}^\infty |\langle v, e_i \rangle|^2 \leq ||v||^2 \]

for all \( v \in V \).

Proof: Denote \( v_N = \sum_{i=1}^N \langle v, e_i \rangle e_i \). Then

\[ ||v - v_N||^2 = \langle v - \sum \langle v, e_i \rangle e_i, v - \sum \langle v, e_i \rangle e_i \rangle \]

\[ = \langle v, v \rangle - \sum \langle v, e_i \rangle \overline{\langle v, e_i \rangle} - \sum \langle v, e_i \rangle \langle e_i, v \rangle + \sum |\langle v, e_i \rangle|^2 \]

\[ = ||v||^2 - \sum |\langle v, e_i \rangle|^2 \]
So \( \sum_{i=1}^{N} |\langle \nu, e_i \rangle|^2 = \| \nu \|^2 - \| \nu - \nu_N \|^2 \leq \| \nu \|^2 \).

Now let \( N \to \infty \).

What does \( \sum_{i=1}^{\infty} \lambda_i e_i \) mean? When does it converge?

If \( x \in V \), \( e_i \in V \), we say that \( \sum_{i=1}^{\infty} x_i \) converges to \( x \) if

\[
\sum_{i=1}^{n} x_i \to x \quad \text{as} \quad n \to \infty.
\]

(\( \forall \epsilon > 0 \), \( \exists N \) s.t. \( n \geq N \Rightarrow \| \sum_{i=1}^{n} x_i - x \| < \epsilon \).)

In a Hilbert space, there is a precise characterization of when \( \sum \lambda_i e_i \) converges.

Thus let \( H \) be a Hilbert space and let \( \{ e_1, e_2, \ldots \} \) be an orthonormal sequence. Then

\[
\sum_{i=1}^{\infty} \lambda_i e_i \text{ converges} \iff \sum |\lambda_i|^2 < \infty.
\]
Proof: Denote

\[ x_n = \sum_{i=1}^{n} \lambda_i e_i, \quad r_n = \sum_{i=1}^{n} |\lambda_i|^2. \]

Then (assuming \( m < n \))

\[ \| x_n - x_m \|^2 = \| \sum_{i=m+1}^{n} \lambda_i e_i \|^2 \]

\[ = \sum_{i=m+1}^{n} |\lambda_i|^2 \]

\[ = |r_n - r_m| \]

Therefore \( \{x_n\} \) is Cauchy \( \iff \) \( r_n \) is Cauchy.

\[ \Rightarrow \]

\( \{x_n\} \) converges \( \iff \) \( \{x_n\} \) is Cauchy

\( \Rightarrow \) \( \{r_n\} \) is Cauchy

\( \Rightarrow \) \( \{r_n\} \) converges (since \( \mathbb{R} \) is complete)

\[ \Leftarrow \]

\( \{r_n\} \) converges \( \iff \) \( \{r_n\} \) is Cauchy

\( \Rightarrow \) \( \{x_n\} \) is Cauchy

\( \Rightarrow \) \( \{x_n\} \) converges (since \( \mathbb{H} \) is complete).
Complete Orthonormal Sequences

Let \( \{e_i, e_2, \ldots \} \) be an ONS in a Hilbert Space \( \mathcal{H} \).

We would like to write

\[
\nu = \sum <\nu, e_i> e_i
\]

for arbitrary \( \nu \in \mathcal{H} \). Is this possible?

By Bessel's inequality,

\[
\sum_{i=1}^{\infty} |<\nu, e_i>|^2 \leq ||\nu||^2 < \infty
\]

\[
\Rightarrow \sum <\nu, e_i> e_i \text{ converges. But does it converge to } \nu?
\]

Examples

1. \( e_1 = (0, 1, 0, \ldots) \)
2. \( e_2 = (0, 0, 1, 0, \ldots) \)
3. \( e_3 = (0, 0, 0, 1, 0, \ldots) \)

\[ \vdots \]

in \( L^2 \)

If \( \nu = (1, 0, 0, \ldots) \) then \( \sum <\nu, e_i> e_i = 0 \neq \nu \)

1. \( e_i = e^{i(2\pi r)t} \), \( r \in \mathbb{Z} \), in \( L^2(-\pi, \pi) \)
An ON set is said to be complete if
\[ \forall i \in \mathbb{N}, \quad u \perp e_i \quad \forall i \implies u = 0. \]

**Theorem**  If \( \{e_1, e_2, \ldots\} \) is an ONS in a Hilbert space \( H \), then TFAE:

1. \( \{e_i\} \) is complete
2. \( H = \sum \langle \nu, e_i \rangle e_i \quad \forall \nu \in H \)
3. \( \|\nu\|^2 = \sum |\langle \nu, e_i \rangle|^2 \quad \forall \nu \in H \)

**Proof**

(1 \( \implies \) 2) Let \( u = \nu - \sum_{i=1}^{\infty} \langle \nu, e_i \rangle e_i \).

Then
\[
\langle u, e_j \rangle = \langle \nu, e_j \rangle - \sum_{i=1}^{\infty} \langle \nu, e_i \rangle \langle e_i, e_j \rangle
= \langle \nu, e_j \rangle - \lim_{n \to \infty} \sum_{i=1}^{n} \langle \nu, e_i \rangle \langle e_i, e_j \rangle
= \langle \nu, e_j \rangle - \lim_{n \to \infty} \sum_{i=1}^{n} \langle \nu, e_i \rangle \langle e_i, e_j \rangle
= \langle \nu, e_j \rangle - \sum_{i=1}^{\infty} \langle \nu, e_i \rangle \langle e_i, e_j \rangle = 0
\]
\[ \implies u \perp e_j \quad \forall j \implies u = 0. \]

**Lemma:** If \( x_n \to x \), then \( \langle x_n, y \rangle \to \langle x, y \rangle \).

**Proof:** \[ |\langle x_n, y \rangle - \langle x, y \rangle| = |\langle x_n - x, y \rangle| \leq \|x_n - x\| \cdot \|y\| \]
\[ \to 0 \quad \text{as} \ n \to 0. \quad \text{Cauchy-Schwarz} \]

Now apply with \( x_n = \sum_{i=1}^{n} \langle \nu, e_i \rangle e_i \)
(2 ⇝ 3)

Lemma: If \( x_n \to x \), then \( \|x_n\| \to \|x\| \).

By \( \Delta \) ineq:
\[
\|x\| \leq \|x - x_n\| + \|x_n\| \Rightarrow \|x\| - \|x_n\| \leq \|x - x_n\|
\]
\[
\|x_n\| \leq \|x - x_n\| + \|x\| \Rightarrow \|x_n\| - \|x\| \leq \|x - x_n\|
\]

So \( \|x\| - \|x_n\| \leq \|x - x_n\| \to 0 \) as \( n \to \infty \).

Then
\[
\|x\|^2 = \| \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \|^2
\]
\[
= \lim_{n \to \infty} \| \sum_{i=1}^{n} \langle x, e_i \rangle e_i \|^2
\]
\[
= \lim_{n \to \infty} \| \sum_{i=1}^{n} |\langle x, e_i \rangle|^2
\]
\[
= \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2
\]

(3 ⇝ 1) Suppose (3) holds but \( \mathcal{E} \) is not complete.

Then \( \exists v \neq 0 \) s.t. \( v \perp e_i \ \forall i \). Then
\[
0 = \|v\|^2 = \sum |\langle v, e_i \rangle|^2 = 0 \quad \text{a contradiction.}
\]
A Hilbert space is said to be separable if it contains a complete orthonormal set indexed by $\mathbb{N}$ or finite.

Examples:

- $\ell^2$

- $L^2(\mathbb{R})$ - Hermite polynomials
  - Wavelets

- $L^2((-\pi, \pi)) = \mathcal{F} \cap \mathcal{F}$ (proof of completeness in Mallat)

- Legendre polynomials

Complete orthonormal sequences are very similar to orthonormal Hamel bases, except that we allow infinite linear combos. In fact, complete orthonormal sequences are examples of Hilbert bases. A Hilbert basis is a maximal orthonormal set. Some Hilbert bases are uncountable, so not every Hilbert space is separable. Every Hilbert basis has the same dimension, so we can define the Hilbert dimension. For finite dimensional spaces this agrees with the Hamel dimension, but otherwise it generally does not.
Parseval's Theorem

If $X$ is a Hilbert space with a complete orthonormal set $\{e_i, e_2, \ldots\}$, then

$$
\langle x, y \rangle = \sum_{i=1}^{\infty} \langle x, e_i \rangle \overline{\langle y, e_i \rangle}
$$

Proof:

$$
\langle x, y \rangle = \left\langle \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i , \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \right\rangle
$$

$$
= \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i , \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \rangle
$$

$$
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle x, e_i \rangle \overline{\langle y, e_j \rangle} \langle e_i, e_j \rangle
$$

$$
= \sum_{i=1}^{\infty} \overline{\langle y, e_i \rangle} \langle x, e_i \rangle
$$

And of course we already know

$$
||x||^2 = \sum ||x, e_i||^2
$$

In $\mathbb{C}^n$ or $\ell^2$ with standard bases, this doesn't tell us anything we didn't already know.

But with different bases, or in different spaces, we do get interesting results.
Example  \( H = L^2(\mathbb{R}), \quad \langle f, g \rangle = \int f(t) g^*(t) \, dt \)

\[
S_T := \left\{ f \in L^2(\mathbb{R}) \mid \hat{f}(\omega) = 0 \quad \text{for} \quad |\omega| > \frac{\pi T}{2} \right\}
\]

\[
h_T(t) = \frac{\sin \frac{\pi t}{T}}{\frac{\pi t}{T}}
\]

Then \( \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{T}} h_T(t-nT) \) is a complete orthonormal sequence for \( S_T \).

\[
\langle \frac{1}{\sqrt{T}} h_T(t-nT), \frac{1}{\sqrt{T}} h_T(t-mT) \rangle
\]

\[
= \frac{1}{T} \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} T \cdot \frac{1}{\sqrt{T}} \hat{h}(\omega) \cdot e^{-j n T \omega} \cdot \frac{1}{\sqrt{T}} \hat{h}(\omega) \cdot e^{j m T \omega} \, d\omega
\]

\[
= \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{-(n-m)T \omega} \, d\omega
\]

\[
= \delta[n-m].
\]

From the Whittaker sampling Theorem,

\[
f(t) = \sum_{n=-\infty}^{\infty} f(nt) \cdot h_T(t-nT)
\]

\[
= \sum_{n=-\infty}^{\infty} \left( \sqrt{\frac{T}{\pi}} f(nt) \right) \cdot \frac{1}{\sqrt{T}} h_T(t-nT)
\]

\[
= \text{Fourier coefficient}
\]

\[
f(nT) = \frac{1}{T} \langle f(t), h_T(t-nT) \rangle
\]

and \( \sum_{n=1}^{\infty} \sum_{n=-1}^{-1} \leq \int f(t) (t+nT) \, dt \)

\[
\Rightarrow \text{Completeness}
\]

\[
\tilde{v} = \sum \lambda_i e_i, \quad \text{e.g., Gauss,}
\]

\[
\Rightarrow \lambda_i = \langle e_i, \tilde{v} \rangle
\]

\[
= \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \hat{f}(\omega) e^{j n T \omega} \, d\omega
\]
Example: \( H = L^2(\pi, \pi) \) \( \quad \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) g^*(\omega) \, d\omega \)

We can use the preceding example (with \( T = 1 \)) to show that \( \sum_{n \in \mathbb{Z}} e^{-i\omega n} \) is a complete ONB for \( H \).

\[
\langle e^{-i\omega}, e^{-i\omega} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n-m)\omega} \, d\omega
\]

\[
= \begin{cases} 
1 & \text{if } n = m \\
\frac{-1}{2\pi(n-m)} \left[ e^{-i(n-m)\pi} - e^{i(n-m)\pi} \right] = 0 & \text{otherwise}
\end{cases}
\]

What about completeness? Let \( \hat{f} \) be \( L^2(-\pi, \pi) \).

Extend \( \hat{f} \) to \( H \): \( \hat{f}(\omega) = 0 \) for \( |\omega| > \pi \).

Let \( f(t) \) be the inverse CTFT of \( \hat{f}(\omega) \). Since \( f \in S_2 \) we know

\[
f(t) = \sum_{n=-\infty}^{\infty} f[n] \, h_1(t-n)
\]

where

\[
f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{-i\omega n} \, d\omega
\]

\[
= \langle \hat{f}, e^{-i\omega} \rangle
\]
Then

\[
\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f[n] \ast h_n(t-n)(\omega)
\]

\[
= \sum_{n=-\infty}^{\infty} f[n] e^{-j\omega n} \quad \text{(DTFT)}
\]

\[
= \sum_{n=-\infty}^{\infty} \langle \hat{f}, e^{-j\omega n} \rangle e^{-j\omega n}
\]

\[
\Rightarrow \{ e^{-j\omega n} \}_n \in \mathbb{Z} \ \text{is complete.}
\]

We can apply Parseval's theorem to conclude

\[
\sum_{n} |f[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 d\omega
\]

where \( \hat{f}(\omega) = \text{DTFT of } f(t) \)

Also note that

\[
f[n] = \langle \hat{f}(\omega), e^{-j\omega n} \rangle
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{j\omega n} d\omega
\]

establishes the DTFT inversion formula.
PROJECTIONS IN HILBERT SPACE

Previously we have seen that if \( V \) is an IPS and \( S \subseteq V \) is a finite dim. subspace, then

\[ V = S \oplus S^\perp \]

and we can define the orthogonal projection onto \( S \).

We will now generalize this result. Our generalization requires

- \( V \) is a Hilbert space
- \( S \) is a closed subspace

A subspace \( S \) is said to be closed if

\[ x_n \to x, \ x_n \in S \ \forall n \implies x \in S. \]

When \( \dim(S) < \infty \), \( S \) is always closed. When \( \dim(S) = \infty \), this is not the case.

Example \( V = l^2(\mathbb{N}) \), \( x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots) \)

\( S = \{ \text{finitely many nonzero coordinates} \} \)

\( x = (1, \frac{1}{2}, \ldots) \)

\[ \| x_n - x \|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \to 0 \text{ as } n \to \infty \]

\( \implies \) not closed.
Theorem (Projection Theorem - general version)

Let \( \mathcal{H} \) be a Hilbert space, and \( S \subseteq \mathcal{H} \) a closed subspace. Then \( \mathcal{H} = S \oplus S^\perp \).

Proof: book

Implications

- can define orthogonal projections
- nearest point property
- orthogonality principle

as we have already seen, these follow as long as you can define orthogonal projections.

Example \( \mathcal{H} = L^2(\mathbb{R}), \quad S_\pi = \{ f \in \mathcal{H} \mid \hat{f}(w) = 0 \text{ for } |w| > \frac{\pi}{2} \}. \)

Is \( S_\pi \) closed? If \( f_n \to f, \quad f_n \in S_\pi \), then \( \hat{f}_n \to \hat{f} \), and \( \hat{f}_n(w) = 0 \text{ for } |w| > \frac{\pi}{2} \), so \( \hat{f}(w) = 0 \text{ for } |w| > \frac{\pi}{2} \implies f \in S_\pi \).

For arbitrary \( f \in \mathcal{H} \), what is the proj of \( f \) onto \( S_\pi \)?

If \( g \in S_\pi \), then

\[
\| f - g \|^2 = \int_0^{\frac{\pi}{2}} |\hat{f}(w)|^2 \text{d}w + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} |\hat{f}(w) - \hat{g}(w)|^2 \text{d}w
\]

minimized when \( \hat{g}(w) = \hat{f}(w) \cdot \mathbf{1}_{[0, \frac{\pi}{2}]}(w) + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} |\hat{f}(w)|^2 \text{d}w \)

So projection = ideal low-pass filter.
Can we represent the projection in terms of \( \{ e_1, e_2, \ldots \} \)?

**Projections and Fourier Series**

Suppose \( \{ e_1, e_2, \ldots \} \) is an orthonormal sequence in \( X \) which is complete w.r.t. the closed subspace \( S \), i.e. if \( s \in S \), \( s = e_i \forall i \), then \( s = 0 \).

Let \( \nu \in X \).

Set \( s = \sum_{i=1}^{n} \langle \nu, e_i \rangle e_i \). Since \( S \) is closed, \( s \in S \). Also, \( \forall j \),

\[
\langle \nu - s, e_j \rangle = \langle \nu - \sum_{i=1}^{n} \langle \nu, e_i \rangle e_i, e_j \rangle
\]

\[= \langle \nu, e_j \rangle - \sum_{i=1}^{n} \langle \nu, e_i \rangle \langle e_i, e_j \rangle\]

\[= \langle \nu, e_j \rangle - \langle \nu, e_j \rangle = 0\]

\[\Rightarrow \ \nu - s \in S^\perp\]

So \( \nu = s + (\nu - s) \)

\[\Rightarrow \sum \langle \nu, e_i \rangle e_i \text{ is the projection onto } S.\]
Example, cont'd. In the case of bandlimited signals,

\[(\mathcal{F}f(t)) = \sum_{n \in \mathbb{Z}} \langle f, \frac{1}{\sqrt{T}} h_T(t - nT) \rangle \cdot \frac{1}{\sqrt{T}} h_T(t - nT)\]

\[= \sum_{n \in \mathbb{Z}} \frac{1}{T} \langle f(t), h_T(t - nT) \rangle \cdot h_T(t - nT)\]

\[= \frac{1}{T} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot T \cdot \frac{1}{2\pi} \chi(\omega) \cdot e^{i\omega u} du\]

\[= \frac{1}{2\pi} \int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} \hat{f}(\omega) \cdot e^{i\omega u} du\]

\[= \hat{f}(nT), \quad \hat{f} = \text{low-pass filtered version of } f\]

The Infinite Weiner Smoother

Let \(y[n] = s[n] + w[n], \quad n \in \mathbb{Z},\) be an observed signal. The goal is to estimate \(s[n], \quad n \in \mathbb{Z}.\)

Assume:

- \(x[n]\) is zero mean, WSS with ACF \(r_x[k]\)
  \[= E[x[n]x[n-k]]\]

- \(s[n]\) is zero mean, WSS with ACF \(r_s[k]\)

- \(x[n]\) and \(s[n]\) are jointly WSS with cross-correlation \(r_{xs}[k] = E[s[n]x[n-k]]\)

- \(r_x, r_s, r_{xs} \in \ell^2(\mathbb{Z})\)
Students didn't get this. Leave it for 564. See Yonina Eldar's work for other potential apps. of this theory. (Still, could briefly preview 564, set up problem)
We seek a linear estimator of $s[n]$ (n fixed):

$$\hat{s}[n] = \sum_{k \in \mathbb{Z}} h[k] \cdot x[n-k]$$

which minimizes

$$\|s[n] - \hat{s}[n]\|^2 = \mathbb{E}[|s[n] - \hat{s}[n]|^2]$$.

$$H = \{ \text{zero mean variables } \nu / \text{ finite variance} \}$$

$$S = \left\{ \sum_{k} h[k] \cdot \nu[n-k] \in H \mid \nu[n] \in H \forall n, h[n] \in l^2(\mathbb{Z}) \right\}$$

Facts: $H$ is a Hilbert space and $S$ is a closed subspace.

Therefore, by the orthogonality principle, for each $l \in \mathbb{Z}$,

$$0 = \langle s[n] - \hat{s}[n], x[n-l] \rangle$$

$$= \langle s[n] - \sum_{k} h[k] \cdot x[n-k], x[n-l] \rangle$$

$$= r_{sx}(l) - \sum_{k} h[k] \cdot r_{x}[l-k] \quad \forall l \in \mathbb{Z}$$
\[ h \text{ is independent of } \pi, \text{ (LTI filter)} \]

Furthermore we can solve for \( h \) in the Fourier domain:

\[ \hat{r}_{sx}(\omega) = \hat{h}(\omega) \cdot \hat{r}_{xx}(\omega) \quad \forall \omega \in [\pi, \pi] \]

\[ \Rightarrow \quad \hat{h}(\omega) = \frac{\hat{r}_{sx}(\omega)}{\hat{r}_{xx}(\omega)} \quad \text{cross spectral density} \]

\[ \text{spectral density} \]

If \( s[n] \) and \( w[n] \) are independent, then

\[ r_{sx} = r_{ss}, \quad r_{xx} = r_{ss} + r_{ww} \]

\[ \Rightarrow \quad \hat{h}(\omega) = \frac{\hat{r}_{ss}(\omega)}{\hat{r}_{ss}(\omega) + \hat{r}_{ww}(\omega)} \]

\[ \text{low frequency signal} \]

\[ \text{high frequency noise} \]

\[ \Rightarrow \text{low-pass filter to estimate } s \]
EIGENVALUES AND EIGENVECTORS

Let \( V \) be a vector space over a field \( K \), and let \( L : V \to V \) be linear. If \( \lambda \in K \) and \( 0 \neq \mathbf{v} \in V \) are such that

\[ L \mathbf{v} = \lambda \mathbf{v}, \]

we say \( \lambda \) is an eigenvalue of \( L \), and \( \mathbf{v} \) is an eigenvector corresponding to \( \lambda \).

We will focus on \( V = \mathbb{C}^N \) (or \( \mathbb{R}^N \)) so that \( L \) is an \( N \times N \) matrix, which we will usually denote \( A \).

How to determine eigenvalues/vectors? Note

\( \lambda \) is an eigenvalue of \( A \) \( \iff \) \( \exists \mathbf{v} \neq 0 \) s.t. \( A \mathbf{v} = \lambda \mathbf{v} \)

\( \iff \) \( \exists \mathbf{v} \neq 0 \) s.t. \( (A - \lambda I) \mathbf{v} = 0 \)

\( \iff \) \( A - \lambda I \) is singular (non-invertible)

\( \iff \) \( \det (A - \lambda I) = 0 \)

\( \iff \) \( \lambda \) is root of the poly eqn. \( \det (A - \lambda I) = 0 \).

Since \( \det (A - \lambda I) \) is a degree \( N \) poly, there are \( N \) roots \( \lambda_1, \ldots, \lambda_N \in \mathbb{C} \).
**Example** 1

\[
A = \begin{bmatrix}
1 & 0.5 \\
0.5 & 1
\end{bmatrix}
\]

\[
\det(A - \lambda I) = \det\left( \begin{bmatrix}
1 - \lambda & 0.5 \\
0.5 & 1 - \lambda
\end{bmatrix} \right) = (1-\lambda)^2 - 0.25 = 0
\]

\[
\Rightarrow \lambda = 1 \pm 0.5
\]

\[
\Rightarrow \lambda_1 = 1.5, \lambda_2 = 0.5
\]

\[
\lambda_1 = 1.5 \\
\begin{bmatrix}
-0.5 & 0.5 \\
0.5 & -0.5
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{12}
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

\[
\Rightarrow v_{11} = -v_{12}
\]

Choose \( v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\
\frac{-1}{\sqrt{2}} \end{bmatrix} \) as a normalized eigenvector.

\[
\lambda_2 = 0.5 \\
\begin{bmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
v_{21} \\
v_{22}
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

\[
\Rightarrow v_{21} = -v_{22}
\]

Choose \( v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\
\frac{-1}{\sqrt{2}} \end{bmatrix} \).

**Note:** Eigenvectors corresponding to distinct eigenvalues are orthogonal.
Example 2

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

\[ \det(A - \lambda I) = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2 = 0 \]

\[ \Rightarrow \lambda = \frac{5 \pm \sqrt{33}}{2} \]

\[ \mathbf{v}_1 = \begin{bmatrix} 0.416 \\ 0.909 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -0.825 \\ 0.566 \end{bmatrix} \]

\[ \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0.172 
eq 0 \]

Example 3

\[ A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \]

\[ \det(A - \lambda I) = (1-\lambda)(4-\lambda) + 6 = \lambda^2 - 5\lambda + 10 = 0 \]

\[ \Rightarrow \lambda = \frac{5 \pm \sqrt{-15}}{2} \notin \mathbb{R} \]
Example 4

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2 \times 2} \]

Then \( Ax = x \) \( \forall x \in \mathbb{C}^2 \)

\[ \Rightarrow \lambda = 1 \text{ is the only eigenvalue, and all vectors are eigenvectors} \]

\[ \det(A - \lambda I) = (1 - \lambda)^2 = 0 \]

\[ \Rightarrow \lambda = 1 \text{ (multiplicity 2).} \]

Example 5

Characteristic Polynomial

The polynomial \( p(\lambda) = \det(A - \lambda I) \) is a degree \( N \) polynomial. The coefficient of \( \lambda^N \) is \((-1)^N\), and therefore

\[ p(\lambda) = \frac{N}{i=1} \left( \lambda - \lambda_i \right) \]

If we set \( \lambda = 0 \) we get

\[ \det(A) = \frac{N}{i=1} \lambda_i \quad \Rightarrow \quad A \text{ is invertible if and only if } \lambda_i \neq 0 \text{ for all } i \]

The coefficient of \( \lambda^N \) is

\[ \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} a_{ii} = \text{tr}(A) \]

Terminology: The set of eigenvalues of a linear transformation is called its spectrum.
Note that if we group common factors, \( p(x) \)

\[ p(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_n)^{e_n} \]

where \( n \leq N \), \( e_i \geq 1 \), and \( \lambda_1, \ldots, \lambda_n \)

are distinct. We refer to \( e_i \) as the

**algebraic multiplicity** of \( \lambda_i \).

---

**Eigenspaces**

Let \( A \in \mathbb{C}^{n \times n} \) and let \( \lambda \) be an eigenvalue of \( A \).

The set

\[ E_{\lambda} = \{ \mathbf{v} \in \mathbb{C}^n \mid A\mathbf{v} = \lambda\mathbf{v} \} \]

is called the **eigenspace** corresponding to \( \lambda \).

Note \( E_{\lambda} = \ker (A - \lambda I) \) is a subspace.

**Fact 1** geometric multiplicity of \( \lambda \) \( \leq \) algebraic multiplicity of \( \lambda \).

**Proof:** See Mardia, Kent, & Bibby.
Let $A, B \in \mathbb{C}^{n \times n}$. We say $A$ and $B$ are similar if there exists an invertible matrix $T$ such that

$$A = TBT^{-1}.$$ 

**Exercise**  
Show that if $A, B$ are similar, then they have the same eigenvalues.

**Soln:** Suppose $Bv = \lambda v$. Set $w = Tv$.

Then $Aw = T(Bv) = \lambda Tv = \lambda w$.

A is said to be **diagonalizable** if it is similar to a diagonal matrix, i.e., there exists an invertible matrix $T$ such that

$$A = T\Lambda T^{-1},$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and $\lambda_i$ is an eigenvalue of $A$.

Not all matrices are diagonalizable.
Diagonalizable matrices are nice to work with in many settings. We can often translate to the diagonalized coordinate system and computations and concepts become easier.

**Application: Matrix Powers**

If \( A \) is diagonalizable, then

\[
A^k = T \Lambda^k T^{-1} \quad \text{for} \quad k = 1, 2, ..., n
\]

\[
= T \Lambda^k T^{-1}
\]

\[
= T \begin{bmatrix}
\lambda_1^k \\
\lambda_2^k \\
\vdots \\
\lambda_n^k
\end{bmatrix} T^{-1}
\]

Similarly, we can define

\[
\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{T \Lambda^k T^{-1}}{k!}
\]

\[
= T \left( \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \right) T^{-1}
\]

\[
= T \begin{bmatrix}
\sum \frac{\lambda_1^k}{k!} \\
\vdots \\
\sum \frac{\lambda_n^k}{k!}
\end{bmatrix} T^{-1} = T \begin{bmatrix}
e^{\lambda_1} \\
\vdots \\
e^{\lambda_n}
\end{bmatrix} T^{-1}
\]
Example  Suppose a system evolves according to
\[ x[t+1] = A \cdot x[t]. \] Then \[ x[t] = A^t \cdot x[0]. \]

Now \[ A^t = T \Lambda^t T^{-1} = T \begin{bmatrix} \lambda_1^t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^t \end{bmatrix} T^{-1} \]

so the system is stable provided \(|\lambda_i| \leq 1 \forall i\).

When is a matrix diagonalizable, and how can we determine \(T\)?

Observation  \( A \) is diagonalizable \(\iff\) \( \exists \) a basis of eigenvectors.

Proof: (\(\Rightarrow\)) If \( A \) is diagonalizable, then \( A = T \Lambda T^{-1} \).

Write \( T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \). From \( A T = T \Lambda \),

reading off each column, we see that \( A v_i = \lambda_i v_i \) for each \( i \). Since \( T \) is invertible, \( \{v_1, \ldots, v_n\} \) are LI and hence constitute a basis. (\(\Leftarrow\)). Suppose \( \{\overline{v_1}, \ldots, \overline{v_n}\} \) is a basis of \( \mathbb{C}^n \). Then \( A \overline{v_i} = \lambda_i \overline{v_i} \forall i \), and hence \( A \cdot T = \Lambda \cdot T \) where \( T = \begin{bmatrix} \overline{v_1} & \cdots & \overline{v_n} \end{bmatrix} \).

Since \( \{\overline{v_1}, \ldots, \overline{v_n}\} \) is LI, \( T^{-1} \) exists \(\Rightarrow\) \( A = T \Lambda T^{-1} \).
So when do the eigenvalues form a basis?

**Theorem** If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of a linear transformation $L$, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are corresponding eigenvectors, then $\sum \lambda_i \mathbf{v}_i = \mathbf{0}$ is LI.

**Corollary** If $A \in \mathbb{C}^{n \times n}$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $A$ is diagonalizable.

**Proof:** Suppose not, and that $\mathbf{r}_1 \mathbf{v}_1 + \cdots + \mathbf{r}_m \mathbf{v}_m = \mathbf{0}$ is the shortest nontrivial LC (all $r_i \neq 0$), renumbering the $\mathbf{v}_i$ if necessary.

Then

$$r_1 \lambda_1 \mathbf{v}_1 + r_2 \lambda_1 \mathbf{v}_2 + \cdots + r_m \lambda_1 \mathbf{v}_m = \mathbf{0} \quad (1)$$

and

$$r_1 \lambda_1 \mathbf{v}_1 + r_2 \lambda_2 \mathbf{v}_2 + \cdots + r_m \lambda_m \mathbf{v}_m = \mathbf{0} \quad (2)$$

by taking $L$ of both sides. Then (1) - (2) is

$$r_2 (\lambda_1 - \lambda_2) \mathbf{v}_2 + \cdots + r_m (\lambda_1 - \lambda_m) \mathbf{v}_m.$$

Since the $\lambda_i$ are distinct, we have a shorter nontrivial LC = \mathbf{0}, a contradiction. Note: If $\lambda_i$ are not distinct, say $\lambda_1 = \lambda_2$, then $m = 2$ because $\lambda_1 \mathbf{v}_1 - \lambda_2 \mathbf{v}_1 = \mathbf{0}$ and argument fails.
THE SPECTRAL THEOREM

The spectral theorem is an extremely important and powerful result on the diagonalizability of Hermitian matrices. Note: \( A = A^\dagger \Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle \).

**Proposition**

If \( A \) is an \( N \times N \) Hermitian matrix, then

(i) the eigenvalues of \( A \) are real

(ii) eigenvectors corresponding to distinct eigenvalues are orthogonal.

**Proof:** Suppose \( A \mathbf{v} = \lambda \mathbf{v}, \quad \mathbf{v} \neq \mathbf{0} \). Then

(i) \[ \lambda \| \mathbf{v} \|^2 = \langle \mathbf{v}^\dagger A \mathbf{v} \rangle = \langle \mathbf{v}^\dagger A^\dagger \mathbf{v} \rangle = \langle (A \mathbf{v})^\dagger \mathbf{v} \rangle = \langle \lambda \mathbf{v} \rangle = \lambda \| \mathbf{v} \|^2 \quad \Rightarrow \quad \lambda = \bar{\lambda}. \]

(ii) Suppose \( A \mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad A \mathbf{v}_2 = \lambda_2 \mathbf{v}_2, \quad \lambda_1 \neq \lambda_2 \). Then

\[ \langle A \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, A \mathbf{v}_2 \rangle = \lambda_1 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \lambda_2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \]

\[ \Rightarrow \quad (\lambda_1 - \lambda_2) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0 \quad \Rightarrow \quad \mathbf{v}_1 \perp \mathbf{v}_2 \quad \text{since} \quad \lambda_1 \neq \lambda_2. \]
Therefore, if the eigenvalues of $A$ are distinct, then $A$ is diagonalizable and there exists an orthonormal basis of eigenvectors with real eigenvalues.

This is also true even when the eigenvalues are not distinct.

**Spectral Theorem.** If $A$ is Hermitian, then there exists an orthonormal basis of $\mathbb{C}^n$ of eigenvectors of $A$.

**Proof:** Let $\lambda_i$ be an eigenvalue, and $\mathbf{v}_i$ a corresponding normalized eigenvector. Define

$$S_i := \text{span}\left(\mathbf{v}_i\right)$$

If $\mathbf{v} \in S_i$, then

$$\langle A\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, A\mathbf{v} \rangle$$

$$= \langle \mathbf{v}, \lambda_i \mathbf{v} \rangle$$

$$= \lambda \langle \mathbf{v}, \mathbf{v} \rangle$$

$$= 0$$

$$\Rightarrow A\mathbf{v} \in S_i$$

There we may consider $A_i : S_i \rightarrow S_i$, the restriction of $A$ to $S_i$. This is also a Hermitian matrix, so we can recursively construct a basis of $S_i \cong \mathbb{C}^{n-1}$ to obtain a basis for $\mathbb{C}^n$. 

A linear trans. \( L : V \to V \), \( V \) an IPS, is self-adjoint \( \iff \langle x, y \rangle = \langle y, Lx \rangle \forall x, y \in V \).

**Spectral Thm.** Suppose \( V \) is a fin. dim. complex IPS, and \( L : V \to V \) is self-adjoint.

Then \( \exists \lambda \) the eigenvalues of \( L \) are real, and \( V \) has an ONB consis. of eigenvectors of \( L \).

**Proof:** Let \( \lambda \) be an eigenvalue of \( L \), and \( u \), a consp. normalized eigenvector. Define
\[
S_1 = \text{span} \{ \mathbf{u} \}
\]

If \( \mathbf{v} \in S_1 \), then
\[
\langle L \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, L \mathbf{w} \rangle
\]
\[
= \langle \mathbf{v}, \lambda \mathbf{w} \rangle
\]
\[
= \lambda \langle \mathbf{v}, \mathbf{w} \rangle
\]
\[
= 0
\]
\[
\Rightarrow \ L \mathbf{v} \in S_1. \ Thus \ consider
\]
\[
L_1 : S_1 \to S_1, \ the \ restriction \ of \ L \ to \ S_1.
\]
This is also self-adjoint. By induction, \( \exists \) ONB of \( S_1 \), say \( \{ \mathbf{u}_2, \ldots, \mathbf{u}_n \} \), consisting of eigenvectors of \( L_1 \).

But EV of \( L_1 \) are EVs of \( L \), and \( S_1 \perp \mathbf{u}_1 \), so \( \{ \mathbf{u}_2, \ldots, \mathbf{u}_n \} \) is an ONB of EVs of \( L \).
We also need to establish the existence of at least one eigenvalue/e-vector of \( L \). Let \( \Phi \) be an isomorphism \( \Phi : V \rightarrow \mathbb{C}^N \). Then \( \Phi \circ L \circ \Phi^{-1} : \mathbb{C}^N \rightarrow \mathbb{C}^N \) is linear and thus has a matrix representation, say \( A \). We know \( A \) has \( N \) eigenvalues from charpoly and found thin of alg. Say \( A \xi = \lambda \xi \). Then

\[
\Phi \circ L \circ \Phi^{-1} \xi = \lambda \xi
\]

\[
\Leftrightarrow \quad L (\Phi^{-1} \xi) = \lambda (\Phi^{-1} \xi)
\]

so \( \lambda \) is an eigenvalue, \( \Phi^{-1} \xi \) an ejevector.

Matrix Interp

\[
A = A^H \quad \Rightarrow \quad A = U \Lambda U^H
\]

where \( U^H U = U U^H = I \) etc.,

\[
\Lambda = \text{diag} (\lambda_1, \ldots, \lambda_n), \, \lambda_i \in \mathbb{R}.
\]
Spectral Thm. matrix form

From the spectral theorem, we know that

\[ A \cdot U = U \cdot \Lambda \cdot U \]

where \( U^H U = I_{n \times n} \) and \( \Lambda = \text{diag} \,( \lambda_1, \ldots, \lambda_n ) \), \( \lambda_i \in \mathbb{R} \).

Claim: \( U \cdot U^H = I_{n \times n} \). Need to show

\[ UU^H \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{C}^n. \] Write \( \mathbf{v} = U \cdot \mathbf{w} \).

Then \( U \cdot U^H \mathbf{v} = UU^H U \mathbf{w} \)

\[ = U \mathbf{w} \]

\[ = \mathbf{v} \quad \square. \]

If \( U^H U = UU^H = I_{n \times n} \), we say \( U \) is unitary. Unitary matrices are norm preserving:

\[ \| U \mathbf{x} - U \mathbf{y} \|^2 = (U \mathbf{x} - U \mathbf{y})^H (U \mathbf{x} - U \mathbf{y}) \]

\[ = (\mathbf{x} - \mathbf{y})^H U^H U (\mathbf{x} - \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|. \]

So if \( A^H = A \), then

\[ A = U \Lambda U^H \]

for some unitary \( U \), real diagonal \( \Lambda \).
\[ A x = U \Lambda U^H x \]

\[ = U A \begin{bmatrix} u_1^H x \\ \vdots \\ u_N^H x \end{bmatrix} = U \begin{bmatrix} \lambda_1 u_1^H x \\ \vdots \\ \lambda_N u_N^H x \end{bmatrix} \]

\[ = \sum_{i=1}^{N} \lambda_i \langle u_i, x \rangle u_i^H x \]

\[ = \left( \sum_{i=1}^{N} \lambda_i u_i u_i^H \right) x \]

---

**Example**

\[ A = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix} \]

\[ U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1.5 & 0 \\ 0 & .5 \end{bmatrix} \]
If $A^T = A$, then there exists a basis of $\mathbb{R}^n$ consisting of orthogonal real eigenvectors.

Thus, we can write

$$A = U \Lambda U^T,$$

where $U^T U = U U^T = I$, $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$, $\lambda_i \in \mathbb{R}$.

The proof is the same as for complex matrices, except now it requires some effort to show that there exist eigenvalues with real eigenvectors.

If $U^T U = U U^T = I$, $U \in \mathbb{R}^{n \times n}$, we say $U$ is an orthogonal matrix. If $n=2$,

$$U = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ for some } \theta \in [0, 2\pi).$$

So $A = U \Lambda U^T$

= rotate, scale, rotate back
Proof that every real, symmetric matrix has if \( A \in \mathbb{R}^{n \times n} \), 
\( A^T = A \), then \( A \) has \( N \) real eigenvalues, and the corresponding real eigenvectors.

Since \( A^T = A \), \( A^H = A \), and thus the complex eigenvalues \( \lambda_1, \ldots, \lambda_N \) of \( A \) are real. Suppose \( \lambda \) is an eigenvalue of \( A \). Let's find a real eigenvector. Let \( v \) be a complex eigenvector. Two cases:

(i) \( v \) is purely imaginary: Then set
\[
\tilde{v} = i \cdot v.
\]
Then \( \tilde{v} \in \mathbb{R}^N \) and
\[
A \tilde{v} = i \cdot Av = i \cdot \lambda v = \lambda \tilde{v}.
\]

(ii) \( v \) is not purely imaginary. Then set
\[
\tilde{v} = v + \overline{v} \in \mathbb{R}^N, \quad \tilde{v} \neq 0.
\]
Then
\[
A \tilde{v} = Av + A\overline{v} = \lambda v + \overline{\lambda v}
= \lambda v + \overline{\lambda v}
= \lambda (v + \overline{v})
= \lambda \tilde{v}.
\]
POSITIVE (SEMI-) DEFINITE MATRICES

Let $A \in \mathbb{C}^{n \times n}$ and $A^H = A$. Notice that for any $x \in \mathbb{C}^n$, $x^H A x \in \mathbb{R}$, because

$$(x^H A x)^H = x^H A^H x = x^H A x. \quad (1 \times 1)$$

We say $A$ is positive definite if

$x^H A x > 0 \quad \forall x \neq 0$.

We say $A$ is positive semi-definite (or non-negative definite) if

$x^H A x \geq 0 \quad \forall x$.

Notation:

$A > 0 \quad (PD)$

$A \geq 0 \quad (PD)$
Eigenvalues of PD/PSD Matrices

**Theorem**

A is PD (PSD) \( \iff \) the eigenvalues of A are all positive (nonnegative).

**Proof:** By the spectral theorem, \( \exists \lambda_i \in \mathbb{R}, u_i \in \mathbb{C}^N \) s.t.

\[
A u_i = \lambda_i u_i, \quad i = 1, \ldots, N.
\]

(\( \Rightarrow \)) \( \lambda_i = \langle u_i \mid A u_i \rangle = \langle u_i \mid u_i \rangle \Rightarrow 0 \text{ if } A \text{ PD}
\]

(\( \Leftarrow \)) Suppose \( x \neq 0 \). Then

\[
\langle x^H A x \rangle = \langle x^H \left( \sum_{i=1}^{N} \lambda_i u_i u_i^H \right) x \rangle
\]

\[
= \sum_{i=1}^{N} \lambda_i \langle x^H u_i \cdot u_i^H x \rangle
\]

\[
= \sum_{i=1}^{N} \lambda_i \| u_i^H x \| ^2
\]

\[
\begin{cases} > 0 & \text{if } \lambda_i > 0 \quad \forall i \in \mathbb{R} \\ \geq 0 & \text{if } \lambda_i \leq 0 \quad \forall i \in \mathbb{R} \end{cases}
\]
Implications

1. If $A$ is PSD, then $\det A = \prod \lambda_i \geq 0$

2. $A$ is PD $\Rightarrow$ $\det A > 0$

3. $A$ is PD $\Rightarrow$ $A$ is invertible (since $\det A \neq 0$, and since the rank of a diagonalizable matrix is the number of nonzero eigenvalues)

4. If $A$ is PD, $A = U\Lambda U^H$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ for all $i$. Then

$$A^{-1} = U\Lambda^{-1}U^H, \quad \Lambda^{-1} = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$$

because

$$U\Lambda U^H \cdot U\Lambda^{-1}U^H = U\Lambda \Lambda^{-1}U^H = UU^H = I_{n \times n}$$

(true for any nonsingular Hermitian matrix)

and therefore $A^{-1}$ is also PD.

Application: Covariance matrix of multivariate Gaussian.
\(5\) If \(A\) is PSD, then \(\exists\) a full rank matrix \(B\) s.t.
\[
A = B^H B \quad \text{(matrix square root)}.
\]

\[
A = \begin{bmatrix}
\begin{array}{c|c}
A_1 & A_2 \\
\hline
\end{array}
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_p
\end{bmatrix}
\begin{bmatrix}
A_1^H \\
\vdots \\
A_p^H
\end{bmatrix}
\]

\(\Rightarrow\) \(B = A_1 \begin{bmatrix}
\sqrt{\lambda_1} \\
\vdots \\
\sqrt{\lambda_p}
\end{bmatrix}\)

where \(\lambda_1, \ldots, \lambda_p > 0\).

\(6\) If \(A = (a_{ij})\) is PD (PSD), then the diagonal entries of \(A\), \(a_{ii}\), are positive (non-negative).

The converse is not true.
Grammians

Let \( v_1, \ldots, v_N \in V \), an IPS. Set

\[
G = \begin{bmatrix}
\langle v_1, v_1 \rangle & \cdots & \langle v_1, v_N \rangle \\
\langle v_2, v_1 \rangle & \cdots & \langle v_2, v_N \rangle \\
\vdots & \ddots & \vdots \\
\langle v_N, v_1 \rangle & \cdots & \langle v_N, v_N \rangle
\end{bmatrix} \in \mathbb{C}^{N \times N}
\]

Arises in projection, least squares: \( G = A^H A \)

**Theorem:** \( G \) is PSD, and \( G \) is PD \( \iff \) \( v_1, \ldots, v_N \) are LI.

**Proof:** Let \( x = [x_1 \ldots x_N]^T \in \mathbb{C}^N \). Then

\[
x^H G x = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \overline{x_j} \langle v_j, v_i \rangle
\]

\[
= \sum_{i} x_i \langle \sum \overline{x_j} v_j, v_i \rangle
\]

\[
= \langle \sum \overline{x_j} v_j, \sum_i \overline{x_i} v_i \rangle
\]

\[
\geq 0
\]

with equality iff \( \sum \overline{x_i} v_i = 0 \).
Corollary: 
\[ |\langle v_1, v_2 \rangle| \leq \sqrt{\langle v_1, v_1 \rangle \cdot \langle v_2, v_2 \rangle}, \text{ w/ equality if } v_1, v_2 \text{ are LI.} \]

Pf: Consider 
\[ G = \begin{bmatrix} 
\langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle \\
\langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle 
\end{bmatrix} \]

\[ G \text{ is PSD} \Rightarrow \det G \geq 0 \]

\[ \det G = \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle - \langle v_2, v_1 \rangle^2 \langle v_1, v_2 \rangle \geq 0 \]

w/ equality iff \( v_1, v_2 \) LI.
THE MULTIVARIATE GAUSSIAN DISTRIBUTION

The random vector \( \mathbf{X} \in \mathbb{R}^n \) has a MVG dist. if it has the joint density

\[
f_{\mu,\Sigma}(\mathbf{x}) = \left(\frac{2\pi}{\Sigma} \right)^{\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right\}
\]

where \( \mu \in \mathbb{R}^n \), and \( \Sigma \) is P.D. Note that \( \Sigma^{-1} \) is also P.D.

Using the spectral theorem and properties of P.D. matrices, we can prove several important properties:

- \( \int_{\mathbb{R}^n} f_{\mu, \Sigma}(\mathbf{x}) \, d\mathbf{x} = 1 \)
- \( \left\{ \mathbf{x} : f_{\mu, \Sigma}(\mathbf{x}) = c \right\} \) is an ellipse
- \( \mathbb{E} \left[ (\mathbf{X} - \mu)(\mathbf{X} - \mu)^T \right] = \Sigma \)
- Generate realizations of \( \mathbf{X} \) from independent univariate Gaussian RVs.
Contours of $f_{\mu \Sigma}$ \iff\ Contours of $(\chi - \mu)^T \Sigma^{-1} (\chi - \mu)$

To ease visualization, let's focus on the case $N=2$.

We'll consider 3 cases that increase in generality.

\[ \Sigma = \text{multiple of identity} \]

\[
\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{pmatrix}
\]

\[
(\chi - \mu)^T \Sigma^{-1} (\chi - \mu) = \frac{(\chi_1 - \mu_1)^2}{\sigma_1^2} + \frac{(\chi_2 - \mu_2)^2}{\sigma_2^2} = c
\]

$\Rightarrow$ circle

$f_{\mu \Sigma} = 2d$ bell surface, radially symmetric, circular contours
\[
\Sigma = \text{diagonal} \\
\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \quad \Rightarrow \quad \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{pmatrix} \\

(x - \mu)^T \Sigma^{-1} (x - \mu) = \frac{(x_1 - \mu_1)^2}{\sigma_1^2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2} = C \\

\Rightarrow \quad \text{ellipse} \\

\text{axes are coordinate axes} \\
\text{slice along any axis = bell curve} \\

\Sigma = \text{arbitrary PD matrix} \\
\Sigma = U \Lambda U^T \quad \Rightarrow \quad \Sigma^{-1} = U \Lambda^{-1} U^T \\
(x - \mu)^T \Sigma^{-1} (x - \mu) = (x - \mu)^T U \Lambda U^T (x - \mu) \\
= [U^T (x - \mu)]^T \Lambda [U^T (x - \mu)] \\
= (\tilde{x} - \tilde{\mu})^T \Lambda (\tilde{x} - \tilde{\mu})
where \( \tilde{x} = U^T x \), \( \tilde{\mu} = U^T \mu \)

and \( U^T \) is a rotation

\[
\frac{(x_1 - \tilde{\mu}_1)^2}{\lambda_1} + \frac{(x_2 - \tilde{\mu}_2)^2}{\lambda_2} = c
\]

\Rightarrow \text{rotated ellipse}

Note: \( U^T x = \begin{bmatrix} u_1^T x \\ \vdots \\ u_N^T x \end{bmatrix} \) = expansion coefficients of \( x \)

in the basis \( \{u_1, \ldots, u_N\} \).

\Rightarrow u_i \text{ are the axes of the ellipses,}
\quad \lambda_i \text{ determine relative length}

\( N > 2 \)

\text{contours are arbitrary ellipsoids}
Theorem: If \( X \sim \mathcal{N}(\mu, \Sigma) \), then

\[
E[(X-\mu)(X-\mu)^T] = \Sigma.
\]

Proof:
First, consider the case \( \mu = 0 \), \( \Sigma = I_{N \times N} \).

Then

\[
E[X X^T] = \begin{bmatrix}
E[X_1^2] & E[X_1X_2] \\
E[X_2X_1] & E[X_2^2]
\end{bmatrix}
\]

We want to show this is \( I \).
Consider

\[
E[X_1^2] = \int_{\mathbb{R}^N} x_1^2 f_{\mu, I}(x) \, dx
\]

\[
= \int x_1^2 (2\pi)^{-N/2} e^{-\frac{1}{2}x_1^T I x_1} \, dx_1
\]

\[
= \int x_1^2 (2\pi)^{-1/2} e^{-\frac{1}{2}x_1^2} \, dx_1, \quad \int (2\pi)^{-1/2} e^{-\frac{1}{2}x_2^2} \, dx_2
\]

\[
\cdots \int (2\pi)^{-1/2} e^{-\frac{1}{2}x_N^2} \, dx_N
\]

\[
= (\text{Variance of } N(0,1)) \cdot 1 \cdots 1
\]

\[
= 1.
\]
Similarly, $E[\chi_i^2] = 1 \quad \forall i$.

Now consider

$$E[\chi_1 \chi_2] = \int \chi_1 \chi_2 \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}(\chi_1^2 + \chi_2^2)} \, d\chi$$

$$= \int \chi_1 \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}\chi_1^2} \, d\chi_1 \cdot \int \chi_2 \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}\chi_2^2} \, d\chi_2$$

$$= 0$$

Similarly, $E[\chi_i \chi_j] = 0 \quad \forall i \neq j$.

Now consider general $X \sim N(\mu, \Sigma)$.

Fact: If $X \sim N(\mu, \Sigma)$ and $Y = AX + b$, then $Y \sim N(A\mu + b, A\Sigma A^T)$.

Proof: Use characteristic function, $E(e^{it^T Y})$.

Set $Y = \Lambda^{-1/2} U^T(X - \mu)$, where $\Sigma = U\Lambda U^T$.

Then $Y \sim N(0, I)$, so

$$E[XX^T] = E[U\Lambda^{1/2} Y Y^T U^{1/2}]$$

$$= U\Lambda^{1/2} E[YY^T] U^{1/2} = U\Lambda^{1/2} I U^{1/2}$$

$$= U\Lambda U^T = \Sigma.$$
Suppose you wish to generate a realization of $X \sim N(m, R)$, but you only have access to a univariate, standard normal RNG, such as Matlab’s `randn`.

1. Generate $Y \sim N(0, I)$:
   ```matlab
   >> Y = randn (N, 1);
   ``

2. Compute spectral decompostion of $R$:
   ```matlab
   >> [U, L] = eig (R);
   ``

3. Transform $X = UL^{\frac{1}{2}}Y + m$:
   ```matlab
   >> X = U * sqrt (L) * Y + m;
   ```

Then $X \sim N (UL^{\frac{1}{2}} \cdot 0 + m, UL^{\frac{1}{2}} \cdot I \cdot L^{\frac{1}{2}}U^T)$

$\sim N (m, R)$
EIGENFILTERS

\[ f[n] \xrightarrow{ \oplus } h \xrightarrow{ } y[n] \]

- \( f[n] \): 0 mean, wss random process (signal)
- \( v[n] \): 0 mean, white noise process, \( E[v[n]v[n]] = \sigma^2 \)
- \( h \): FIR filter, length \( m \)

Denote

\[
\begin{bmatrix}
  f[n] \\
  f[n-m+1]
\end{bmatrix}, \quad
\begin{bmatrix}
  v[n] \\
  v[n-m+1]
\end{bmatrix}
\]

Then, \( y[n] = h^H(f[n] + v[n]) \)

Output power due to signal:

\[
E[|h^Hf[n]|^2] = E[h^Hf[n] \cdot f[n] \cdot h^H] = h^H R h
\]

Output power due to noise:

\[
E[|h^Hv[n]|^2] = \sigma^2 h^H h.
\]

Maximize ratio:

\[
\max_{h \neq 0} \frac{h^H R h}{\sigma^2 h^H h} = \frac{1}{\sigma^2}
\]

\[ \Rightarrow h = \text{eigenvector of } R \text{ corresponding to largest eigenvalue}. \]
Next time we'll have a packet on the Rayleigh quotient, pre-PCA, and include this as an example.
Motivation 1: Low rank approximation

Given \( x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^N, p < N \). Seek \( A \in \mathbb{R}^{N \times p}, \mu \in \mathbb{R}^N \), \( y^{(1)}, \ldots, y^{(n)} \in \mathbb{R}^p \) s.t.

\[
x^{(i)} \approx \mu + A \cdot y^{(i)}, \quad i = 1, \ldots, n.
\]

\( N = 2 \\
p = 1 \\
A = [a] \quad (1 \times 1) \)
Mathematically, we want to minimize

$$\sum_{i=1}^{n} \| x^{(i)} - \mu - Ay^{(i)} \|^2$$

w.r.t. \( \mu \in \mathbb{R}^N \), \( y^{(i)} \in \mathbb{R}^p \), and \( A \in \mathbb{R}^{N \times p} \), \( A^TA = I \).

**Motivation 2: Max Variance Subspace**

Given \( x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^N \), \( p < N \), construct \( y^{(1)}, \ldots, y^{(n)} \in \mathbb{R}^p \) as follows:

1. Find \( a_1 \in \mathbb{R}^N \), \( \| a_1 \| = 1 \) s.t.

   $$y^{(i)}_1 := a_1^T (x^{(i)} - \bar{x})$$

   has maximum variance (sample)

2. Having found \( a_1, a_2, \ldots, a_{k-1} \), find \( a_k \in \mathbb{R}^N \), \( \| a_k \| = 1 \), \( a_k \perp a_l, l < k \) s.t.

   $$y^{(i)}_k := a_k^T (x^{(i)} - \bar{x})$$

   has maximum variance (sample)
It turns out that both of the problems have the same solution: Define the sample mean
\[ \bar{x} := \frac{1}{n} \sum_{i=1}^{n} x^{(i)} \]
and sample covariance matrix
\[ S := \frac{1}{n} \sum_{i=1}^{n} (x^{(i)} - \bar{x})(x^{(i)} - \bar{x})^T. \]

Let \( S = U \Lambda U^T \)
\( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0. \)
(Why is \( S \geq 0 \)?)

**Theorem** The solutions to both the low-rank approx and max variance approx is given by
\[ a_i = u_i, \quad \mu = \bar{x}, \quad y^{(i)} = A^T(x^{(i)} - \bar{x}) \]

For \( x \in \mathbb{R}^n \), we say \( y_k = \frac{1}{\lambda_k} (x - \bar{x}) \)
is the \( k \)th principal component of \( x \).
We will verify this for the max variance formulation.

$$\text{var} \left( \sum_{k=1}^{n} y_k^{(i)} \right) = \frac{1}{n} \sum_{i=1}^{n} (y_k^{(i)} - \overline{y}_k)^2$$

where

$$y_k^{(i)} = q_k^T (x^{(i)} - \overline{x})$$

$$\overline{y}_k = \frac{1}{n} \sum_{i=1}^{n} y_k^{(i)} = \frac{1}{n} \sum_{i=1}^{n} \left( a_k^T (x^{(i)} - \overline{x}) \right)$$

$$= a_k^T (\overline{x} - \overline{x}) = 0$$

So

$$\text{var} \left( \sum_{k=1}^{n} y_k^{(i)} \right) = \frac{1}{n} \sum_{i=1}^{n} a_k^T (x^{(i)} - \overline{x}) (x^{(i)} - \overline{x}) a_k$$

$$= a_k^T S a_k$$

Since $S > 0$, the result follows from the following more general result.
Let $W \geq 0$, $W \in \mathbb{C}^{N \times N}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of $W$, $v_1, \ldots, v_N$ be the corresponding eigenvectors. Then
\[
\max_{\|a\|=1} a^H Wa = \lambda_1,
\]
and the maximum is achieved by $v_1$. In addition,
\[
\max_{\|a\|=1, a \in \mathbb{C}^N} a^H Wa = \lambda_k,
\]
and the maximum is achieved by $v_k$.

Proof: Write $W = U \Lambda U^H$. Let $a \in \mathbb{C}^N$, $\|a\|=1$.

Then $a^H Wa = a^H \Lambda a = (U^H a)^* \Lambda (U^H a)$.

Set $b = U^H a$. Then $\|b\|=1$, and
\[
\max_{\|a\|=1} a^H Wa = \max_{\|b\|=1} b^H \Lambda b.
\]

Now $b^H \Lambda b = \sum_{j=1}^{N} \lambda_j |b_j|^2$. This is maximized when $b = [1, 0 \ldots 0]^T \Rightarrow a = Ub = v_1$.

Uniqueness? If $\lambda_1 > \lambda_2$, then maximizer is (complex) unit scalar multiple of $v_1$. Else, any vector in eigenspace of $\lambda_1$. 

Rayleigh Quotient
Similarly, if \( a \perp \xi_{2}, \ldots, \xi_{k}, 3 \), then

\[
b = U^{H}a = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{k} \\ \vdots \\ b_{n} \end{bmatrix}, \quad \text{so} \quad b^{H}A b = \sum_{1}^{p} \lambda_{j} \|b_{j}\|^2
\]

is maximized when \( b_{k} = 1 \Rightarrow a = \xi_{k} \).

Similar comments on uniqueness.

The quantity \( \frac{1_{1} + \cdots + 1_{p}}{1_{1} + \cdots + 1_{n}} \) is the \% total

variation explained by the first \( p \) PCs,

used to choose \( p \).

**Simulation Versions**

Now suppose we have a RV \( X \in \mathbb{R}^{n} \) as opposed to

a random sample. The previous results generalize to this

setting. Here the problems are (assume \( X \) is zero mean, \( E[XX] = R \))

- \( \min \ E[\|X - AY\|^2] \)
- \( \max \ a^{T}Ra \)

\[ A^{T}A = I_{n} \]

\( Y = kX \), \( k \in \mathbb{R}^{p \times n} \)

Solution: \( A = \) first \( p \) eigenvectors of \( R \), \( K = A^{T} \)
The proofs are essentially the same.

**Karhunen–Loève Transform**

Let \( X \in \mathbb{R}^n \) be a zero mean \( RV \) with \( \mathbb{E}[XX^T] = R \).

Write \( R = U \Lambda U^T \), orthogonal, \( \Lambda_1 \geq \cdots \geq \Lambda_N \geq 0 \).

Set \( Y = U^T X \). Then

\[
\mathbb{E}[Y Y^T] = U^T R U = \Lambda \quad \text{(uncorrelated)}
\]

\[ \Rightarrow \quad U^T \] is a "whitening filter,"

called the KLT. The first \( p \) variables of \( Y \) give the best \( p \)-dim approximation to \( X \)

in the sense of minimizing \( \mathbb{E}[\| X - A \begin{bmatrix} \hat{Y} \\ \vdots \\ \hat{Y}^p \end{bmatrix} \|_2^2] \)

for some \( A \). The KLT is often used as a

theoretical tool in communication/compression theory.
THE SINGULAR VALUE DECOMPOSITION

Theorem] Let $A \in \mathbb{C}^{m \times n}$. There exist unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(m,n)$, s.t.

$$A = U \Sigma V^H$$

where $\Sigma \in \mathbb{R}^{m \times n}$, $\Sigma = \text{diag} (\sigma_1, \ldots, \sigma_p)$.

If $A \in \mathbb{R}^{m \times n}$, then $U, V$ are real, orthogonal matrices.

$m \leq n$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{bmatrix} \begin{bmatrix} V^H \end{bmatrix}$$

$m \geq n$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{bmatrix} \begin{bmatrix} V^H \end{bmatrix}$$
The SVD exists for arbitrary matrices.

\( \sigma_1, \ldots, \sigma_p \) are called the \textit{singular values} of \( A \).

\( U = [u_1, \ldots, u_m] \) \( \rightarrow \) \textit{left singular vector}

\( V = [v_1, \ldots, v_n] \) \( \rightarrow \) \textit{right ""}

Lemma \hspace{1cm} \text{Let } A \in \mathbb{C}^{m \times n}. \text{ Then } A^H A \text{ and } AA^H \text{ are PSD, and they have the same positive eigenvalues, with the same multiplicities.}

\[ x^H A^H A x = \| A x \|^2 \geq 0 \]

\[ x^H A A^H x = \| A^H x \|^2 \geq 0 \]

Suppose \( A^H A \nu = \lambda \nu, \lambda > 0, \nu \neq 0 \). Set \( u = A \nu \). Then \( AA^H u = A A^H A \nu = A \lambda \nu = \lambda A \nu = \lambda u \). Similarly, suppose \( AA^H u = \lambda u \).

Set \( \nu = A^H u \). Then \( A^H A \nu = A^H A A^H u = A^H \lambda u = \lambda A^H u = \lambda \nu \). What did we use \( \lambda > 0 \)?

If \( \lambda = 0 \), then \( u = A \nu = \lambda \nu = 0 \), so \( u \) is not an eigenvector.

\text{Multiplicities: Since } A^H A \text{ and } AA^H \text{ are diagonalizable, geo. mult} = \text{alg. mult.} \text{ Also, from construction, if } v_1, v_2 \text{ are in the same eigenspace, then } \langle Av_1, Av_2 \rangle = 0, \text{ so eigenspaces map to eigenspaces.}
Lemma 1 Let $A \in \mathbb{C}^{m \times n}$. Then

1. $R(A)^{\perp} = N(A^{H})$
2. $R(A^{H})^{\perp} = N(A)$.

Proof of 1: $N(A^{H}) \subseteq R(A)^{\perp}$

Let $x \in N(A^{H})$ and $y \in R(A)$, so that $y = Az$ for some $z \in \mathbb{C}^{n}$.

Then $\langle x, y \rangle = \langle x, Az \rangle = \langle A^{H}x, z \rangle = \langle 0, z \rangle = 0$.

$R(A)^{\perp} \subseteq N(A^{H})$: Suppose $x \perp y \forall y \in R(A)$.

Then $x \perp Az \forall z \in \mathbb{C}^{n}$. That is

$0 = \langle x, Az \rangle = \langle A^{H}x, z \rangle \forall z \in \mathbb{C}^{n}$

$\Rightarrow A^{H}x = 0 \Rightarrow x \in N(A^{H})$.

Proof of 2: Apply 1 with $A \rightarrow A^{H}$.

Proof of SVD Theorem

Let $A^{H}A = V \Lambda V^{H}$ be the spectral decomposition of $A^{H}A$, with positive eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r$, $r \leq p$ and $V = [v_1 \ldots v_n]$. For $i = 1, \ldots, r$, set

$u_i := \frac{1}{\sqrt{\lambda_i}} A v_i$
Notice
\[
\langle u_i, u_j \rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} (A u_i)^H (A u_j) = \lambda_i^{\frac{1}{2}} A^H A u_j
\]
\[
\frac{\lambda_i}{\sqrt{\lambda_i \lambda_j}} u_i^H u_j = \delta_{i,j}, \quad 1 \leq i,j \leq r.
\]
Let \( u_{r+1}, \ldots, u_m \in \mathbb{C}^n \) be such that
\( U = [u_1, \ldots, u_m] \) is unitary. Then
\[
A A^H = \sum_{i=1}^{r} \lambda_i u_i u_i^H + \sum_{i=r+1}^{m} 0 \cdot u_i u_i^H
\]
\[
= U \begin{bmatrix} \lambda_1 & \cdots & \lambda_r \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \end{bmatrix} U^H
\]
We will show
\[
U^H AV = \text{diag}(\sigma_1, \ldots, \sigma_p), \quad \sigma_i = \sqrt{\lambda_i}
\]
The \((i,j)\) element of \( U^H AV \) is
\[
\epsilon_i^H (U^H AV) \epsilon_j = (U \epsilon_i)^H A V \epsilon_j = \epsilon_i^H A \epsilon_j.
\]
Case 1: \( i \leq r \):
\[
U_i^H A u_j = \frac{1}{\sqrt{\lambda_i}} (A u_i)^H A u_j = \frac{1}{\sqrt{\lambda_i}} \epsilon_i^H A^H A u_j
\]
\[
= \epsilon_i^H u_j = \sqrt{\lambda_i} S_{i,j}.
\]
Here, use 2nd lemma to show \( u_1, \ldots, u_m \in E_0 = N(AA^H) \) to improve this argument.
Case 2: \( i > r \):

\[ A \cdot A^H u_i = 0 \implies A^H u_i \in N(A) \]

Obviously, \( A^H u_i \in R(A^H) \)

So \( A^H u_i \in N(A) \cap R(A^H) = \{0\} \)

because \( N(A) \), \( R(A^H) \) are orthogonal compliments.

Thus \( A^H u_i = 0 \). Therefore

\[ u_i \cdot A^j = (A^H u_i)^H v_j = 0. \]

For the case of real \( A \), use the same argument, but apply the spectral theorem for real symmetric matrices.

Observations:

- Right singular vectors = eigenvectors of \( A^H A \)
- Left " " = " " \( A A^H \)
- Positive singular values = square roots of positive eigenvalues of \( A^H A / A A^H \).

This suggests a method for computing the SVD, but not a very efficient one.
This is now redundant,
based on 1st
sticky above.
Claim: $N(A) = N(A^H A)$.

(C) $\mathbf{x} \in N(A) \Rightarrow A\mathbf{x} = 0 \Rightarrow A^H A \mathbf{x} = 0 \Rightarrow \mathbf{x} \in N(A^H A)$

(>) $\mathbf{x} \in N(A^H A) \Rightarrow A^H A \mathbf{x} = 0 \Rightarrow \mathbf{x}^H A^H A \mathbf{x} = 0$

$\Rightarrow \|A\mathbf{x}\|^2 = 0 \Rightarrow A \mathbf{x} = 0 \Rightarrow \mathbf{x} \in N(A)$.

Therefore

$$\text{rank } (A) = n - \dim (N(A))$$

$$= n - \dim (N(A^H A))$$

$$= \text{rank } (A^H A)$$

$$= r \; (# \text{ of non-zero eigenvalues})$$

$$= \# \text{ of non-zero singular values}.$$
Minimun Norm Solution for Underdetermined L.S.

Recall the least squares problem:

$$\min_{\theta \in \mathbb{F}^p} \| x - A\theta \|^2$$

where $x \in K^N$, $A \in K^{N \times p}$, $K = \mathbb{R}$ or $\mathbb{C}$.

If $\text{rank}(A) = p$, the solution is $\hat{\theta} = (A^TA)^{-1}A^T x$.

What if $\text{rank}(A) < p$? We could just eliminate redundant columns. But suppose we don't want to do that. Then there are infinitely many minimizers $\{ \theta = \theta_0 + v \mid v \in \text{N}(A) \}$, where $\theta_0$ is a solution.

$\dim > 0$ if $\text{rank}(A) < p$.

Let's find the solution with minimum norm.

Diagonal Case
First suppose \( A = \Sigma \) where

\[
\Sigma = \begin{pmatrix}
\sigma_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_r
\end{pmatrix}_{N \times p}
\]

\( \sigma_1 \geq \cdots \geq \sigma_r > 0 \). Notice \( \Pi_A x = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ \vdots \\ 0 \end{pmatrix} \)

Then a least squares solution is a solution to

\[
\begin{pmatrix}
\sigma_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_r
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\vdots \\
\theta_p
\end{pmatrix} =
\begin{pmatrix}
x_1 \\
\vdots \\
x_r \\
0
\end{pmatrix}
\]

Then one solution is \( \theta_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ 0 \cdots 0 \end{pmatrix}^T \).

An arbitrary solution has the form \( \theta = \theta_0 + \theta_1 \), where \( \theta_1 \in N(A) \), i.e. \( \theta_1 = \begin{pmatrix} 0 & \cdots & 0 & \theta_{r+1} & \cdots & \theta_p \end{pmatrix}^T \).

Now \( \| \theta \|^2 = \| \theta_0 \|^2 + \| \theta_1 \|^2 > \| \theta_0 \|^2 \),

so the minimum norm solution is

\[
\theta_0 = \begin{pmatrix}
\frac{1}{\sigma_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{\sigma_r}
\end{pmatrix}_{p \times N}
\begin{pmatrix}
x_1 \\
\vdots \\
x_r \\
0
\end{pmatrix} = \Sigma^+ x
\]
General Case

Let \( A = U \Sigma V^H \) (SVD). Then

\[
\min_\theta \| x - A \theta \| = \min_\theta \| x - U \Sigma V^H \theta \|
\]

\[
= \min_\theta \| U^H x - \Sigma V^H \theta \|
\]

Now set \( \phi = V^H \theta \) (change of variables), \( y = U^H x \). Then

\[
\hat{\phi} = \Sigma^+ y = \Sigma^+ U^H x
\]

\[
\therefore V^H \hat{\theta} = \Sigma^+ U^H x
\]

\[
\Rightarrow \hat{\theta} = V \Sigma^+ U^H x
\]

This holds for all \( A \), even if \( \text{rank}(A) = p \) or if \( A \) is invertible.
Consider the vector space $\mathbb{C}^{m \times n}$ regarded as $m \times n$ matrices w/ complex entries. This space can be equipped w/ a norm in several ways.

An example of an "operator norm" we will consider is the $\ell_2$ norm

$$\|A\|_{\ell_2} = \sup_{\|x\|_{\ell_2} = 1} \|Ax\|_{\ell_2}$$

**Fact:** $\|A\|_{\ell_2}$ = largest singular value of $A$.

Given a matrix $A$ with rank $r$, we could ask what matrix $B$ of rank $k < r$ minimized $\|A - B\|_{\ell_2}$.

**Theorem** Let $A \in \mathbb{C}^{m \times n}$, $\text{rank}(A) = r$, and $k < r$.

Let $A = U \Sigma V^H$ be the SVD of $A$. Set

$$A_k := \sum_{i=1}^{k} \sigma_i u_i v_i^H = U \Sigma_k V^H$$

where $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k)$. Then

$$\|A - A_k\|_{\ell_2} = \min_{B: \text{rank}(B) = k} \|A - B\|_{\ell_2} = \sigma_{k+1}$$
Proof: \( A - A_k = U \cdot (\Sigma - \Sigma_k) V^H \) so

\[ \| A - A_k \|_2 = \text{largest sing. val.} = \sigma_{k+1}. \]

Let \( B \in \mathbb{C}^{m \times n} \), \( \text{rank} (B) = k \). Let \( z \in \mathbb{C}^n \) s.t.
\[ \| z \|_2 = 1 \] and

\[ z \in \text{N}(B) \cap \text{span} \{ v_1, \ldots, v_{k+1} \} \neq \emptyset \]

\[ \uparrow \]
\[ \text{dim} = n - k \]
\[ \text{dim} = k + 1 \]

Then
\[ \| A - B \|_2^2 > \| (A - B) z \|_2^2 = \| A z \|_2^2 \]

\[ = \left\| \sum_{i=1}^{k+1} \sigma_i \cdot (v_i^H z) \cdot v_i \right\|^2 \]

\[ = \sum_{i=1}^{k+1} \sigma_i^2 - (v_i^H z)^2 \geq \sigma_{k+1}^2 \] \( \blacksquare \)

Question: Is \( B \) unique? Yes, if \( \sigma_k > \sigma_{k+1} \).

**Effective Rank**

Plot of singular values:

Numerically stable pseudocode: first set small singular values to zero.
TOTAL LEAST SQUARES

Given \((x_1, y_1), \ldots, (x_m, y_m)\) find \(a, b\) s.t.

\[ y_i = ax_i + b \]

LE: vertical errors
TLS: distance to line

Motivation

- Stability for large slope
- Uncertainty in \(x_i\)'s.

\[ \text{distance}^2 = (\Delta x)^2 + (\Delta y)^2 \]

so minimize distance \(\leftrightarrow\) minimize total error in \(x\) and \(y\).
Homogeneous case (see book for inhomogeneous case)

\[(x_1, y_1), \ldots, (x_m, y_m), \quad x_i \in \mathbb{C}^n, \quad a \in \mathbb{C}^n\]

Seek \[a, \Delta x_1, \ldots, \Delta x_m, \Delta y_1, \ldots, \Delta y_m\] s.t.

\[a^H (x_i + \Delta x_i) = y_i + \Delta y_i\]

and \[\sum_{i=1}^m \|\Delta x_i\|^2 + 1\|y_i\|^2\] is minimal.

In matrix notation:

\[
\begin{align*}
X &= \begin{bmatrix} x_1^H \\ \vdots \\ x_m^H \end{bmatrix}, \quad \Delta X &= \begin{bmatrix} \Delta x_1^H \\ \vdots \\ \Delta x_m^H \end{bmatrix} \\
Y &= \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \Delta Y &= \begin{bmatrix} \Delta y_1 \\ \vdots \\ \Delta y_m \end{bmatrix}
\end{align*}
\]

Want: \[(X + \Delta X) \cdot a = (Y + \Delta Y)\]

or \[
\begin{bmatrix}
X + \Delta X & y + \Delta y
\end{bmatrix}
\begin{bmatrix}
a \\
-1
\end{bmatrix} = 0
\]

or \[
\begin{bmatrix}
X \\
y + \Delta y
\end{bmatrix}
\begin{bmatrix}
a \\
-1
\end{bmatrix} = 0
\]

or \[
\left(\begin{bmatrix}
X \\
y
\end{bmatrix} + \begin{bmatrix}
\Delta X \\
\Delta y
\end{bmatrix}\right)\begin{bmatrix}
a \\
-1
\end{bmatrix} = 0
\]
Need $C + \Delta$ to be rank deficient

and $\Delta$ to be as small as possible

$$\Rightarrow \Delta = \frac{\sigma \cdot U \cdot V^H}{k+1}$$

Let

$$C = \sum_{i=1}^{k+1} \sigma_i \cdot u_i \cdot v_i^H$$

be the SVD of $C$. Then

$$\Delta = -\sigma_{k+1} \cdot u_{k+1} \cdot v_{k+1}^H$$

causes $C + \Delta$ to be the optimal rank deficient approx to $C$.

If $\sigma_k > \sigma_{k+1}$

Note $\bigvee C + \Delta \big) = \text{span} \{ u_{k+1} \}$, so

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \alpha \cdot u_{k+1} \text{ for some } \alpha.$$ 

If $u_{k+1}$'s last component is 0, no sol'n.

Else, normalize so that last entry is -1.

If $\sigma_k = \sigma_{k+1}$, see book.

Also see book for proof of minimum total distance property.
THE ORTHOGONAL PROCRUSTES PROBLEM

Find rotation which minimizes sum of squared errors.

General Problem: Given $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathbb{C}^n$, find $Q \in \mathbb{C}^{n \times n}$ s.t. $QQ^* = I$ and

$$\sum_{i=1}^{m} \| y_i - Qx_i \|^2$$

is minimal.
Denote

$$A = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}, \quad B = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

and \( C = A - QB \). Then

$$\sum \| y_i - Qx_i \|^2 = \sum \ c_{ij}^2$$

$$= \| C \|^2_F = \text{tr} \left( C C^H \right)$$

$$= \text{tr} (A^H A^H) + \text{tr} (B^H B^H) - \text{tr} (QB A^H) - \text{tr} (AB^H Q^H)$$

$$\Rightarrow \text{need to maximize} \quad \text{tr} (QB A^H) + \text{tr} (AB^H Q^H)$$

**Theorem** The maximum is achieved by

$$Q = VU^H$$

where \( BA^H = U\Sigma V^H \)
Proof: \[ BA^H = U \Sigma V^H \]
\[ A B^H = V \Sigma^T U^H \]

Denote \[ Z = V^H Q U, \text{ so} \]
\[ Q = V Z U^H \]
\[ Q^H = U Z^H V \]

(change of variable - \( Z \) is still unitary)

Then \[ \text{tr} (QBA^H) + \text{tr} (AB^H Q^H) \]
\[ = \text{tr} (VZU^H \cdot U \Sigma V^H) + \text{tr} (V \Sigma^T U^H U Z^H V^H) \]
\[ = \text{tr} (Z \Sigma) + \text{tr} (\Sigma^T Z^H) \]
\[ = \text{tr} (Z \Sigma) + \text{tr} (Z^H \Sigma) \]
\[ = \text{tr} ((Z + Z^H) \Sigma) \]

Spectral Theorem for Normal Operators (include)

If \[ D^H D = D D^H, \text{ then } \exists \ W \text{ unitary, } \]
\[ \Pi \text{ diagonal (w/ possibly complex entries) s.t. } \]
\[ D = W \Pi W^H \]
If $D^*D = DD^*$, we say $D$ is normal.

Examples: Hermitian, skew-Hermitian, unitary ($D = -D^*$)

Thus, let us write $Z = W\Gamma W^*$. Then

$$Z^* = W\overline{\Gamma} W^*$$

and

$$\text{tr} \left( (Z + Z^*) \Sigma \right) = \text{tr} \left( W \left( \Gamma + \overline{\Gamma} \right) W^* \Sigma \right)$$

$$= \text{tr} \left( (\Gamma + \overline{\Gamma}) \Sigma \right)$$

$$= \sum (\sigma_i + \overline{\sigma_i}) \sigma_i$$

Now the eigenvalues of unitary matrices have magnitude 1.

To see this, if $Z\nu = \lambda\nu$, then

$$\|\nu\|^2 = \|Z\nu\|^2 = \|\lambda\nu\|^2 = |\lambda|^2 \|\nu\|^2.$$ 

Thus $\sum (\sigma_i + \overline{\sigma_i}) \sigma_i \leq 2 \sum \sigma_i$.

Equality is achieved iff $\sigma_i = 1$ for all $i \Rightarrow Z = I$

$\Rightarrow A = VU^*$. 
Incorrect — see next page
Proof that $\text{tr} \left( (Z + Z^H) \Sigma \right) \leq 2 \sum_{i=1}^{n} \sigma_i$

By the spectral theorem for normal matrices, $Z = W \Gamma W^H$

where $WW^H = W^H W = I$ and $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)$, $\gamma_i \in \mathbb{C}$.

Denote

$$W = \begin{bmatrix}
w_{11} & w_{12} & \cdots & w_{1n} \\
w_{21} & w_{22} & \cdots & w_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
w_{n1} & w_{n2} & \cdots & w_{nn}
\end{bmatrix} = \begin{bmatrix}
w_1 & \cdots & w_n
\end{bmatrix} = \begin{bmatrix}
w_1^H \\
w_2^H \\
\vdots \\
w_n^H
\end{bmatrix}$$

Note that $\|w_i\| = \|w_i^H\| = 1 \forall i$ since $W$ is unitary.

Then

$$\text{tr} \left( (Z + Z^H) \Sigma \right) = \text{tr} \left( W (\Gamma + \Gamma^H) W^H \Sigma \right)$$

$$= \text{tr} \left( \sum_{j=1}^{n} (\gamma_j + \overline{\gamma}_j) w_j w_j^H \Sigma \right)$$

$$= \sum_{j=1}^{n} (\gamma_j + \overline{\gamma}_j) \text{tr} \left( w_j w_j^H \Sigma \right) = \sum_{j=1}^{n} (\gamma_j + \overline{\gamma}_j) \text{tr} \left( w_j^H \Sigma w_j \right)$$

$$= \sum_{j=1}^{n} (\gamma_j + \overline{\gamma}_j) \sum_{i=1}^{n} \sigma_i |w_{ij}|^2$$

$$= \sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} (\gamma_j + \overline{\gamma}_j) |w_{ij}|^2$$

$$\leq 2 \sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} |w_{ij}|^2 = 2 \sum \sigma_i \|w_i\|^2$$

$$= 2 \sum_{i=1}^{n} \sigma_i$$

(since $|\gamma_i| = 1 \Rightarrow \gamma_i + \overline{\gamma}_i = 2 \text{Re} \gamma_i \gamma_i^* \leq 2$)