LINEAR DISCRIMINANT ANALYSIS

Generative Models for Classification

The Bayes classifier requires knowledge of the distribution on (X, Y). However, this distrib. is often unknown, and we only have training data $(x_1, y_1), \ldots, (x_n, y_n)$, $x_i \in \mathbb{R}^d$, $y_i \in \{1, \ldots, K\}$.

A <u>generative model</u> is an assumption about the true form of this unknown distribution.
Generative models are typically <u>parametric</u>.
To build a classifier, we may estimate the model parameters using the training data, a plug the result into the formula for the Bayes classifier. For this reason, such methods are also called "plug-in" methods.

$\boxed{\text{LDA}}$ In LDA, we assume

$$X \mid Y = k \sim N(\mu_k, \Sigma), \quad k = 1, \ldots, K$$

Here $N(\mu, \Sigma)$ is the <u>multivariate Gaussian/normal</u> distribution with parameters μ, Σ, and pdf

$$\phi(x;\mu,\Sigma) = (2\pi)^{-\frac{d}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\}$$

<u>Note</u>: each class has the same covariance matrix Σ

<u>Parameter Estimation</u>

$$\hat{\pi}_k = \frac{|\{i : y_i = k\}|}{n}$$

$$\hat{\mu}_k = \frac{1}{|\{i : y_i = k\}|} \sum_{i : y_i = k} x_i$$

$$\hat{\Sigma} = \frac{1}{n} \sum_{k=1}^{K} \sum_{i : y_i = k} (x_i - \hat{\mu}_k)(x_i - \hat{\mu}_k)^T$$

"pooled covariance estimate"

$\boxed{\text{Interpretations}}$ The LDA classifier is then

$$f(x) = \arg\max_k \ \hat{\pi}_k \cdot \phi(x; \hat{\mu}_k, \hat{\Sigma})$$

$$= \arg\max_k \ \log \hat{\pi}_k + \log \phi(x; \hat{\mu}_k, \hat{\Sigma})$$

$$= \arg\min_k \ \underbrace{(x - \hat{\mu}_k)^T \hat{\Sigma}^{-1} (x - \hat{\mu}_k)} - 2\log \hat{\pi}_k$$

squared Mahalanobis distance between x and $\hat{\mu}_k$

Case $K = 2$

$$(x - \hat{\mu}_1)^T \hat{\Sigma}^{-1} (x - \hat{\mu}_1) - 2\log \hat{\pi}_1 \ \underset{2}{\overset{1}{\lessgtr}} \ (x - \hat{\mu}_2)^T \hat{\Sigma}^{-1} (x - \hat{\mu}_2) - 2\log \hat{\pi}_2$$

$$\iff a^T x + b \ \underset{2}{\overset{1}{\gtrless}} \ 0$$

$\boxed{\text{linear classifier}}$

where

$$a = \hat{\Sigma}^{-1} (\hat{\mu}_1 - \hat{\mu}_2)$$

$$b = -\frac{1}{2} \hat{\mu}_1^T \hat{\Sigma}^{-1} \hat{\mu}_1 + \frac{1}{2} \hat{\mu}_2 \hat{\Sigma}^{-1} \hat{\mu}_2 + \log \frac{\hat{\pi}_1}{\hat{\pi}_2}$$

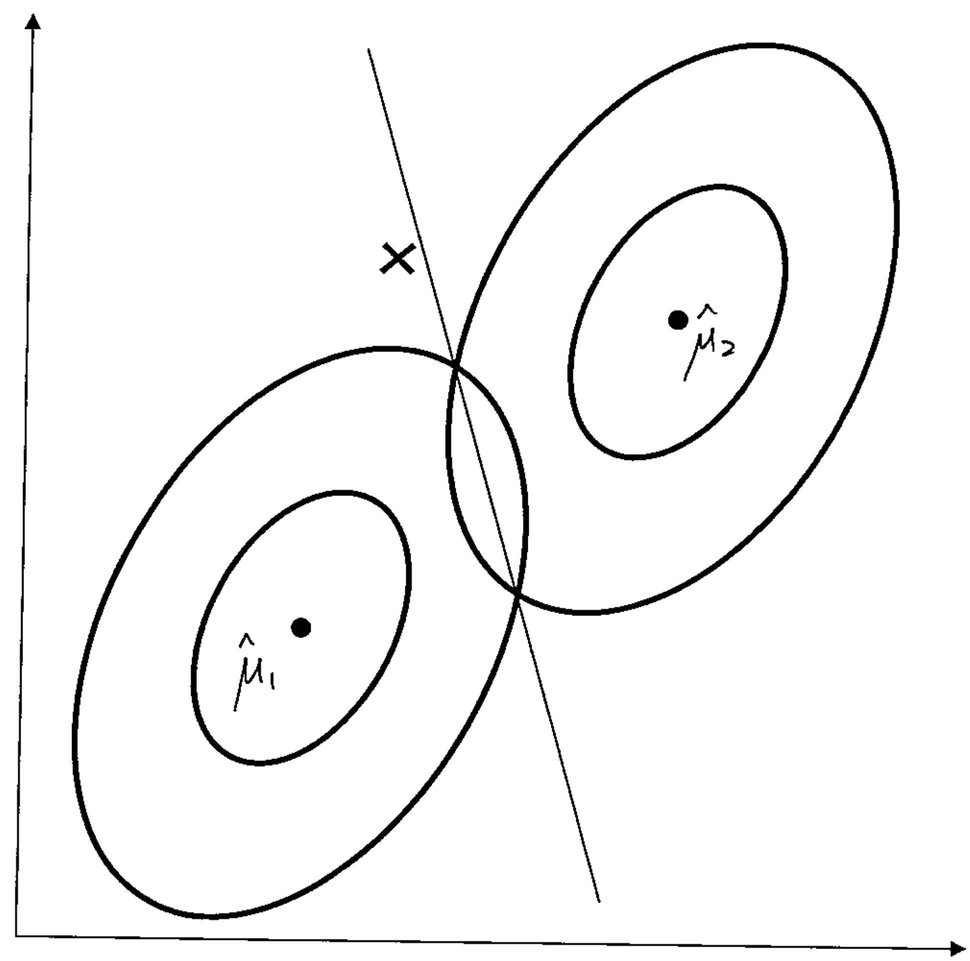
Denote the Mahalanobis distance

$$d_M(x; \mu, \Sigma) = \sqrt{(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

Recall that the contour

$$\{ x : d_M(x ; \mu, \Sigma) = c \}$$

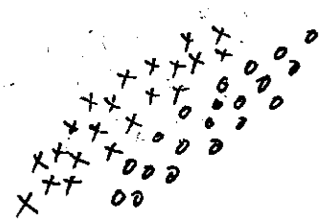
(A) is an



This picture assumes $\hat{\pi}_1 = \hat{\pi}_2$

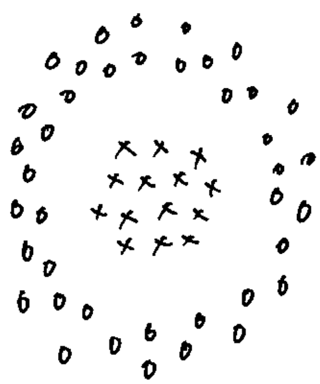
Would LDA be appropriate:

(i)

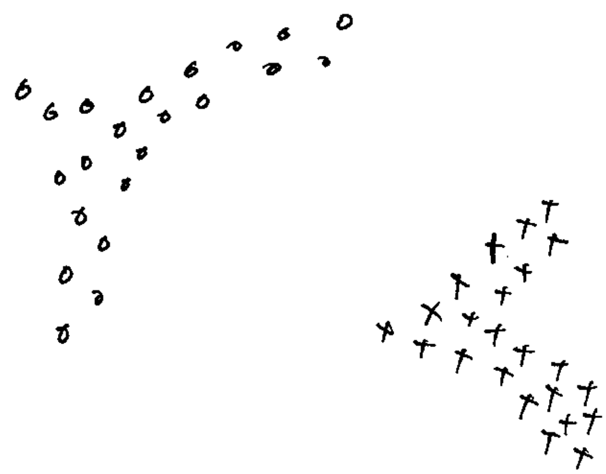


(ii)

(iii)

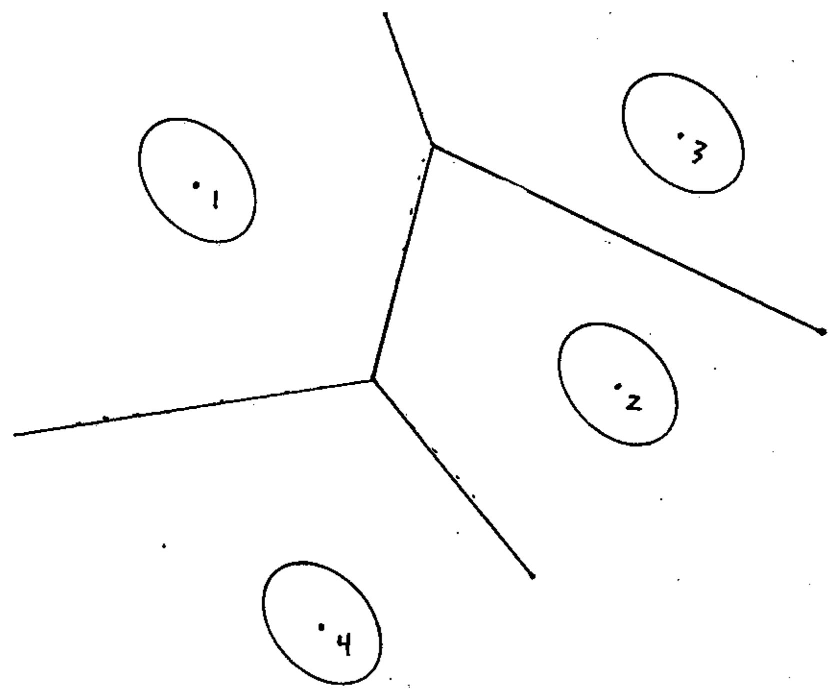


(iv)



Case $K > 2$

The decision regions are convex polytopes (insersections of linear half-spaces)



Issues w/ LDA

- The number of parameters to be estimated is

B

if n is too small, we could

- Generative model is not valid

Example (of a structured covariance matrix)

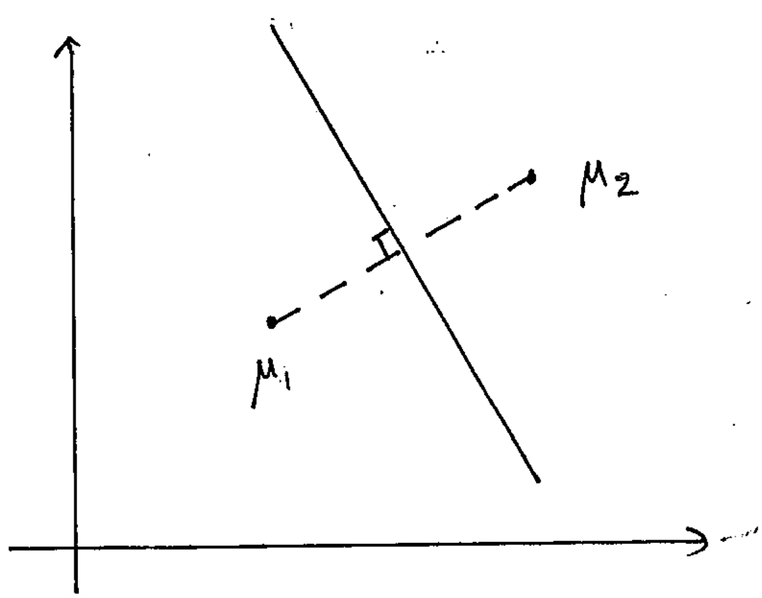
ⓒ $\Sigma = \sigma^2 I \implies \hat{\sigma}^2 =$

Then LDA becomes (assuming $K=2$, $\hat{\pi}_1 = \hat{\pi}_2$)

$$\frac{1}{\hat{\sigma}^2}\|x - \hat{\mu}_1\|^2 \underset{1}{\overset{2}{\gtrless}} \frac{1}{\hat{\sigma}^2}\|x - \hat{\mu}_2\|^2$$

$$\iff \|x - \hat{\mu}_1\|^2 \underset{1}{\overset{2}{\gtrless}} \|x - \hat{\mu}_2\|^2$$

which is called the __________ __________ classifier.



$\boxed{\text{QDA}}$ Generative model with

$$X \mid Y=k \sim N(\mu_k, \Sigma_k), \quad k=1,\dots,K$$

$$\hat{\Sigma}_k = \frac{1}{|\{i : y_i = k\}|} \sum_{i : y_i = k} (x_i - \hat{\mu}_k)(x_i - \hat{\mu}_k)^T$$

Decision boundaries are now <u>quadratic</u>

$\boxed{\text{Key}}$ A. ellipse

B. $$\underbrace{Kd}_{\text{means}} + \underbrace{\frac{d(d+1)}{2}}_{\text{covariance}} + \underbrace{K-1}_{\text{class prior probs.}}$$

could

- first apply PCA
- assume a structured covariance matrix

C. $$\hat{\sigma}^2 = \frac{1}{d} \operatorname{tr}\{\hat{\Sigma}\}$$

nearest centroid classifier