LOCALLY LINEAR REGRESSION

Consider a regression setting with training data (x_i, y_i), $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$

↑ scalar (for now)

and assume the relationship between x and y is nonlinear

Furthermore, suppose we are not willing to adopt a polynomial model, which we previously saw could be fit using _____ _____.

(A)

Ideas

- Local averaging

$$\hat{f}(x) = \text{avg. } y_i \text{ among } x_i$$
$$\text{s.t. } |x - x_i| < \delta$$

(B) =

- Weighted local averaging

$$\hat{f}(x) = \text{weighted avg. of } y_i$$
$$\text{based on } |x - x_i|$$

Known as the Nadayara-Watson estimate →

=

Note: local averaging is the special case where

Unfortunately, local averaging suffers near the boundaries of the data set, and in other sparsely populated regions.

These problems can be alleviated by ...

Locally Linear Regression

Here's the algorithm:

Input: $(x_i, y_i)_{i=1}^n$, $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$

$x \in \mathbb{R}^d$

Solve: $\min_{\beta(x), \beta_0(x)} \sum_{i=1}^n k_\sigma(x_i - x)\left[y_i - \beta(x)^T x_i - \beta_0(x)\right]^2$

$\downarrow$

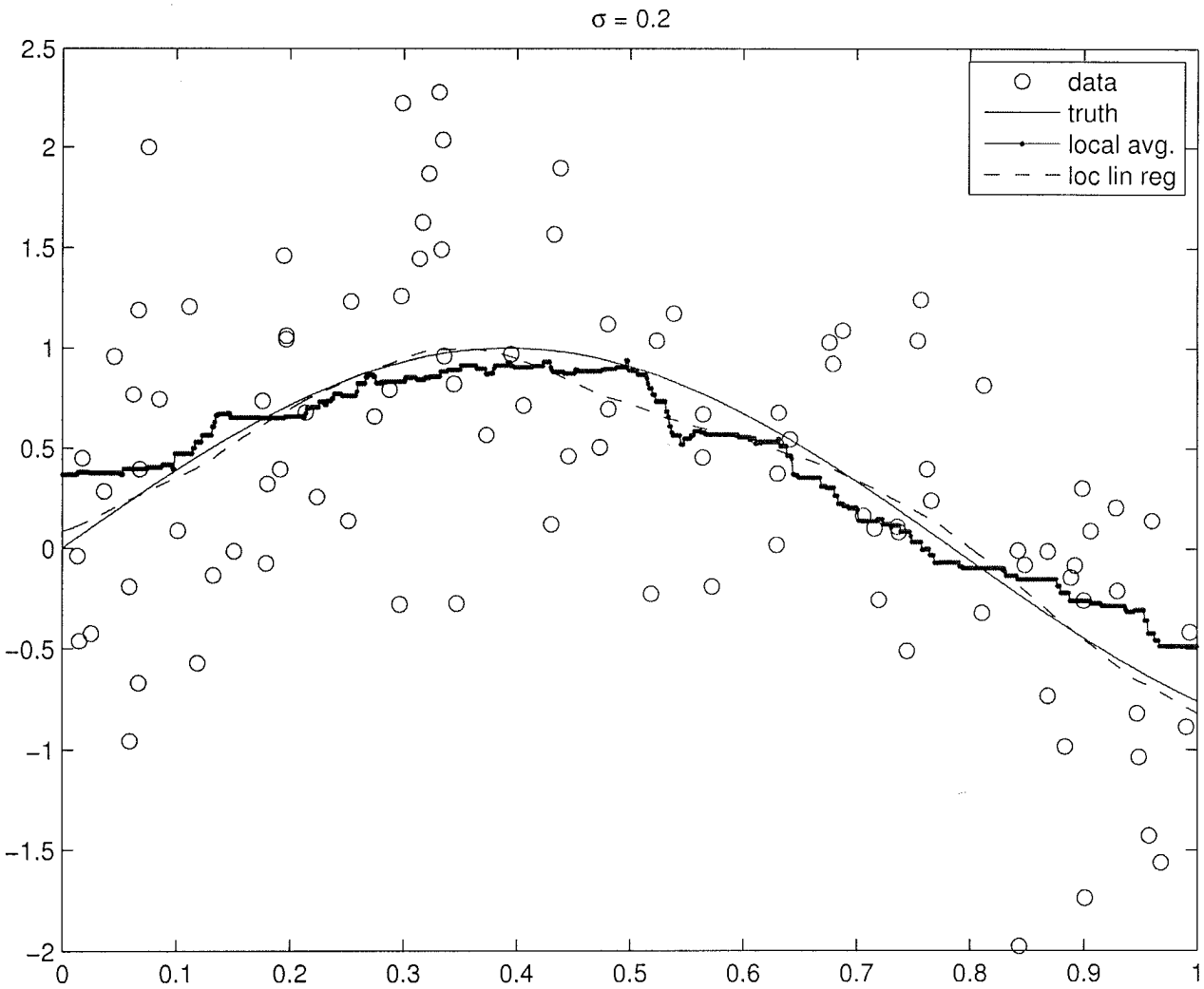
$(\hat{\beta}(x), \hat{\beta}_0(x))$

Output: $\hat{f}(x) = \hat{\beta}(x)^T x + \hat{\beta}_0(x)$

Note: A weighted least squares problem must be solved at each new x!

Relative to local averaging, LLR is

- smooth
- more accurate at boundaries



Weighted Least Squares

$$\sum_{i=1}^{n} w_i \left(y_i - \beta^T x_i - \beta_0\right)^2$$

$$= \left(\underline{y} - A\theta\right)^T \cdot W \cdot \left(\underline{y} - A\theta\right)$$

where

$$\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \theta = \begin{bmatrix} \beta_0 \\ \beta \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & x_1^{(1)} & \cdots & x_1^{(d)} \\ 1 & x_2^{(1)} & \cdots & x_2^{(d)} \\ & \vdots & & \\ 1 & x_n^{(1)} & \cdots & x_n^{(d)} \end{bmatrix}, \quad W = \begin{bmatrix} w_1 & & 0 \\ & w_2 & \\ 0 & & \ddots \\ & & & w_n \end{bmatrix}$$

How can we minimize

$$\left(\underline{y} - A\theta\right)^T W \left(\underline{y} - A\theta\right)$$

w.r.t. θ?

$$(\underline{y} - A\theta)^T W (\underline{y} - A\theta) = (\underline{y} - A\theta)^T W^{\frac{1}{2}} W^{\frac{1}{2}} (\underline{y} - A\theta)$$

$$= (W^{\frac{1}{2}}\underline{y} - W^{\frac{1}{2}}A\theta)^T (W^{\frac{1}{2}}\underline{y} - W^{\frac{1}{2}}A\theta)$$

$$= (\tilde{\underline{y}} - \tilde{A}\theta)^T (\tilde{\underline{y}} - \tilde{A}\theta), \qquad \tilde{y} = W^{\frac{1}{2}}y, \ \tilde{A} = W^{\frac{1}{2}}A$$

$$= \| \tilde{\underline{y}} - \tilde{A}\theta \|^2$$

(c) $\Rightarrow \hat{\theta} =$

$=$

Applying this to LLR, we have

$$\hat{f}(x) = \hat{\theta}(x)^T \cdot \begin{bmatrix} 1 \\ x \end{bmatrix}$$

$$= \underline{y}^T \cdot W(x) A \left(A^T W(x) A\right)^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$$

where

$$W(x) =$$

Issues

- Selecting the kernel: unlike density estimation, the kernel need not integrate to 1.

Kernels with <u>finite support</u> may be preferable to the Gaussian kernel, which has infinite support.

Examples:

$$K_\sigma(y) = D\left(\frac{\|y\|}{\sigma}\right)$$

Epanechnikov

where

$$D(t) = \begin{cases} \frac{3}{4}(1-t^2) & \text{if } |t| \le 1 \\ 0 & \text{else} \end{cases}$$

Tri-cube

or

$$D(t) = \begin{cases} (1-t^3)^3 & \text{if } |t| \le 1 \\ 0 & \text{else} \end{cases}$$

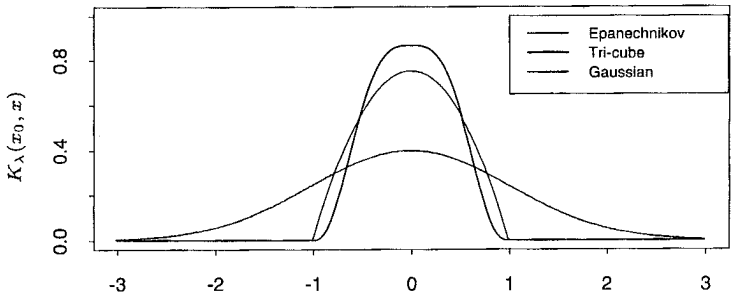
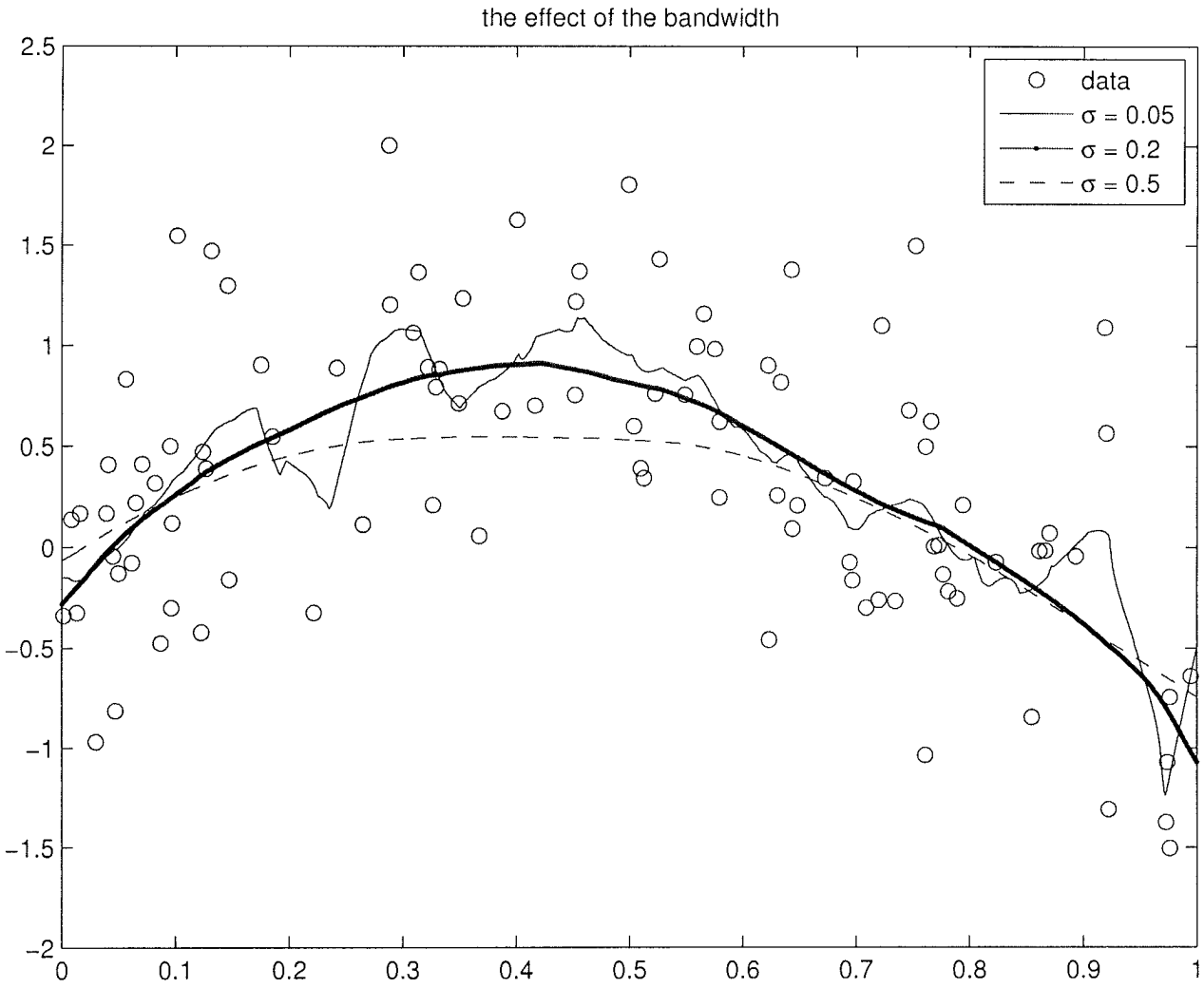


FIGURE 6.2. *A comparison of three popular kernels for local smoothing. Each has been calibrated to integrate to 1. The tri-cube kernel is compact and has two continuous derivatives at the boundary of its support, while the Epanechnikov kernel has none. The Gaussian kernel is continuously differentiable, but has infinite support.*

- bandwidth selection (bias-variance tradeoff)



Key

A. Least squares

B. $$\hat{f}(x) = \frac{1}{|\{i : |x_i - x| < \delta\}|} \sum_{i : |x_i - x| < \delta} y_i$$

$$\hat{f}(x) = \frac{\sum_{i=1}^{n} k_\sigma(x - x_i) \cdot y_i}{\sum_{i=1}^{n} k_\sigma(x - x_i)}$$

$$k_\sigma(x - x_i) = 1_{\{|x - x_i| < \delta\}}$$

C. $$\hat{\theta} = (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T \tilde{y}$$

$$= (A^T W A)^{-1} A^T W y$$

$$W(x) = \begin{bmatrix} k_\sigma(x_1 - x) & & 0 \\ & k_\sigma(x_2 - x) & \\ 0 & & \ddots \\ & & & k_\sigma(x_n - x) \end{bmatrix}$$