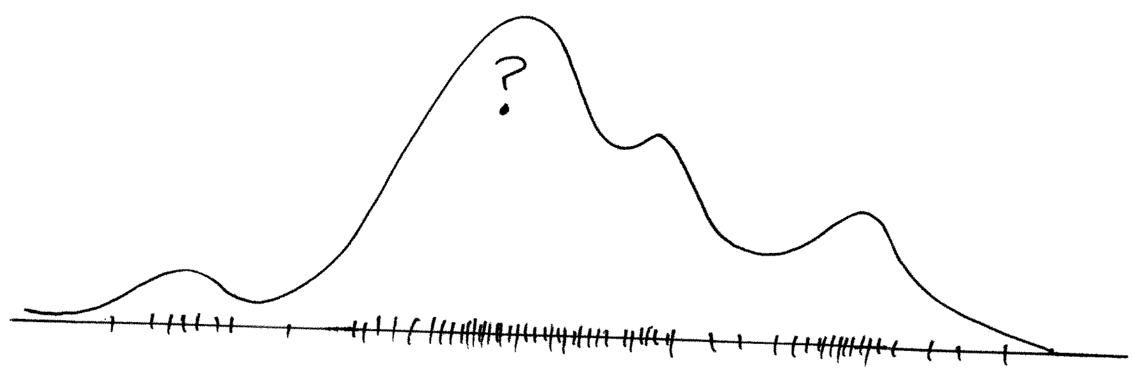
KERNEL DENSITY ESTIMATION

Density Estimation

In density estimation problems, we are given a random sample

$$x_1, \ldots, x_n$$

from an unknown density $f(x)$



The objective is to estimate f.

Before examining this task let's first see why it is important.

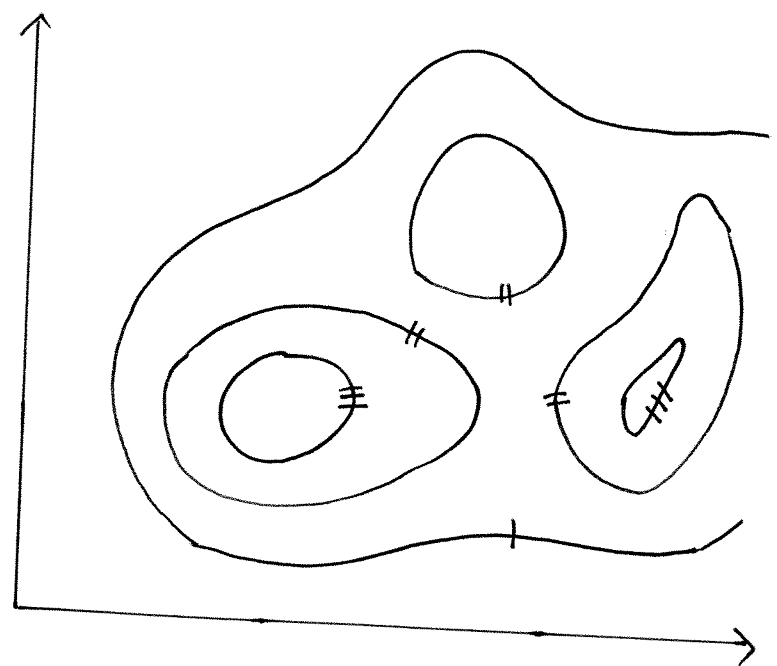
Classification From the formula for the Bayes classifier, a "plug-in" classifier has the form

$$\hat{f}(x) = \arg\max_{k} \; \hat{\pi}_k \hat{g}_k(x)$$

where $\hat{g}_k$ is an estimate of the class-conditional density

Clustering

The contours of a density can serve to define clusters in a natural way: All x in the same "connected component" of a contour are in the same cluster.



Novelty Detection Given a random sample $x_1, \dots, x_n$ from the same nominal distribution, we can estimate its density f, and use

$$\hat{f}(x) \gtrless \gamma$$

to detect whether a future observation comes from the nominal distribution or a new one

Kernel Density Estimation

A kernel density estimate has the form

Ⓐ $\hat{f}(x) :=$

where $k_\sigma(y)$ is called a <u>kernel</u>

<u>Example</u> Gaussian kernel

$k_\sigma(y) =$

<u>Remarks</u>

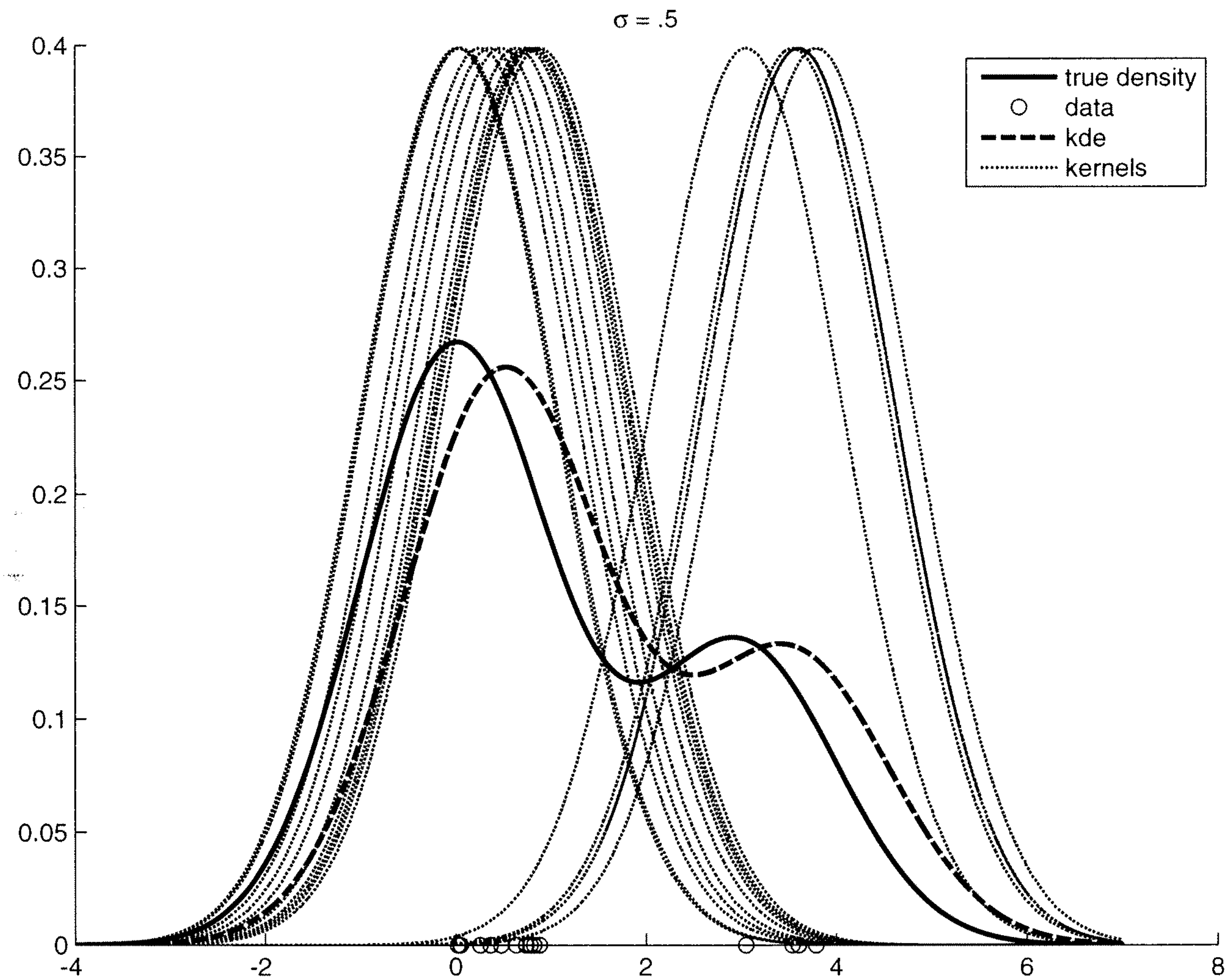
1) Another term for a KDE is a

2) A KDE is nonparametric. Why?

3) The Gaussian kernel is the most common.

4) The parameter σ is called the

KDE = average of "local" density estimates $k_\sigma(x - x_i)$



Kernels

A kernel function should satisfy

B

1)

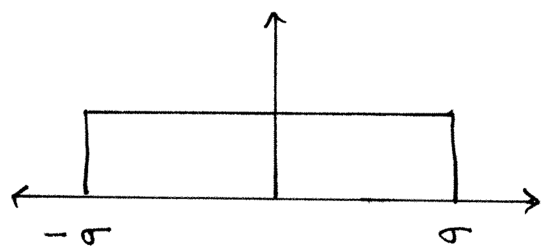
2)

3)

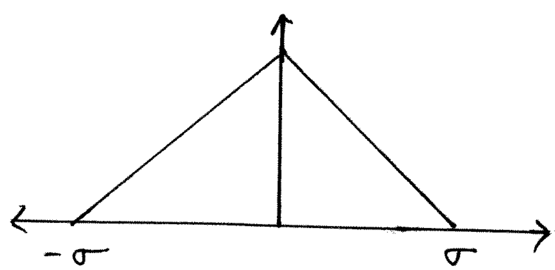
Examples

- Uniform kernel

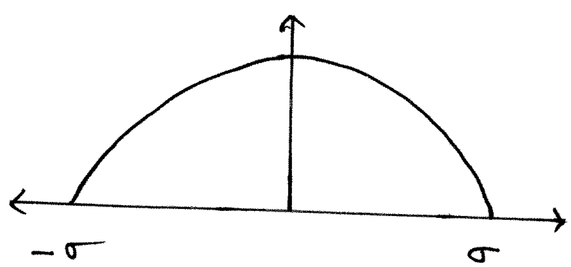
$$k_\sigma(y) = c_\sigma \mathbb{1}_{\{\|y\| \le \sigma\}}$$



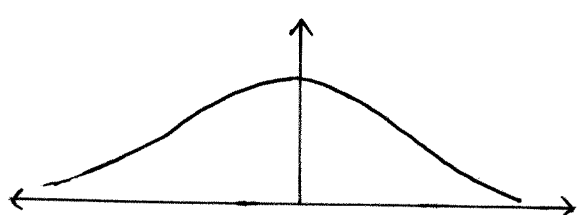
- Triangular kernel



- Epanichnikov kernel
(parabolic)

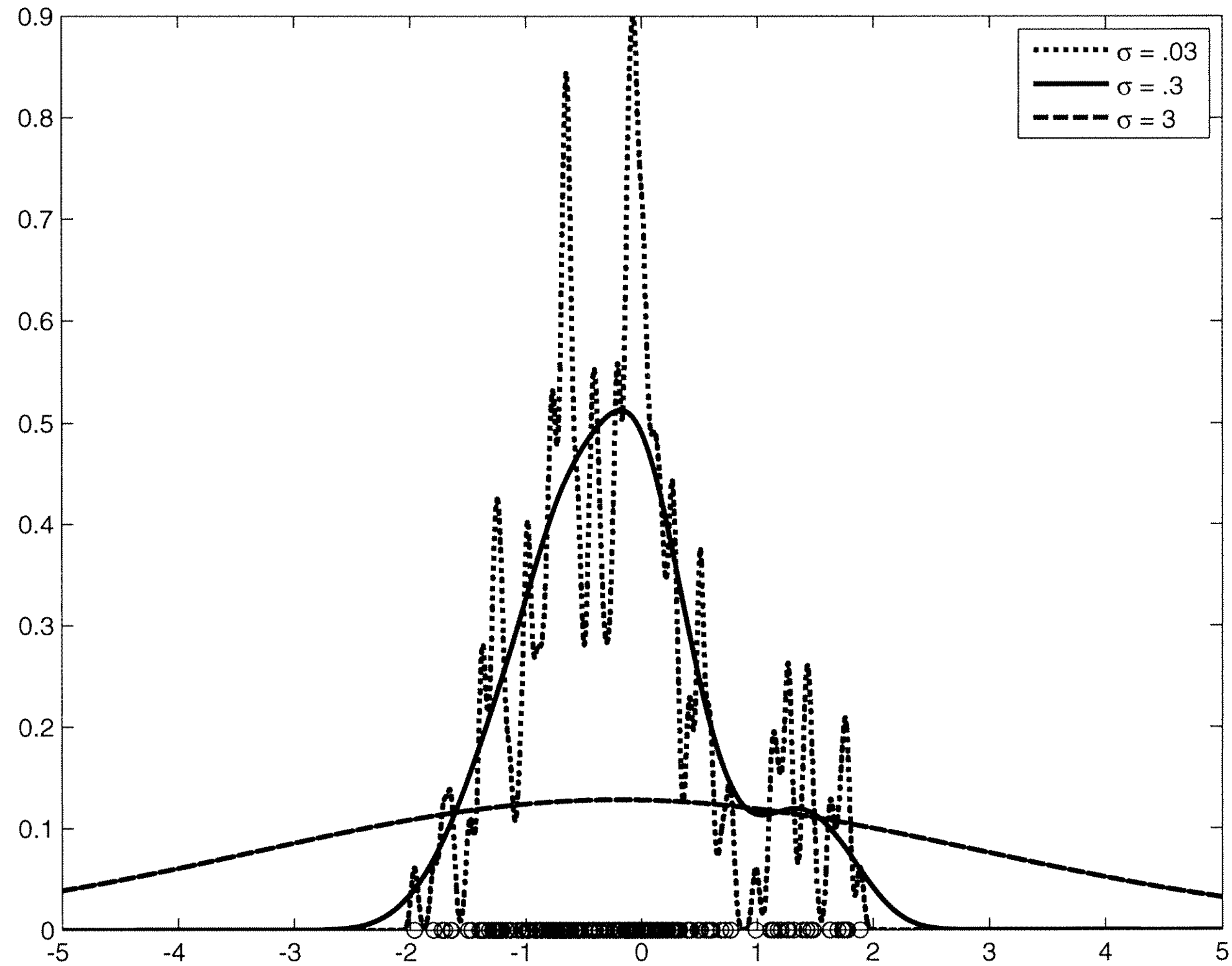


- Cauchy kernel



The accuracy of a KDE depends critically on the

(c) ________________



Automatically setting σ is a nontrivial problem to which we will return later in the course.

Theorem Let $\hat{f}_\sigma(x)$ be a KDE based on the kernel k_σ. Suppose $\sigma = \sigma_n$ is such that

- $\sigma_n \longrightarrow 0$ as $n \to \infty$
- $n \cdot \sigma_n^d \to \infty$ as $n \to \infty$.

Then

$$E\left\{ \int \left| \hat{f}_\sigma(x) - f(x) \right| dx \right\} \longrightarrow 0$$

as $n \to \infty$, regardless of the true density f.

Proof: Devroye and Lugosi, Combinatorial Methods in Density Estimation.

$\boxed{\text{Key}}$

A. $\hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} K_\sigma(x - x_i)$

$$K_\sigma(y) = (2\pi\sigma^2)^{-\frac{d}{2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\}$$

$$= \phi(y;\ 0,\ \sigma^2 I)$$

Parzen window, $\sigma =$ bandwidth

B. 1) $\int K_\sigma(y)\,dy = 1$

2) $K_\sigma(y) \geq 0$

3) $K_\sigma(-y) = K_\sigma(y)$

C. bandwidth