EIGENDECOMPOSITIONS &
THE SPECTRAL THEOREM

The Spectral Theorem

Definition: If $U \in \mathbb{C}^{N \times N}$ is such that

$$U^H U = UU^H = I_{N \times N}$$

then $U$ is said to be unitary.

If $U \in \mathbb{R}^{N \times N}$ is such that

$$U^T U = UU^T = I_{N \times N}$$

then $U$ is said to be orthogonal.

Intuitively, such matrices are distance preserving since

$$\|Ux - Uy\|^2 = (Ux - Uy)^H (Ux - Uy)$$

$$= (x - y)^H U^H U (x - y)$$

$$= \|x - y\|^2$$

\[\text{slightly confusing since the columns of } U \text{ are in fact orthonormal}\]
Unitary and orthogonal matrices effect a change of coordinate system.

\textbf{Theorem} \hspace{1em} \text{(Spectral Theorem)}

If $A \in \mathbb{C}^{N \times N}$ is Hermitian, then there exist a unitary matrix $U$ and a real diagonal matrix $\Lambda$ such that

$$A = U \Lambda U^H.$$

If $A \in \mathbb{R}^{N \times N}$ is symmetric, the same result holds where now $U$ is orthogonal.

\textbf{Proof} \hspace{1em} \text{See Moon and Stirling, Mathematical Methods and Algorithms for Signal Processing.}
Suppose $A$ is Hermitian/symmetric. Write

$$A = U\Lambda U^H$$

according to the spectral theorem. Let

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}$$

Since $AU = U\Lambda$, we conclude that the $\lambda_i$ are eigenvalues with $u_i$ the associated eigenvector:

$$Au_i = \lambda_i u_i, \quad i = 1, \ldots, N.$$ 

The spectral theorem also gives rise to the following spectral decomposition of $A$:

$$A = \sum_{i=1}^{N} \lambda_i u_i u_i^H$$
Positive (semi-) definite matrices

Let $A$ be a Hermitian/symmetric $N \times N$ matrix. We say $A$ is \underline{positive definite (PD)} if

$$\mathbf{x} \neq \mathbf{0} \implies \mathbf{x}^H A \mathbf{x} > 0.$$ 

We say $A$ is \underline{positive semi-definite (PSD)} if

$$\forall \mathbf{x} \quad \mathbf{x}^H A \mathbf{x} \geq 0.$$ 

[PSD is also called nonnegative definite]

Exercise \underline{Show that} $A$ is PD (PSD) iff the eigenvalues of $A$ are positive (nonnegative).
Solution  

By the spectral theorem, \( u_1, ..., u_N \) is an orthonormal collection (a basis, in fact) such that 

\[ A u_i = \lambda_i u_i. \]

For each \( i = 1, ..., N \) we have 

\[ \lambda_i = u_i \cdot u_i^H u_i. \]

\[ = u_i^H \cdot \lambda_i u_i. \]

\[ = u_i^H A u_i. \]

\( > 0 \) if \( A \) is PD 

\( \geq 0 \) if \( A \) is PSD.

Conversely, suppose \( \lambda_i > 0 \ (\geq 0) \ \forall i \).

Then for \( x \neq 0 \) we have 

\[ x^H A x = \sum_{i=1}^{n} \lambda_i x^H u_i u_i^H x \]

\[ = \sum_{i=1}^{n} \lambda_i \| u_i^H x \|^2 \]

\( > 0 \) if \( \lambda_i > 0 \ \forall i \)

\( \geq 0 \) if \( \lambda_i > 0 \ \forall i \)