Lattices:
. . . to Cryptography

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Visions of Cryptography
10 December 2013

Agenda

1. The ~~two~~ one main lattice-based OWF
2. Two simple tricks that yield all* of lattice cryptography
3. Lots of applications

A Hard Problem: Short Integer Solution

- Goal: given uniform $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, find short nonzero $\mathbf{z} \in \mathbb{Z}^m$ such that:

$$\underbrace{\begin{pmatrix} \cdots & \mathbf{A} & \cdots \end{pmatrix}}_{m} \begin{pmatrix} \\ \mathbf{z} \\ \\ \end{pmatrix} = \mathbf{0} \in \mathbb{Z}_q^n$$

(When $m \geq n \log q$, short solutions are guaranteed to exist.)

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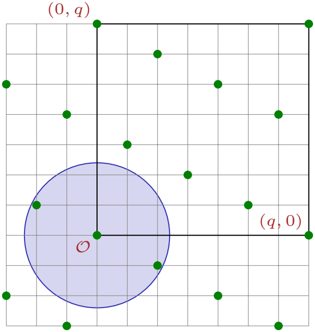

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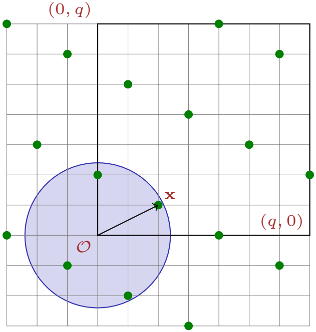
- $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ reduces $\mathbf{x}$ modulo $\mathcal{L}^{\perp}(\mathbf{A})$.



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Worst-Case/Average-Case Connection [Ajtai'96,...,MR'04,GPV'08,MP'13]

Finding solution $\mathbf{z}$ with $\|\mathbf{z}\| \leq \beta \ll q$

(for uniformly random $\mathbf{A}$)

$\Downarrow$

solving $\mathsf{GapSVP}_{\beta\sqrt{n}}$ and $\mathsf{SIVP}_{\beta\sqrt{n}}$ on any n-dim lattice.

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One-Way & Collision-Resistant Hash Function

- Set $m > n \lg q$. Define $f_{\mathbf{A}} : \{0,1\}^m \to \mathbb{Z}_q^n$ as

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- Collision $\mathbf{x}, \mathbf{x}' \in \{0,1\}^m$ where $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}'$ ...

  ... yields solution $\mathbf{z} = \mathbf{x} - \mathbf{x}' \in \{0, \pm 1\}^m$, of norm $\|\mathbf{z}\| \leq \sqrt{m}$.

Another (?) Hard (?) Problem: Learning With Errors

- Wlog, $\mathbf{A} = [\bar{\mathbf{A}} \mid \mathbf{I}_n] \in \mathbb{Z}_q^{n \times (m+n)}$.

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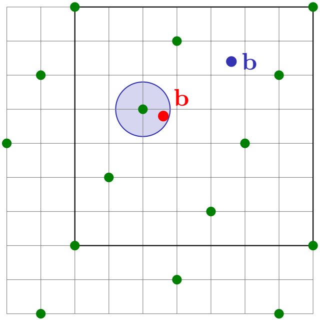

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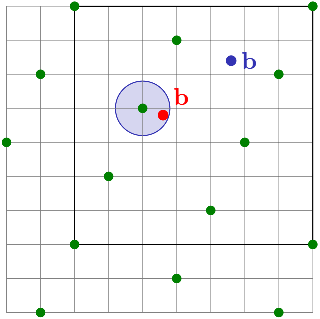
- As hard as worst case problems on m-dim lattices [Regev'05,P'09].



The two amazingly simple tricks behind all of lattice cryptography...

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     Then $[\mathbf{A}' \mid \mathbf{I}_n] \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} = \mathbf{u} + [\mathbf{A} \mid \mathbf{I}_n] \cdot \mathbf{x} = \mathbf{0}$.

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- For *many* solutions, let $\mathbf{U} = -[\mathbf{A} \mid \mathbf{I}_n] \cdot \mathbf{X}$ and $\mathbf{A}' = [\mathbf{U} \mid \mathbf{A}]$.

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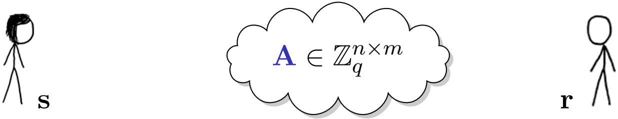
- Of course, we can also multiply on the left:

  Let $\mathbf{u}^t = \mathbf{x}^t \left[\begin{smallmatrix} \mathbf{A} \\ \mathbf{I}_m \end{smallmatrix} \right]$ and $\mathbf{A}' = \left[\begin{smallmatrix} \mathbf{u}^t \\ \mathbf{A} \end{smallmatrix} \right]$.

Key Agreement/Encryption



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$$k_a = \mathbf{s}_1^t \cdot \mathbf{u} + \mathsf{err} \approx \mathbf{s}_1^t \mathbf{A} \mathbf{r}_1$$

$$k_b = \mathbf{v}^t \cdot \mathbf{r}_1 + \mathsf{err} \approx \mathbf{s}_1^t \mathbf{A} \mathbf{r}_1$$

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- A special parity-check matrix: let $\mathbf{g}^t = [1\ 2\ 4\ \cdots\ 2^{k-1} \geq \frac{q}{2}]$ and

$$\mathbf{G} = \begin{bmatrix} \cdots \mathbf{g}^t \cdots & & & \\ & \cdots \mathbf{g}^t \cdots & & \\ & & \ddots & \\ & & & \cdots \mathbf{g}^t \cdots \end{bmatrix} \in \mathbb{Z}_q^{n \times nk}.$$

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  Can generate $(\mathbf{x}, \mathbf{u})$ in two equivalent ways:

$$\text{Gauss} \rightarrow \mathbf{x} \xrightarrow{\ \mathbf{G}\ } \mathbf{u} \quad \equiv \quad \mathbf{x} \xleftarrow{\ \mathbf{G}^{-1}\ } \mathbf{u} \leftarrow \mathbb{Z}_q^n$$

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  (Something similar works for any q.)

Put $\mathbf{G}$ in Public Key $\Rightarrow$ TDF, Signatures, IBE [GPV'08,MP'12]

- Let $\mathbf{A}' = [\mathbf{A} \mid \mathbf{G} - \mathbf{A}\mathbf{R}]$, so $\mathbf{A}' \left[\begin{smallmatrix} \mathbf{R} \\ \mathbf{I} \end{smallmatrix} \right] = \mathbf{G}$. Trapdoor $= \mathbf{R}$.

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Solution: output $\mathbf{x} = \mathbf{p} + \left[\begin{smallmatrix} \mathbf{R} \\ \mathbf{I} \end{smallmatrix} \right] \cdot \mathbf{G}^{-1}(\mathbf{u} - \mathbf{A}'\mathbf{p})$ for 'perturbation' $\mathbf{p}$.

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- Invert SIS: given target $\mathbf{u}$, output $\mathbf{x} = \left[\begin{smallmatrix} \mathbf{R} \\ \mathbf{I} \end{smallmatrix} \right] \cdot \mathbf{G}^{-1}(\mathbf{u})$. Then

$$\mathbf{A}'\mathbf{x} = \mathbf{G} \cdot \mathbf{G}^{-1}(\mathbf{u}) = \mathbf{u}.$$

Problem: $\mathbf{x}$ is 'skewed,' leaks trapdoor $\mathbf{R}$!

Solution: output $\mathbf{x} = \mathbf{p} + \left[\begin{smallmatrix} \mathbf{R} \\ \mathbf{I} \end{smallmatrix} \right] \cdot \mathbf{G}^{-1}(\mathbf{u} - \mathbf{A}'\mathbf{p})$ for 'perturbation' $\mathbf{p}$.

$$\text{Gauss} \to \mathbf{x} \xrightarrow{\mathbf{A}'} \mathbf{u} \quad \equiv \quad \mathbf{x} \xleftarrow{\mathbf{A}'^{-1}} \mathbf{u} \leftarrow \mathbb{Z}_q^n$$

Put $\mathbf{G}$ in Evaluation Key $\Rightarrow$ FHE [BV'11]

- Secret key $\mathbf{s} \in \mathbb{Z}^n$, ciphertext $\mathbf{c} \in \mathbb{Z}_q^n$ is s.t. $\mathbf{s}^t \cdot \mathbf{c} \approx \frac{q+1}{2} \cdot \mu$.

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Error in $\mathbf{C}_\times$ is $\mathbf{e}_1^t \cdot \mathbf{G}^{-1}(\mathbf{C}_2) + \mu_1 \cdot \mathbf{e}_2^t$.

Asymmetry allows homom mult with additive noise growth. [BV'13]

Concluding Thoughts

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Thanks!