# Algebraically Structured LWE, Revisited

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# 'Algebraic' Learning With Errors

- A foundation of efficient lattice crypto: Ring-LWE, Module-LWE, Polynomial-LWE, Order-LWE, Middle-Product-LWE, ...

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  [SSTX'09,LPR'10,LS'15,L'16,PRS'17,RSSS'17,AD'17,RSW'18,BBPS'18,...]
- But these reductions are often difficult to understand and use:
  - Several steps between problems of interest
  - Complex analysis and parameters
  - Frequently large blowup and distortion of error distributions, across different metrics
  - Sometimes non-uniform advice that appears hard to compute

# Prior Hardness of Ring-LWE and MP-LWE

$$\begin{tikzcd} \text{(dual) } \mathcal{O}_K\text{-LWE} \\ \text{worst-case approx-}\mathcal{O}_K\text{-SIVP} \arrow[u, dashed, "{\text{[LPR'10,PRS'17]}}"] \end{tikzcd}$$

# Prior Hardness of Ring-LWE and MP-LWE

(primal) $\mathcal{O}_K$-LWE

[LPR'10,DD'12,RSW'18] complex & non-uniform; expands error

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# Prior Hardness of Ring-LWE and MP-LWE

MP-LWE$_{n,d}$

[RSSS'17] for any $\alpha$ s.t. $d \leq \deg(\alpha) \leq n$, expands error by $\geq d \cdot \mathsf{EF}(\alpha)$

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## Definitions

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- For any order $\mathcal{L} = \mathbb{Z}[\alpha]$ with $d \leq \deg(\alpha) \leq n$,

$$\mathbb{Z}[\alpha]\text{-LWE} \leq \text{MP-LWE}_{n,d}$$

  with error expansion $\|V_\alpha\|$.

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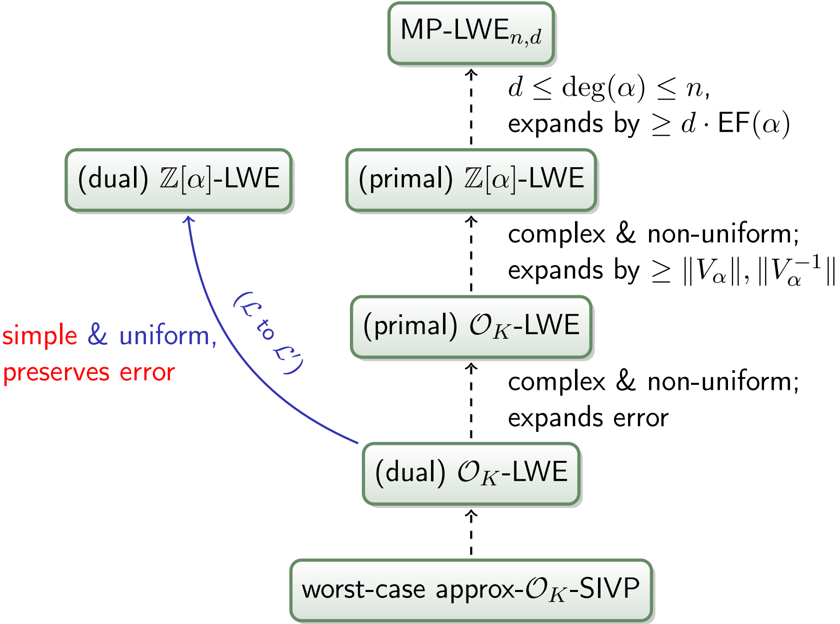

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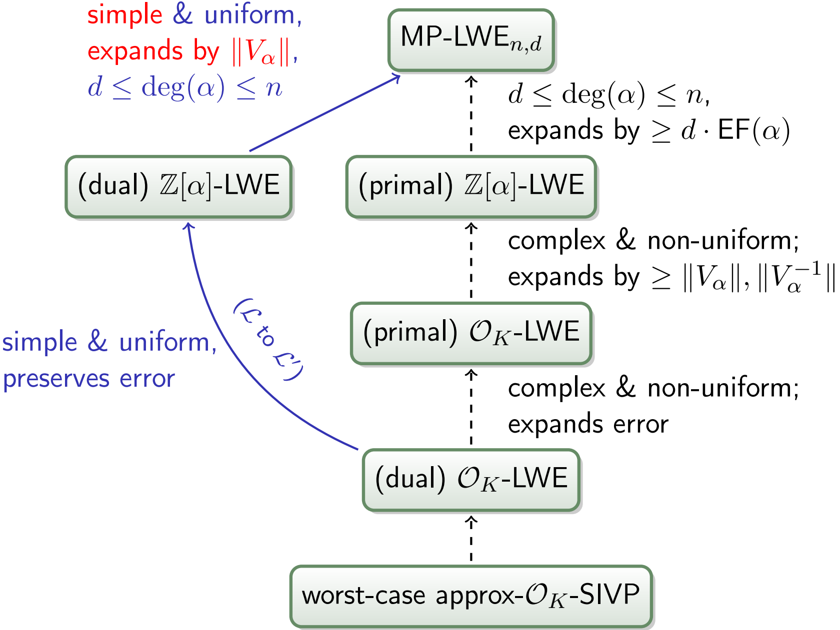

# Ring-LWE and Variants

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## Poly-LWE

- Same, but $R = \mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/f(x)$ and $s, a, s \cdot a \in R_q$ (no dual $R_q^\vee$).

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- Generalizes:

| | |
|---|---|
| Ring-LWE | by taking $\mathcal{L} = \mathcal{O}_K$ to be the full ring of integers |
| Order-LWE | by taking $\mathcal{L} = \mathcal{O}$ to be an order of $K$ |
| Poly-LWE | by taking $\mathcal{L} = \mathbb{Z}[\alpha]^\vee$ for some $\alpha \in \mathcal{O}_K$ |
| Module-LWE | by allowing $a, s$ to be vectors |

# Our $\mathcal{L}$-LWE Reductions

**Theorem 1: $\mathcal{L}$ to $\mathcal{L}'$**

- Let $\mathcal{L}' \subseteq \mathcal{L} \subset K$ be lattices with respective coefficient rings $\mathcal{O}' \subseteq \mathcal{O}$, and $|\mathcal{L}/\mathcal{L}'|$ coprime to $q$. (E.g., $\mathcal{L}' = \mathcal{O}' \subseteq \mathcal{L} = \mathcal{O}$.)

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- Proof: $\mathcal{O}'$ is a rank-$d$ $\mathcal{O}$-module. Keep just first coordinate of $b \approx s \cdot a$.

# Middle-Product-LWE

## MP-LWE

- For $s \in \mathbb{Z}_q^{<n+d-1}[x]$ and $a \in \mathbb{Z}_q^{<n}[x]$, the

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with error expansion $\|V_\alpha\|$ of, e.g., spherical Gaussians.

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- Proof sketch: rest of the talk...

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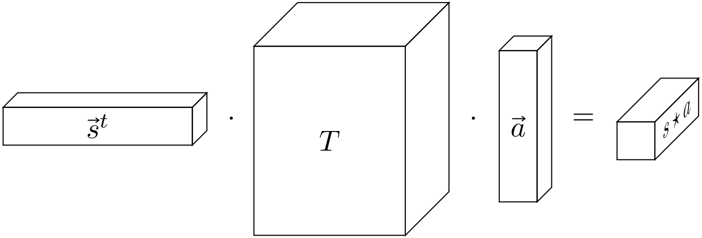

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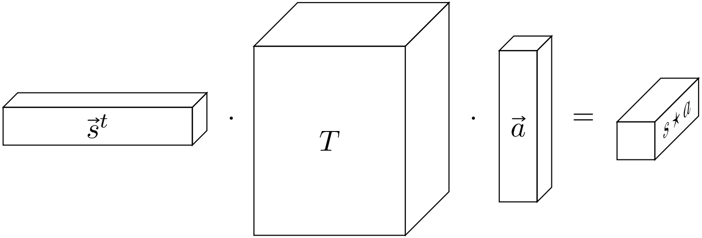
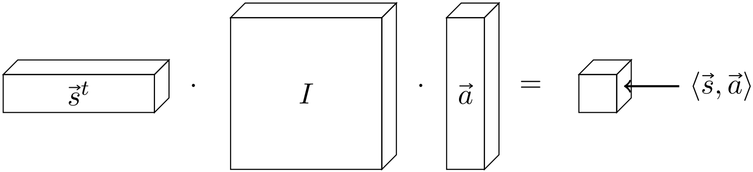
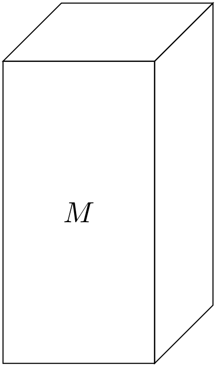



- Plain LWE:



# Middle-Product-LWE$_{n,d}$ Tensor



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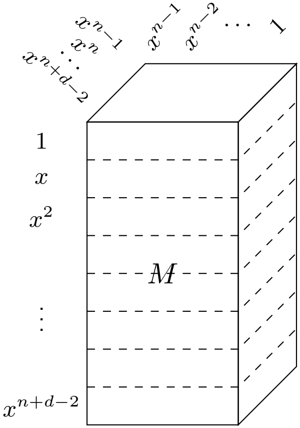

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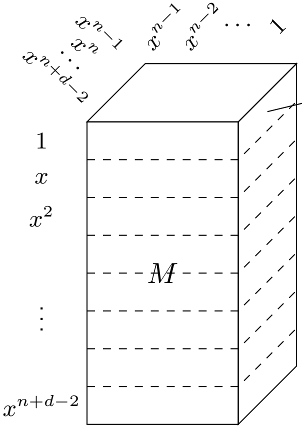


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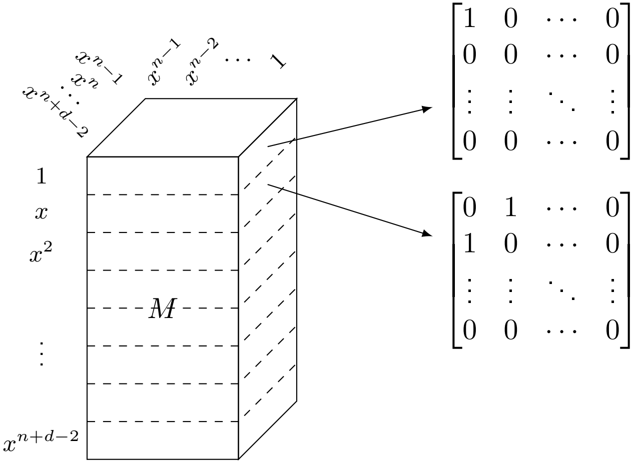

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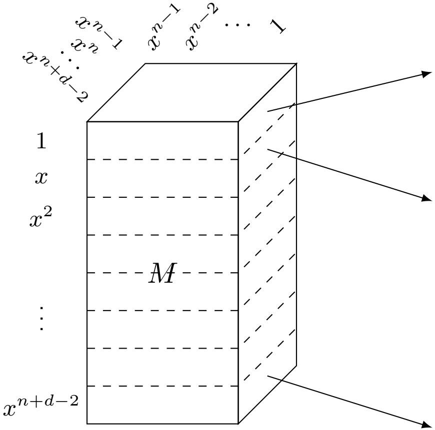


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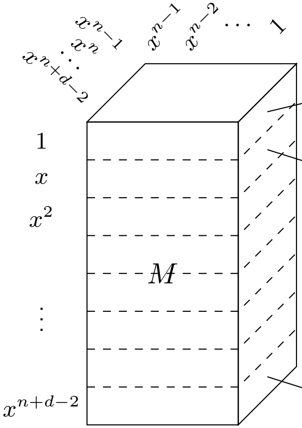


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$$\vdots$$

$$\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

a full basis of $d \times n$ Hankel matrices:

entry $j, k$ given by $j + k$

(constant on anti-diagonals)

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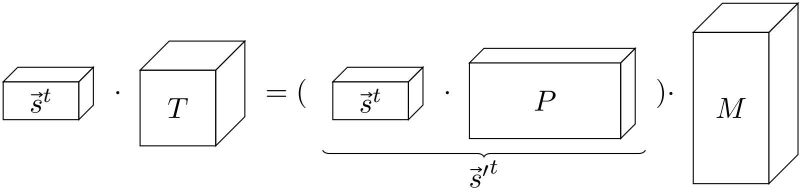

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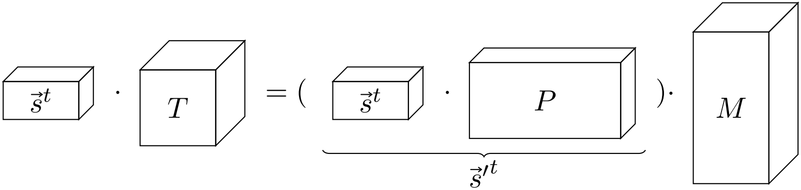


- Generally: $T$-LWE $\leq$ $M$-LWE for any $T, M$ that factor as above.

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Thanks!