Lattice-Based Cryptography

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University of Michigan

QCrypt 2016

Agenda

1. Foundations: lattice problems, SIS/LWE and their applications

2. Ring-Based Crypto: NTRU, Ring-SIS/LWE and ideal lattices

3. Practical Implementations: BLISS, NewHope, Frodo, HElib, $\Lambda \circ \lambda$, ...

4. Along the Way: open questions, research directions

Foundations

Lattice-Based Cryptography

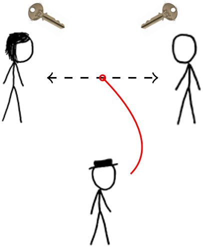
$y = g^x \bmod p$

$N = p \cdot q$

$m^e \bmod N$

$e(g^a, g^b)$

$\Longrightarrow$



(Images courtesy xkcd.org)

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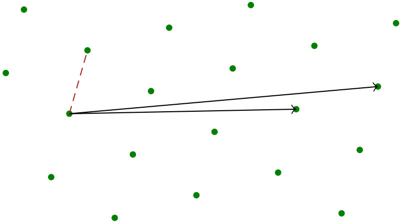

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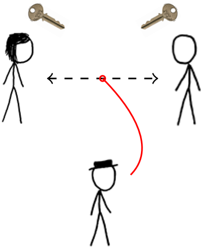

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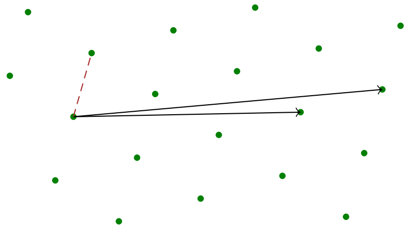

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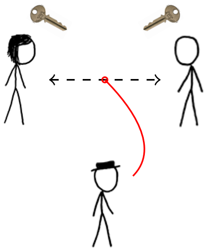

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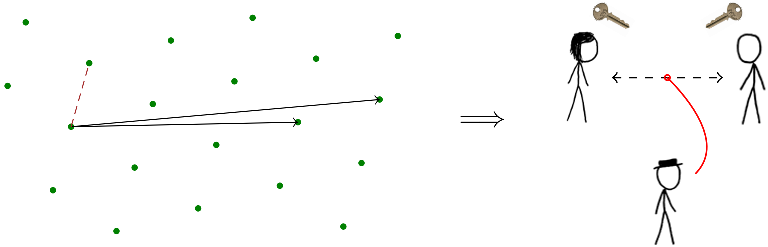

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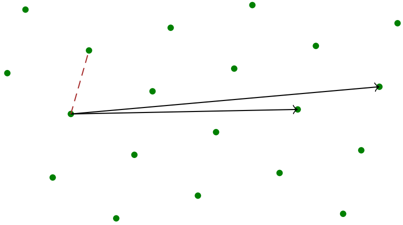

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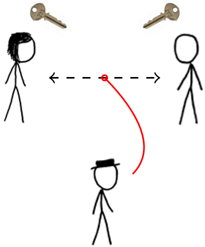

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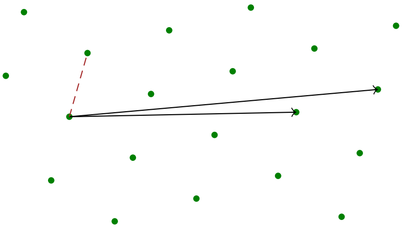

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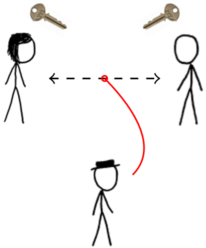

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- Security from mild worst-case assumptions
- Solutions to ‘holy grail’ problems in crypto: FHE and related

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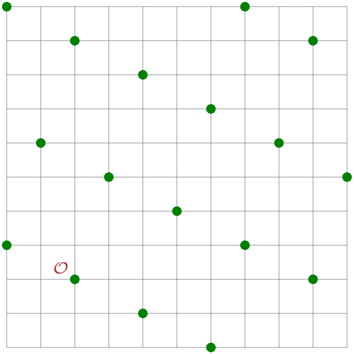

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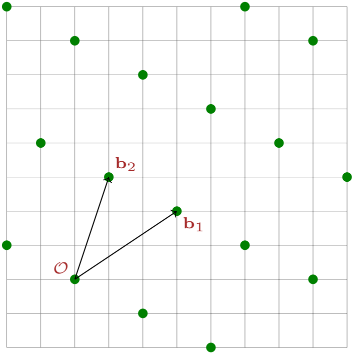

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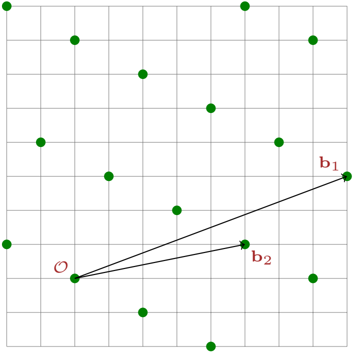

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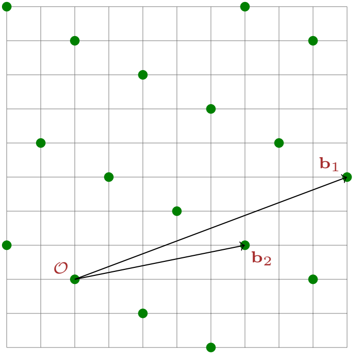

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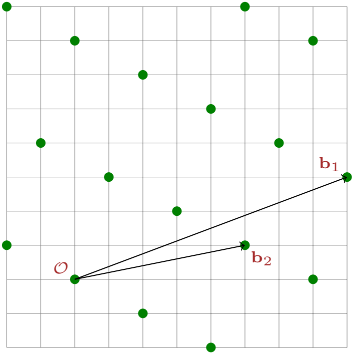

Hard Lattice Problems

- Find/detect 'short' nonzero lattice vectors: (Gap)SVP$_\gamma$, SIVP$_\gamma$
- For $\gamma = \text{poly}(m)$, solving appears to require $2^{\Omega(m)}$ time (and space).

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- Goal: find nontrivial $z_1, \ldots, z_m \in \{0, \pm 1\}$ such that:

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Collision-Resistant Hash Function

- Set $m > n \log_2 q$. Define 'shrinking' $f_{\mathbf{A}} \colon \{0,1\}^m \to \mathbb{Z}_q^n$

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...yields solution $\mathbf{z} = \mathbf{x} - \mathbf{x}' \in \{0, \pm 1\}^m$.

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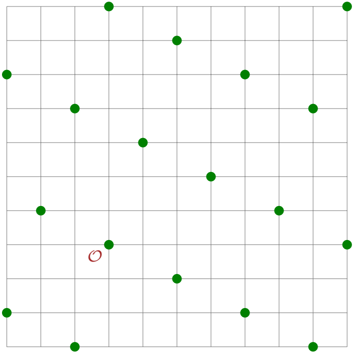

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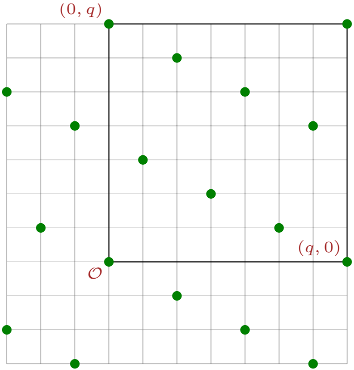

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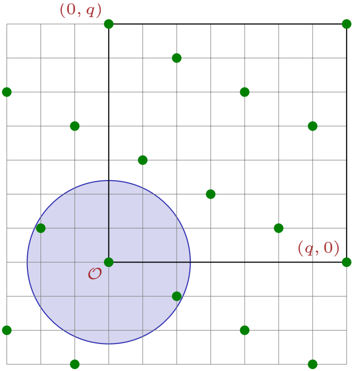

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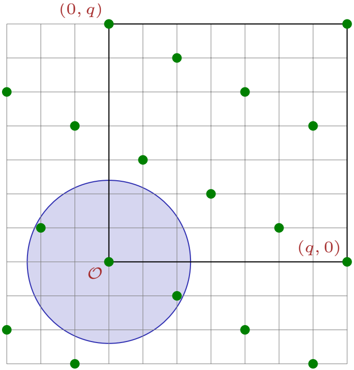

Worst-Case to Average-Case Reduction [Ajtai'96,...]

Finding 'short' ($\|\mathbf{z}\| \leq \beta \ll q$) nonzero $\mathbf{z} \in \mathcal{L}^{\perp}(\mathbf{A})$

(for uniformly random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$)

$$\Downarrow$$

solving $\mathsf{GapSVP}_{\beta\sqrt{n}}, \mathsf{SIVP}_{\beta\sqrt{n}}$ on any n-dim lattice

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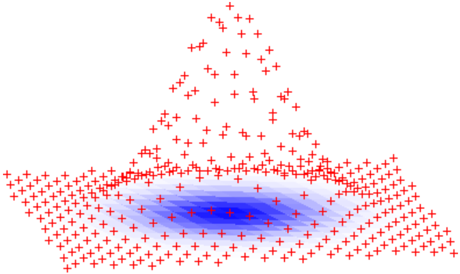

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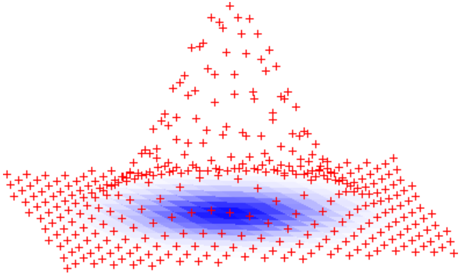

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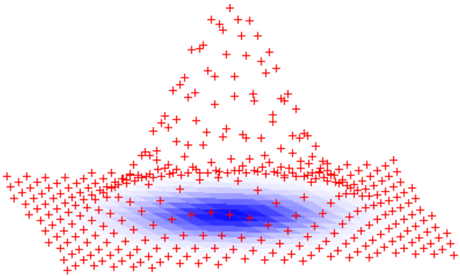

- Verify$(\mathbf{A}, \mu, \mathbf{z})$: check that $\mathbf{A}\mathbf{z} = H(\mu)$ and $\mathbf{z}$ is sufficiently short.
- Security: forging a signature for a new message μ^* requires finding short $\mathbf{z}^*$ s.t. $\mathbf{A}\mathbf{z}^* = H(\mu^*)$. This is SIS: hard!

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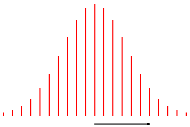

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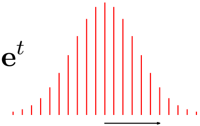

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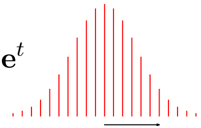

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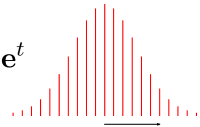

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(n/α)-approx *worst case* lattice problems $\leq$ search-LWE $\leq$ decision-LWE $\leq$ crypto

(quantum [R'05]) [BFKL'93,R'05,...]

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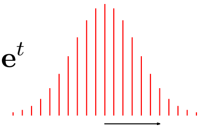

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- Also fully *classical* reductions, for worse params [Peikert'09,BLPRS'13]

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!!! Fully Homomorphic Encryption

!!! Attribute-Based Encryption for arbitrary policies

and much, much more. . .

Key Exchange from LWE [Regev'05,LP'11]

$\mathbf{r} \leftarrow \mathbb{Z}^n$ (error)

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Efficiency from Rings

SIS/LWE are (Sort Of) Efficient

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- Inherently $\geq n^2$ time to encrypt & decrypt an n-bit message.

Wishful Thinking. . .

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- Same ring structures used in NTRU cryptosystem [HPS'98], compact one-way / CR hash functions [Mic'02,PR'06,LM'06,...]

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- **Decision**: <u>distinguish</u> $(a_i \,,\, b_i)$ from uniform $(a_i \,,\, b_i) \in R_q \times R_q$
  (with noticeable advantage)

Hardness of Ring-LWE [LyubashevskyPeikertRegev'10]

- Two main theorems (reductions):

worst-case approx-SVP on *ideal* lattices in R $\leq$ search R-LWE $\leq$ decision R-LWE

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★ If you can break the crypto, then you can distinguish (a_i, b_i) from (a_i, b_i)...

Ideal Lattices

- Say $R = \mathbb{Z}[X]/(X^n + 1)$ for power-of-two n. (Or $R = \mathcal{O}_K$.)
- An ideal $\mathcal{I} \subseteq R$ is closed under $+$ and $-$, and under $\cdot$ with R.

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Error distribution is Gaussian in canonical embedding.

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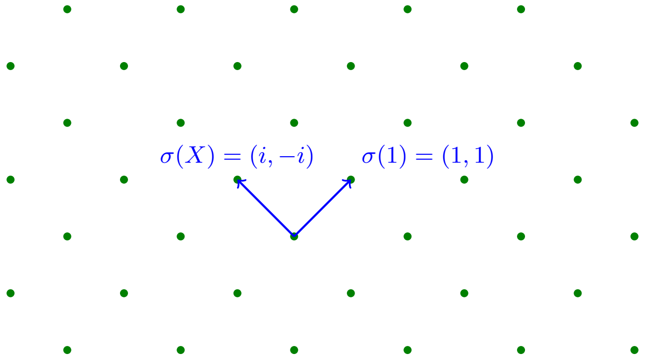

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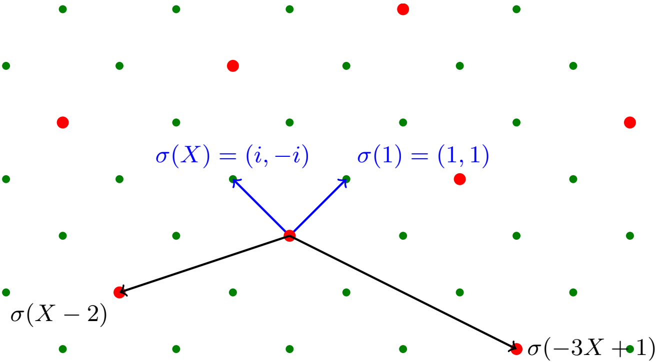

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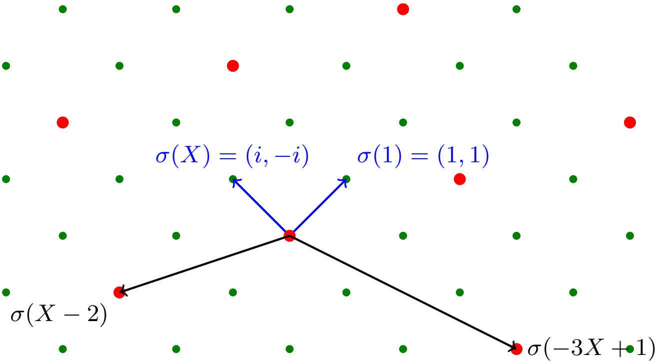

(Approximate) Shortest Vector Problem

- Given (an arbitrary basis of) an arbitrary ideal $\mathcal{I} \subseteq R$, find a nearly shortest nonzero $a \in \mathcal{I}$.

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   - ⋆ There is a $2^{\Omega(\sqrt{n}/\log n)}$ barrier for the main technique. Can it be circumvented?

Implementations

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  About 10x slower than NewHope, but only ≈2x slower than ECDH.

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- Compelling efficiency:

System	Sig (Kb)	PK (Kb)	KSign/sec	KVer/sec
RSA-4096	4.0	4.0	0.1	7.5
ECDSA-256	0.5	0.25	9.5	2.5
BLISS	5.6	7.0	8.0	33

(Conjectured ≥ 128 bits of security, `openssl` implementations.)

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