

On Error Correction in the Exponent

Chris Peikert

MIT Computer Science and AI Laboratory

Theory of Cryptography Conference

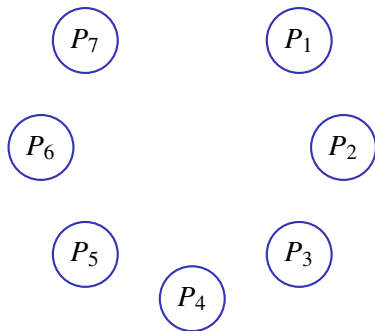
5 March 2006

Error Correction (in the Exponent)

Sharing Secrets (mod q)

- Random $\mathbf{p}(\cdot)$, $\deg(\mathbf{p}) < k$,
s.t. $\mathbf{p}(0) = \text{secret}$.
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(x_1, \dots, x_n) is **Reed-Solomon codewd.**



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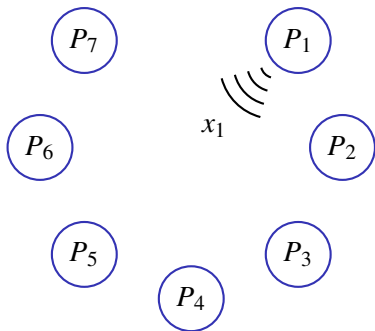
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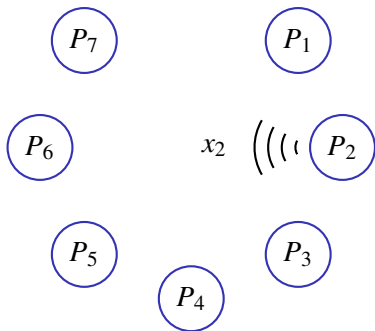
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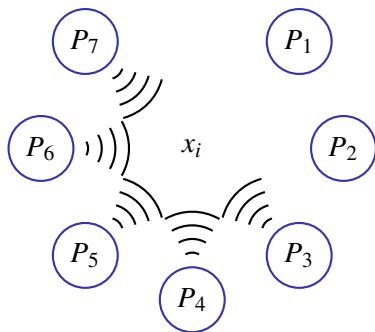
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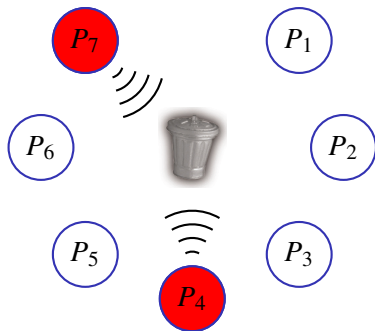
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Error correction: [BeWe86, GuSu98]



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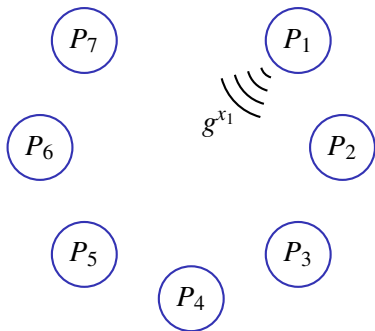
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[CJKR96, PK96, RG03, NPR99, D03, CD04, CG99, BF99, ...]

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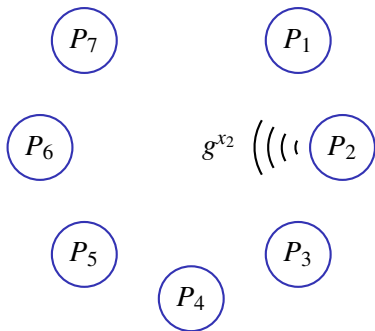
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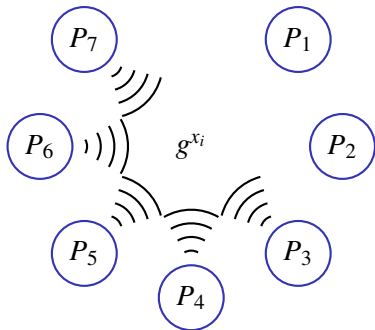
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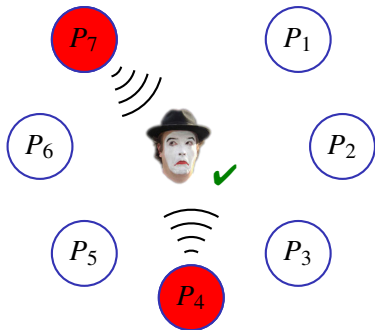
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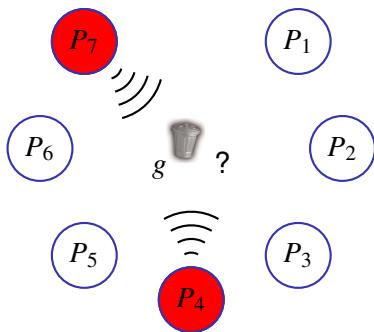
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ERROR CORRECTION: ???

- Guess-and-check: $\frac{n \log n}{k}$ errors



Our Contributions

- ➔ The first **detailed study** of the **complexity of ECE**.

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Unconditional Results

Errors

$$n - \sqrt{nk}$$

$$n - k - k^{1-\epsilon}$$

Complexity

EASY AS DH

HARD AS DLOG

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Unconditional Results

| | <u>Errors</u> | <u>Complexity</u> |
|--------------------------|--------------------------|---------------------|
| Gap | $n - \sqrt{nk}$ | EASY AS DH |
| $\approx \delta \cdot k$ | $n - k - k^{1-\epsilon}$ | HARD AS DLOG |

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Evidence for:

$$\text{DDH} < \text{ECE} \leq \text{DH}$$

A new approach for:

$$\text{DDH} < \text{ECE} \begin{matrix} = & \text{DLOG} \\ & \parallel \\ = & \text{DH} \end{matrix}$$

Theorem

*Decoding (in the exponent) to distance $n - k - k^{1-\epsilon}$ is as **hard as computing discrete logs** in G .*

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Proof Sketch

- 1 Finding a **representation** on uniform $w \in G^n$ is as hard as dlog .
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- Representation on w : nonzero $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_q^n$ s.t.

$$\prod_i w_i^{a_i} = 1.$$

- [Bra93] showed hardness.

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We show $\exists \ell = k + k^{1-\epsilon}$ points $w_i = g^{x_i}$, with x_i on poly of $\text{deg} < k$.

- There are $\binom{n}{\ell}$ *distinct events* (each very rare).
- These events have *limited dependence*.

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 - Decode w to $(g^{x_1}, \dots, g^{x_n})$, where x_i lie on poly of $\text{deg} < k$.
 - There are $\gg k$ points $w_i = g^{x_i}$. $w\text{log}$: w_1, \dots, w_{k+1} .
 - Interpolate in the exponent:

$$w_{k+1} = \prod_{i=1}^k w_i^{\lambda_i} \Rightarrow \text{representation!}$$

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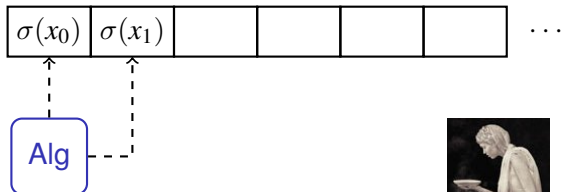
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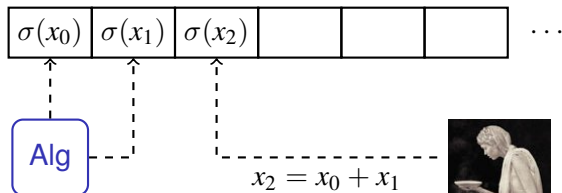
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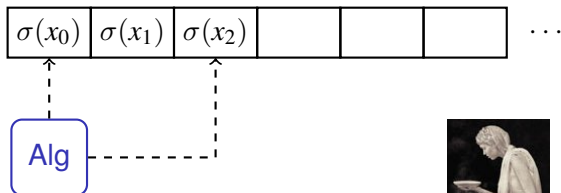
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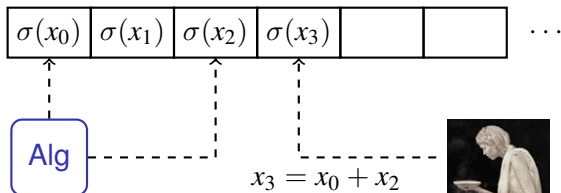
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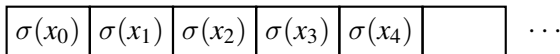
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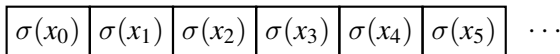


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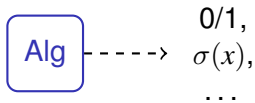
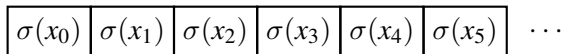
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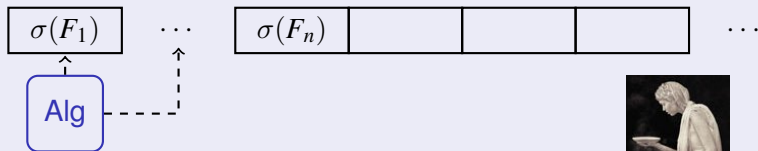
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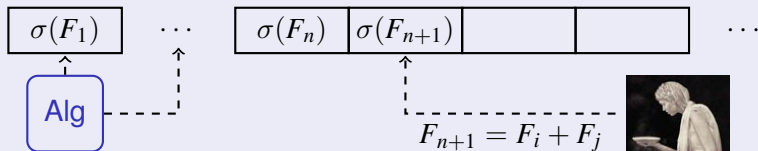
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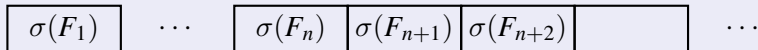
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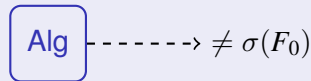
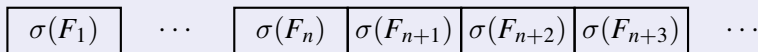
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- Leave \mathbf{p} and \mathbf{e} as **indeterminants**; encode **polynomials** $F(\mathbf{p}, \mathbf{e})$



- Differs from real game** only if $\exists F_i \neq F_j$, but $(F_i - F_j)(\mathbf{p}, \mathbf{e}) = 0$.
Analyze event for **“strange” distribution** of \mathbf{p}, \mathbf{e} .

To Show

For all $F = F_i - F_j \neq 0$, $\Pr[F(\mathbf{p}, \mathbf{e}) = 0]$ is small.

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- 4 By Schwartz's Lemma, $\Pr[F(\mathbf{p}, \mathbf{e}) = 0]$ small.

Question

- Recall: error correction is **easy, given DH oracle**.
- What about **DDH**?

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- Recall: error correction is **easy, given DH oracle**.
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- ☞ **Converse does not appear to hold.**
I.e., error correction seems **strictly harder** than DDH.

Conclusions

- Characterized hardness of ECE for a spectrum of errors.
- Given evidence for $\text{DDH} < \text{ECE}$.
- Suggested a new approach for linking DH and DLOG.

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Thank you!

