Trapdoors for Lattices: Simpler, Tighter, Faster, Smaller

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Lattice-Based Cryptography

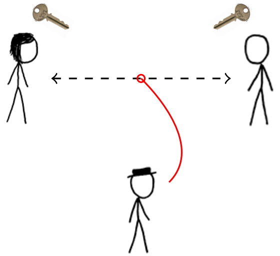
$y = g^x \bmod p$

$m^e \bmod N$

$N = p \cdot q$

$e(g^a, g^b)$

$\Longrightarrow$



(Images courtesy xkcd.org)

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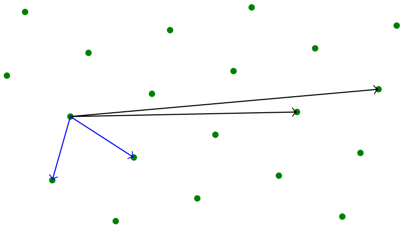

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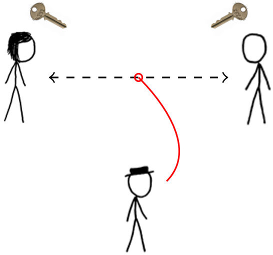

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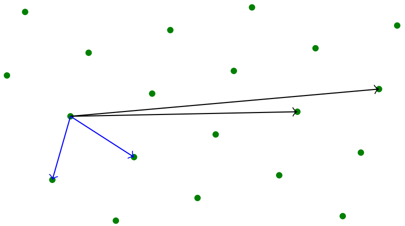

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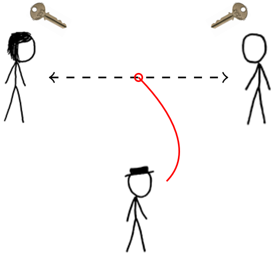

Why?

- Simple & efficient: linear, highly parallel operations
- Resist quantum attacks (so far)
- Secure under worst-case hardness assumptions [Ajtai'96,...]
- Solve 'holy grail' problems like FHE [Gentry'09,...]

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Lattice-Based One-Way Functions

- Public key $\begin{bmatrix} \cdots & \mathbf{A} & \cdots \end{bmatrix} \in \mathbb{Z}_q^{n \times m}$ for $q = \text{poly}(n)$, $m = \Omega(n \log q)$.

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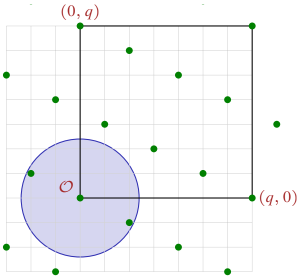

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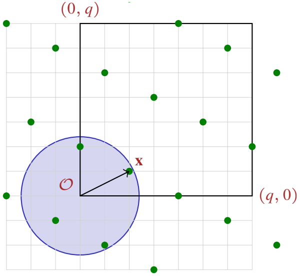

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- $f_{\mathbf{A}}$, $g_{\mathbf{A}}$ in forward direction yield CRHFs, CPA-secure encryption

  ...and not much else.

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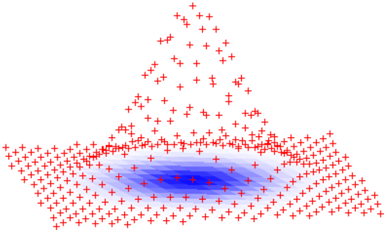

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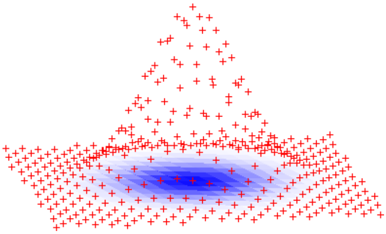
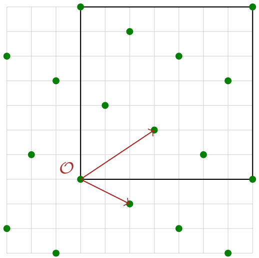
- How? Use a "strong trapdoor" for $\mathbf{A}$: a short basis of $\Lambda^{\perp}(\mathbf{A})$

[Babai'86,GGH'97,Klein'01,GPV'08,P'10]





Applications of Strong Trapdoors

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Other "Black-Box" Applications of f^{-1}, g^{-1}

- Standard model signatures [CHKP'10,R'10,B'10]
- CCA-secure encryption [PW'08,P'09]
- (Hierarchical) ID-based encryption [GPV'08,CHKP'10,ABB'10a,ABB'10b]
- Much more:
  [PVW'08,PV'08,GHV'10,GKV'10,BF'10a,BF'10b,OPW'11,AFV'11,ABVVW'11,...]

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$g_{\mathbf{A}}^{-1}$: [Babai'86] (tight,iterative,fp) vs [Babai'86] (looser,parallel,offline)

$f_{\mathbf{A}}^{-1}$: [Klein'01,GPV'08] (ditto) vs [P'10] (ditto)

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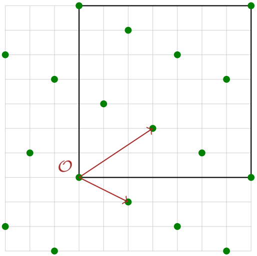

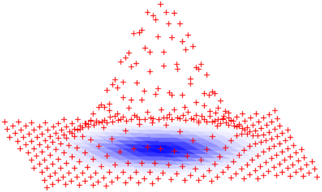

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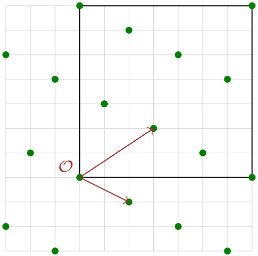

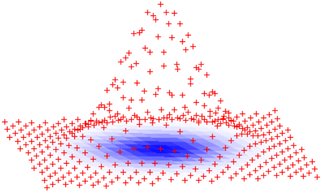

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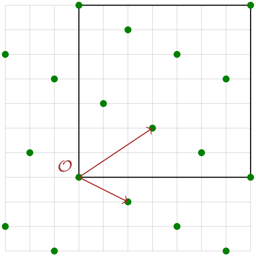

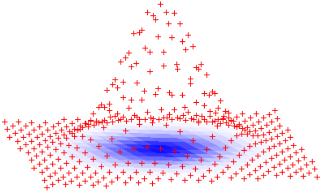

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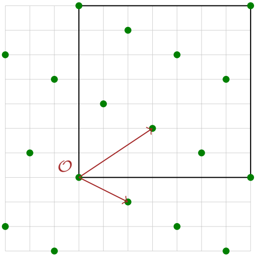

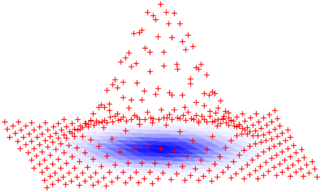

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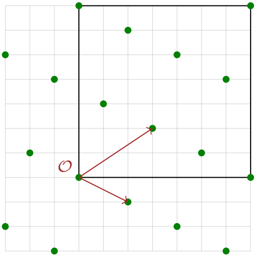

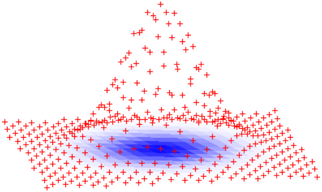

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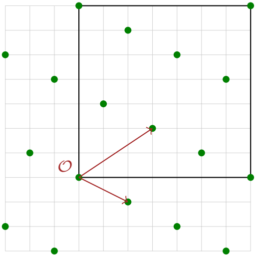

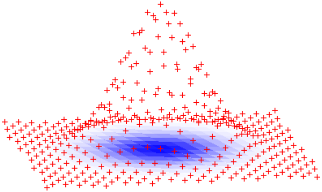

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✔ Better dimension m & quality s

$\implies$ "win-win-win" in security-keysize-runtime

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- ✔ More efficient applications (beyond “black-box” improvements)

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Example parameters for (ring-based) GPV signatures:

	n	q	δ to break	pk size (bits)
Before (fast f^{-1})	436	2^{32}	1.007	$\approx 17 \times 10^6$
Now	284	2^{24}	1.007	$\approx 360 \times 10^3$

Bottom line: $\approx$ 45-fold improvement in key size.

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2. Randomize $\mathbf{G} \leftrightarrow \mathbf{A}$ via a "nice" unimodular transformation.

   (The transformation is the trapdoor!)

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2. Randomize $\mathbf{G} \leftrightarrow \mathbf{A}$ via a “nice” unimodular transformation.
   (The transformation is the trapdoor!)

3. Reduce $f_{\mathbf{A}}^{-1}, g_{\mathbf{A}}^{-1}$ to $f_{\mathbf{G}}^{-1}, g_{\mathbf{G}}^{-1}$ plus pre-/post-processing.

Step 1: Gadget $\mathbf{G}$ and Inversion Algorithms

- Let $q = 2^k$. Define 1-by-k "parity check" vector

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  Also applies to $\mathbf{H} \cdot \mathbf{G}$ for any invertible $\mathbf{H} \in \mathbb{Z}_q^{n \times n}$.

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1. Define semi-random $[\bar{\mathbf{A}} \mid \mathbf{G}]$ for uniform (universal) $\bar{\mathbf{A}} \in \mathbb{Z}_q^{n \times \bar{m}}$.

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Relating New and Old Trapdoors

Given a basis $\mathbf{S}$ for $\Lambda^{\perp}(\mathbf{G})$ and a trapdoor $\mathbf{R}$ for $\mathbf{A}$,

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(But we'll never need to.)

Step 3: Reduce $f_{\mathbf{A}}^{-1}, g_{\mathbf{A}}^{-1}$ to $f_{\mathbf{G}}^{-1}, g_{\mathbf{G}}^{-1}$

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$$\mathbf{b}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right] = \mathbf{s}^t\mathbf{G} + \mathbf{e}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right].$$

Works if each entry of $\mathbf{e}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]$ in $[-\frac{q}{4}, \frac{q}{4})$, e.g. if $\|\mathbf{e}\| < q/(4s_1(\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]))$.

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$$\mathbf{b}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right] = \mathbf{s}^t\mathbf{G} + \mathbf{e}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right].$$

Works if each entry of $\mathbf{e}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]$ in $[-\frac{q}{4}, \frac{q}{4})$, e.g. if $\|\mathbf{e}\| < q/(4s_1(\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]))$.

Sampling Gaussian Preimages

Given $\mathbf{u} = f_{\mathbf{A}}(\mathbf{x}') = \mathbf{A}\mathbf{x}'$, sample $\mathbf{z} \leftarrow f_{\mathbf{G}}^{-1}(\mathbf{u})$ and output $\mathbf{x} = \left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]\mathbf{z}$?

- We have $\mathbf{A}\mathbf{x} = \mathbf{G}\mathbf{z} = \mathbf{u}$ as desired.

Step 3: Reduce $f_{\mathbf{A}}^{-1}, g_{\mathbf{A}}^{-1}$ to $f_{\mathbf{G}}^{-1}, g_{\mathbf{G}}^{-1}$

- Suppose $\mathbf{R}$ is a trapdoor for $\mathbf{A}$ (w/ tag $\mathbf{H} = \mathbf{I}$): $\mathbf{A}\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right] = \mathbf{G}$.

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- Solution: use offline 'perturbation' [P'10] to get spherical Gaussian w/ std dev $\approx s_1(\mathbf{R})$: output $\mathbf{x} = \mathbf{p} + \begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}\mathbf{z}$.

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- Note: $\mathbf{R}'$ is only width$(\mathbf{A}) \times$ width$(\mathbf{G})$ = $m \times n\log q$.

  So size of $\mathbf{R}'$ grows only as $O(m)$, not $\Omega(m^2)$ [CHKP'10].

  Also computationally efficient: $n\log q$ samples, no HNF or ToBasis.

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  Let $msk = sk_\varepsilon = \mathbf{R}_\varepsilon$ and $mpk = \{\mathbf{A}_i\}$ $(d + 1 \text{ vs } \geq 4d + 2)$

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- Security ("puncturing"): Set up mpk, trapdoor $\mathbf{R}$ with tags = id^*.

  Family $\mathcal{H}$ with "invertible differences" from extension ring of $\mathbb{Z}_q$ [DF'94,Fehr'98,ABB'10]

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Questions?