Lattice-Based Cryptography: Constructing Trapdoors and More Applications

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crypt@b-it 2013

Lattice-Based One-Way Functions

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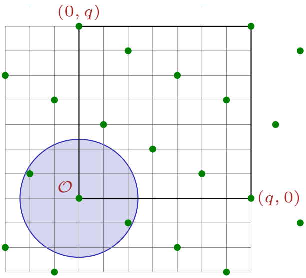

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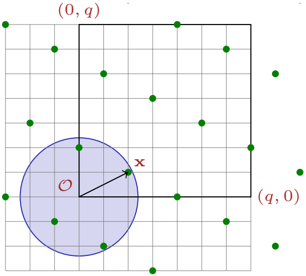

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- $f_{\mathbf{A}}$, $g_{\mathbf{A}}$ in forward direction yield CRHFs, CPA security (w/FHE!)

  ...but not much else.

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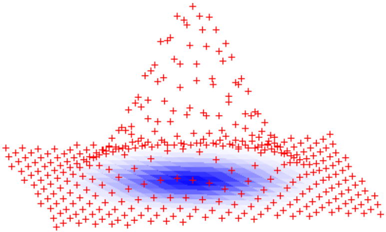
sample random $\mathbf{x} \leftarrow f_{\mathbf{A}}^{-1}(\mathbf{u})$ with prob $\propto \exp(-\|\mathbf{x}\|^2/s^2)$.	find the unique preimage $\mathbf{s}$ (equivalently, $\mathbf{e}$)



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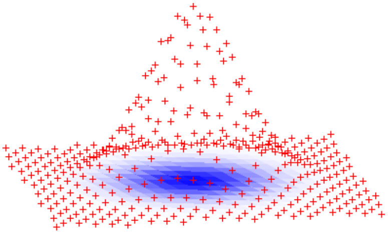
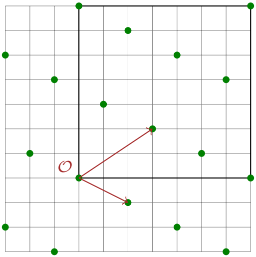
- How? Use a "strong trapdoor" for $\mathbf{A}$: a short basis of $\Lambda^{\perp}(\mathbf{A})$

[Babai'86,GGH'97,Klein'01,GPV'08,P'10]





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Other "Black-Box" Applications of f^{-1}, g^{-1}

- Standard Model (no RO) signatures [CHKP'10,R'10,B'10]
- SM CCA-secure encryption [PW'08,P'09]
- SM (Hierarchical) IBE [GPV'08,CHKP'10,ABB'10a,ABB'10b]
- Many more: OT, NISZK, homom enc/sigs, deniable enc, func enc, ...
  [PVW'08,PV'08,GHV'10,GKV'10,BF'10a,BF'10b,OPW'11,AFV'11,ABVVW'11,...]

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	tight, iterative, fp	looser, parallel, offline
$g_{\mathbf{A}}^{-1}$	[Babai'86]	[Babai'86]
$f_{\mathbf{A}}^{-1}$	[Klein'01,GPV'08]	[P'10]

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- ✔ New kind of trapdoor — not a basis! (But just as powerful.)
- ✔ More efficient applications: CCA, (H)IBE in standard model

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3. Reduce $f_{\mathbf{A}}^{-1}$, $g_{\mathbf{A}}^{-1}$ to $f_{\mathbf{G}}^{-1}$, $g_{\mathbf{G}}^{-1}$ plus pre-/post-processing.

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- Let $q = 2^k$. Define 1-by-k "parity check" vector

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  Also applies to $\mathbf{H} \cdot \mathbf{G}$ for any invertible $\mathbf{H} \in \mathbb{Z}_q^{n \times n}$.

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Relating New and Old Trapdoors

Given a basis $\mathbf{S}$ for $\Lambda^\perp(\mathbf{G})$ and a trapdoor $\mathbf{R}$ for $\mathbf{A}$,
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(But we'll never need to.)

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Inverting LWE Function

Given $\mathbf{b}^t = \mathbf{s}^t\mathbf{A} + \mathbf{e}^t$, recover $\mathbf{s}$ from

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Works if each entry of $\mathbf{e}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]$ in $[-\frac{q}{4}, \frac{q}{4}) \Leftarrow \|\mathbf{e}\| < q/(4s_1(\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]))$.

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Given $\mathbf{u}$, sample $\mathbf{z} \leftarrow f_{\mathbf{G}}^{-1}(\mathbf{u})$ and output $\mathbf{x} = \left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]\mathbf{z} \in f_{\mathbf{A}}^{-1}(\mathbf{u})$?

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- Solution: use offline 'perturbation' [P'10] to get spherical Gaussian w/ std dev $\approx s_1(\mathbf{R})$: output $\mathbf{x} = \mathbf{p} + \left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]\mathbf{z}$.

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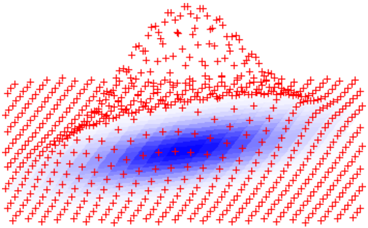

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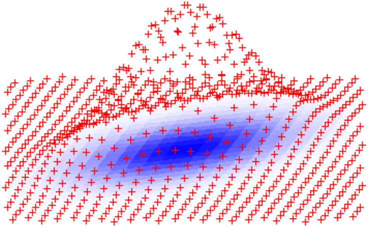
Covariance can be measured — and it leaks $\mathbf{R}$! (up to rotation)



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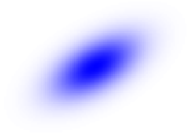
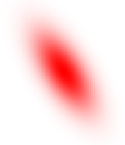
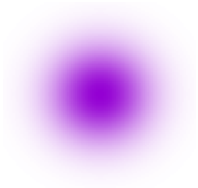
 $+$  $=$ 

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   For $\Sigma_1 = \mathbf{R} \, \mathbf{R}^t$, can use any $\boxed{s > s_1(\mathbf{R}) := \text{max singular val of } \mathbf{R}.}$

‘Convolution’ Sampling Algorithm [P’10]

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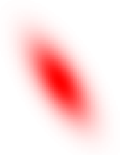

$(s^2\mathbf{I} - \mathbf{R}\mathbf{R}^t)$

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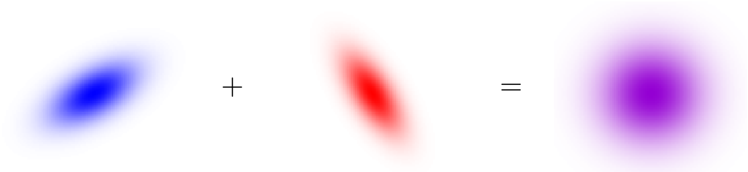

$$\mathbf{R}\mathbf{R}^t \quad + \quad (s^2\mathbf{I} - \mathbf{R}\mathbf{R}^t) \quad = \quad s^2\,\mathbf{I}$$

Convolution* Theorem

Algorithm generates a spherical discrete Gaussian over $\mathcal{L}^{\perp}_{\mathbf{u}}(\mathbf{A})$.

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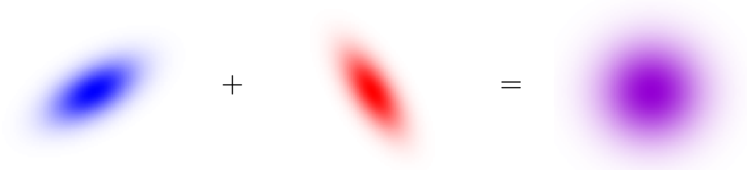
  3. Output $\mathbf{x} = \mathbf{p} + \left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]\mathbf{z}$. (Note: $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{p} + \mathbf{G}\mathbf{z} = \mathbf{u}$.)



$$\mathbf{R}\mathbf{R}^t \quad + \quad (s^2\mathbf{I} - \mathbf{R}\mathbf{R}^t) \quad = \quad s^2\,\mathbf{I}$$

Convolution* Theorem

Algorithm generates a spherical discrete Gaussian over $\mathcal{L}^\perp_{\mathbf{u}}(\mathbf{A})$.

(*technically not a convolution, since step 2 depends on step 1.)

Application: Efficient IBE *a la* [ABB'10]

- Setup: choose $\mathbf{A} = [\bar{\mathbf{A}} \mid -\bar{\mathbf{A}}\mathbf{R}]$. Let $mpk = (\mathbf{A}, \mathbf{u})$, $msk = \mathbf{R}$.
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- Extract$(\mathbf{R}, id)$: map $id \mapsto$ invertible $\mathbf{H}_{id} \in \mathbb{Z}_q^{n \times n}$. [DF'94,...,ABB'10]
  Using $\mathbf{R}$, choose $sk_{id} = \mathbf{x} \leftarrow f_{\mathbf{A}_{id}}^{-1}(\mathbf{u})$, where

$$\mathbf{A}_{id} = \mathbf{A} + [\mathbf{0} \mid \mathbf{H}_{id} \cdot \mathbf{G}] = [\bar{\mathbf{A}} \mid \mathbf{H}_{id} \cdot \mathbf{G} - \bar{\mathbf{A}}\mathbf{R}].$$

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- Encrypt to $\mathbf{A}_{id}$, decrypt using sk_{id} as in 'dual' system [GPV'08].

- Security ("puncturing"): Given target id^* (selective security), set up

$$\mathbf{A} = [\bar{\mathbf{A}} \mid -\mathbf{H}_{id^*} \cdot \mathbf{G} - \bar{\mathbf{A}}\mathbf{R}] \Longrightarrow \mathbf{A}_{id} = [\bar{\mathbf{A}} \mid (\mathbf{H}_{id} - \mathbf{H}_{id^*})\mathbf{G} - \bar{\mathbf{A}}\mathbf{R}].$$

  - ★ $\mathbf{H}_{id} - \mathbf{H}_{id^*}$ is invertible for all $id \neq id^*$, so can extract sk_{id} using $\mathbf{R}$.
  - ★ $\mathbf{A}_{id^*} = [\bar{\mathbf{A}} \mid -\bar{\mathbf{A}}\mathbf{R}]$, so can embed an LWE challenge at id^*.

Trapdoor Delegation [CHKP'10]

- Suppose $\mathbf{R}$ is a trapdoor for $\mathbf{A}$, i.e. $\mathbf{A}\left[\begin{smallmatrix}\mathbf{R}\\\mathbf{I}\end{smallmatrix}\right] = \mathbf{H}\cdot\mathbf{G}$.

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- Note: $\mathbf{R}'$ is only width$(\mathbf{A}) \times$ width$(\mathbf{G}) = m \times n \log q$.

  So size of $\mathbf{R}'$ grows only as $O(m)$, not $\Omega(m^2)$ like a basis does.

  Also computationally efficient: $n \log q$ samples, no HNF or ToBasis.

Hierarchical IBE [CHKP'10,ABB'10]

- Setup(d): choose $\mathbf{A}_0, \ldots, \mathbf{A}_d$ where $\mathbf{A}_\varepsilon = [\mathbf{A}_0 \mid \mathbf{A}_1]$ has trapdoor $\mathbf{R}_\varepsilon$ for tag $\mathbf{0}$. Let $msk = sk_\varepsilon = \mathbf{R}_\varepsilon$ and $mpk = \{\mathbf{A}_i\}$.

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- Encrypt to $\mathbf{A}_{id}$, decrypt using $\mathbf{R}_{id}$ as in [GPV'08].
- Security ("puncturing"): Set up mpk, trapdoor $\mathbf{R}$ with tags $= -id^*$.

Conclusions

- A simple trapdoor that's easy to generate, use, and understand.
- Key sizes and algorithms for "strong" trapdoors are now realistic, with ring techniques (tomorrow)

Selected bibliography for this talk:

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