Problem 1. Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) \(\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle\)

Solution.
\[
2 + 3x + 5x^2
\]

(b) \(\langle 1, 1, 1, 1, 1, \ldots \rangle\)

Solution.
\[
1 + x + x^2 + x^3 + \ldots = \frac{1}{1 - x}
\]

(c) \(\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle\)

Solution.
\[
1 + 2x + 4x^2 + 8x^3 + \ldots = (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots = \frac{1}{1 - 2x}
\]

(d) \(\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle\)

Solution.
\[
1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1 - x^2}
\]

(e) \(\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle\)

Solution.
\[
x^3 + x^4 + x^6 + x^8 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots \ldots) = \frac{x^3}{1 - x}
\]

(f) \(\langle 1, 3, 5, 7, 9, 11, \ldots \rangle\)
Solution.

\[
1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}
\]

\[
\frac{d}{dx} \left(1 + x + x^2 + x^3 + \ldots\right) = \frac{d}{dx} \left(\frac{1}{1-x}\right)
\]

\[
1 + 2x + 3x^2 + 4x^2 + \ldots = \frac{1}{(1-x)^2}
\]

\[
2 + 4x + 6x^2 + 8x^2 + \ldots = \frac{2}{(1-x)^2}
\]

\[
1 + 3x + 5x^2 + 7x^3 + \ldots = \frac{2}{(1-x)^2} - \frac{1}{1-x}
\]

\[
= \frac{1 + x}{(1-x)^2}
\]
Problem 2. Find a closed-form generating function for the sequence

\((t_0, t_1, t_2, \ldots)\)

where \(t_n\) is the number of different ways to compose a bag of \(n\) donuts subject to the following restrictions.

(a) All the donuts are chocolate and there are at least 3.

Solution.

\[
\frac{x^3}{1 - x}
\]

(b) All the donuts are glazed and there are at most 4.

Solution.

\[
\frac{1 - x^5}{1 - x}
\]

(c) All the donuts are coconut and there are exactly 2.

Solution.

\[
x^2
\]

(d) All the donuts are plain and the number is a multiple of 4.

Solution.

\[
\frac{1}{1 - x^4}
\]

(e) The donuts must be chocolate, glazed, coconut, or plain and:

\begin{itemize}
  \item There must be at least 3 are chocolate donuts.
  \item There must be at most 4 glazed.
  \item There must be exactly 2 coconut.
  \item There must be a multiple of 4 plain.
\end{itemize}

Solution.

\[
\frac{x^3}{1 - x} \frac{1 - x^5}{1 - x} \left(\frac{1}{1 - x^2} \frac{1}{1 - x^4}\right) = \frac{x^5(1 + x^2 + x^3 + x^4)}{(1 - x)(1 - x^4)}
\]
Problem 3. [20 points] A previous problem set introduced us to the Catalan numbers: \( C_0, C_1, C_2, \ldots \), where the \( n \)-th of them equals the number of balanced strings that can be built with \( 2n \) paretheses. Here is a list of the first several of them:

| \( n \) | 0 1 2 3 4 5 6 7 8 9 10 11 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( C_n \) | 1 1 2 5 14 42 132 429 1430 4862 16796 58786 208012 |

Then, in lecture we were all amazed to see that the decimal expansion of the irrational number \( 500000 - 1000\sqrt{249999} \)

\[ 1.000001000002000005000014000042000132000429001430004862016796058786208012 \ldots \]

“encodes” these numbers! Now, there are many reasons why one may want to turn to religion, but this revelation is probably not a good one. Let’s explain why.

(a) Let \( p_n \) be the number of balanced strings containing \( n \) ‘s. Explain why the following recurrence holds:

\[
\begin{align*}
  p_0 &= 1, & \text{(the empty string)} \\
  p_n &= \sum_{k=1}^{n} p_{k-1} \cdot p_{n-k}, & \text{for } n \geq 1.
\end{align*}
\]

Solution. Note that every nonempty balanced string consists of a sequence of one or more balanced strings. The first balanced string in the sequence must begin with a ( and end with a “matching” ). That is, any balanced string, \( r_n \), with \( n \geq 1 \) ‘s consists of a balanced string, \( s_{k-1} \), enclosed in brackets and containing \( k - 1 \) ‘s, followed by a balanced string, \( t_{n-k} \), with \( n - k \) ‘s:

\[ r_n = (s_{k-1}) \text{ followed by } t_{n-k} \]

where \( 1 \leq k \leq n \). This observation leads directly to the recurrence.

(b) Now consider the generating function for the number of balanced strings:

\[ P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots . \]

Prove that

\[ P(x) = xP(x)^2 + 1. \]

Solution. We can verify this equation using the recurrence relation.

\[
\begin{align*}
  xP(x)^2 + 1 &= x(p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots)^2 + 1 \\
              &= x(p_0^2 + (p_0p_1 + p_1p_0)x + (p_0p_2 + p_1p_1 + p_2p_0)x^2 + \cdots) + 1 \\
              &= 1 + p_0^2 x + (p_0p_1 + p_1p_1)x^2 + (p_0p_2 + p_1p_1 + p_2p_0)x^3 + \cdots \\
              &= 1 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \\
              &= P(x)
\end{align*}
\]
(c) Find a closed-form expression for the generating function $P(x)$.

**Solution.** Given that $P(x) = xP(x)^2 + 1$, the quadratic formula implies that

$$P(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

If $x$ is small, then $P(x)$ should be about $p_0 = 1$. Therefore, the correct choice of sign is

$$P(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

(d) Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

**Solution.**

$$P(1/1000000) = \frac{1 - \sqrt{1 - 4/1000000}}{2/1000000}$$

$$= 500000 - 500000\sqrt{\frac{249999}{250000}}$$

$$= 500000 - 1000\sqrt{249999}$$

(e) Explain why the digits of this irrational number encode these successive numbers of balanced strings.

**Solution.** Suppose that we symbolically carry out the substitution done in the preceding problem part.

$$P(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots$$

$$P(10^{-6}) = p_0 + p_110^{-6} + p_210^{-12} + p_310^{-18} + \cdots$$

Thus, $p_0$ appears in the units position, $p_1$ appears in the millionths position, $p_2$ appears in the trillionths position, and so forth.
Problem 4. Consider the following recurrence equation:

\[
T_n = \begin{cases} 
1 & n = 0 \\
2 & n = 1 \\
2T_{n-1} + 3T_{n-2} & (n \geq 2)
\end{cases}
\]

Let \( f(x) \) be a generating function for the sequence \( \{T_0, T_1, T_2, T_3, \ldots\} \).

(a) Give a generating function in terms of \( f(x) \) for the sequence:

\[ \langle 1, \ 2, \ 2T_1 + 3T_0, \ 2T_2 + 3T_1, \ 2T_3 + 3T_2, \ldots \rangle \]

Solution. We can break this down into a linear combination of three sequences:

\[
\begin{align*}
\langle 1, \ 2, \ 0, \ 0, \ 0, \ldots \rangle &= 1 + 2x \\
\langle 0, \ T_0, \ T_1, \ T_2, \ T_3, \ldots \rangle &= xf(x) \\
\langle 0, \ 0, \ T_0, \ T_1, \ T_2, \ldots \rangle &= x^2f(x)
\end{align*}
\]

In particular, the sequence we want is very nearly generated by \( 1 + 2x + 2xf(x) + 3x^2f(x) \). However, the second term is not quite correct; we’re generating \( 2 + 2T_0 = 4 \) instead of the correct value, which is 2. We correct this by subtracting \( 2x \) from the generating function, which leaves:

\[ 1 + 2xf(x) + 3x^2f(x) \]

(b) Form an equation in \( f(x) \) and solve to obtain a closed-form generating function for \( f(x) \).

Solution. The equation

\[ f(x) = 1 + 2xf(x) + 3x^2f(x) \]

equates the left sides of all the equations defining the sequence \( T_0, T_1, T_2, \ldots \) with all the right sides. Solving for \( f(x) \) gives the closed-form generating function:

\[ f(x) = \frac{1}{1 - 2x - 3x^2} \]

(c) Expand the closed form for \( f(x) \) using partial fractions.

Solution. We can write:

\[ 1 - 2x - 3x^2 = (1 + x)(1 - 3x) \]

Thus, there exist constants \( A \) and \( B \) such that:

\[ f(x) = \frac{1}{1 - 2x - 3x^2} = \frac{A}{1 + x} + \frac{B}{1 - 3x} \]
Now substituting $x = 0$ and $x = 1$ gives the system of equations:

\[
1 = A + B \\
-\frac{1}{4} = \frac{A}{2} - \frac{B}{2}
\]

Solving the system, we find that $A = 1/4$ and $B = 3/4$. Therefore, we have:

\[
f(x) = \frac{1/4}{1 + x} + \frac{3/4}{1 - 3x}
\]

(d) Find a closed-form expression for $T_n$ from the partial fractions expansion.

**Solution.** Using the formula for the sum of an infinite geometric series gives:

\[
f(x) = \frac{1}{4} \left(1 - x + x^2 - x^3 + x^4 - \ldots\right) + \frac{3}{4} \left(1 + 3x + 3^2x^2 + 3^3x^3 + 3^4x^4 + \ldots\right)
\]

Thus, the coefficient of $x^n$ is:

\[
T_n = \frac{1}{4} \cdot (-1)^n + \frac{3}{4} \cdot 3^n
\]