Notes for Recitation 14

Counting Rules

Rule 1 (Generalized Product Rule). Let $S$ be a set of length-$k$ sequences. If there are:

- $n_1$ possible first entries,
- $n_2$ possible second entries for each first entry,
- $n_3$ possible third entries for each combination of first and second entries, etc.

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

A $k$-to-1 function maps exactly $k$ elements of the domain to every element of the range. For example, the function mapping each ear to its owner is 2-to-1:

- ear 1 → person A
- ear 2 → person A
- ear 3 → person B
- ear 4 → person B
- ear 5 → person C
- ear 6 → person C

Rule 2 (Division Rule). If $f : A \rightarrow B$ is $k$-to-1, then $|A| = k \cdot |B|$. 
The Generalized Product Rule

**Problem 1.** Solve the following counting problems using the generalized product rule.

(a) Next week, I’m going to get really fit! On day 1, I’ll exercise for 5 minutes. On each subsequent day, I’ll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 11, 12. How many such sequences are possible?

**Solution.** The number of minutes on the first day can be selected in 1 way. The number of minutes on each subsequent day can be selected in 4 ways. Therefore, the number of exercise sequences is $1 \cdot 4^6$ by the extended product rule.

(b) An $r$-permutation of a set is a sequence of $r$ distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

\[
\begin{array}{ccc}
(a, b) & (a, c) & (a, d) \\
(b, a) & (b, c) & (b, d) \\
(c, a) & (c, b) & (c, d) \\
(d, a) & (d, b) & (d, c) \\
\end{array}
\]

How many $r$-permutations of an $n$-element set are there? Express your answer using factorial notation.

**Solution.** There are $n$ ways to choose the first element, $n - 1$ ways to choose the second, $n - 2$ ways to choose the third, ..., and there are $n - r + 1$ ways to choose the $r$-th element. Thus, there are:

\[
n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!}
\]

$r$-permutations of an $n$-element set.

(c) How many $n \times n$ matrices are there with distinct entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

**Solution.** There are $p$ ways to choose the first entry, $p - 1$ ways to choose the second for each way of choosing the first, $p - 2$ ways of choosing the third, and so forth. In all there are

\[
p(p - 1)(p - 2) \cdots (p - n^2 + 1) = \frac{p!}{(p - n^2)!}
\]

such matrices. Alternatively, this is the number of $n^2$-permutations of a $p$ element set, which is $p!/(p - n^2)!$. 

The Tao of BOOKKEEPER

**Problem 2.** In this problem, we seek enlightenment through contemplation of the word \textit{BOOKKEEPER}.

(a) In how many ways can you arrange the letters in the word \textit{POKE}?

\textbf{Solution.} There are 4! arrangements corresponding to the 4! permutations of the set \{\textit{P, O, K, E}\}.

(b) In how many ways can you arrange the letters in the word \textit{BO}_1\textit{O}_2\textit{K}? Observe that we have subscripted the \textit{O}'s to make them distinct symbols.

\textbf{Solution.} There are 4! arrangements corresponding to the 4! permutations of the set \{\textit{B, O}_1, \textit{O}_2, \textit{K}\}.

(c) Suppose we map arrangements of the letters in \textit{BO}_1\textit{O}_2\textit{K} to arrangements of the letters in \textit{BOOK} by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

\begin{align*}
O_2&BO_1K \\
KO_2&BO_1 \\
O_1&BO_2K & BOOK \\
KO_1&BO_2 & OBOK \\
BO_1&O_2K & KBOO \\
BO_2&O_1K & \ldots \\
\ldots
\end{align*}

(d) What kind of mapping is this, young grasshopper?

\textbf{Solution.} 2-to-1

(e) In light of the Division Rule, how many arrangements are there of \textit{BOOK}?

\textbf{Solution.} 4!/2

(f) Very good, young master! How many arrangements are there of the letters in \textit{KE}_1\textit{E}_2\textit{PE}_3\textit{R}?

\textbf{Solution.} 6!

(g) Suppose we map each arrangement of \textit{KE}_1\textit{E}_2\textit{PE}_3\textit{R} to an arrangement of \textit{KEEPER} by erasing subscripts. List all the different arrangements of \textit{KE}_1\textit{E}_2\textit{PE}_3\textit{R} that are mapped to \textit{REPEEK} in this way.


(h) What kind of mapping is this?

\textbf{Solution.} 3!-to-1
(i) So how many arrangements are there of the letters in $KEEPER$?
   **Solution.** $6!/3!$

(j) *Now you are ready to face the BOOKKEEPER!*
   How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?
   **Solution.** $10!$

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?
   **Solution.** $10!/2!$

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?
   **Solution.** $10!/(2! \cdot 2!)$

(m) How many arrangements of $BOOKKEEPER$ are there?
   **Solution.** $10!/(2! \cdot 2! \cdot 3!)$

(n) How many arrangements of $VODOODOLL$ are there?
   **Solution.** $10!/(2! \cdot 2! \cdot 5!)$

(o) *(IMPORTANT)* How many $n$-bit sequences contain $k$ zeros and $(n - k)$ ones?
   **Solution.** $n!/(k! \cdot (n - k)!)$
   This quantity is denoted $\binom{n}{k}$ and read “$n$ choose $k$”. You will see it almost every day in 6.042 from now until the end of the term.

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*Remember well what you have learned: subscripts on, subscripts off.*

*This is the Tao of Bookkeeper.*
Problem 3. Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) (IMPORTANT) In how many ways can \( k \) elements be chosen from an \( n \)-element set \( \{x_1, x_2, \ldots, x_n\} \)?

Solution. There is a bijection from \( n \)-bit sequences with \( k \) ones. The sequence \((b_1, \ldots, b_n)\) maps to the subset that contains \( x_i \) if and only if \( b_i = 1 \). Therefore, the number of such subsets is \( \binom{n}{k} \).

(b) How many different ways are there to select a dozen donuts if four varieties are available?

Solution. There is a bijection from selections of a dozen donuts to 15-bit sequences with exactly 3 ones. In particular, suppose that the varieties are glazed, chocolate, lemon, and Boston creme. Then a selection of \( g \) glazed, \( c \) chocolate, \( l \) lemon, and \( b \) Boston creme maps to the sequence:

\[
(g \text{ '0's}) 1 (c \text{ '0's}) 1 (l \text{ '0's}) 1 (b \text{ '0's})
\]

Therefore, the number of selections is equal to the number of 15-bit sequences with exactly 3 ones, which is:

\[
\frac{15!}{3! 12!} = \binom{15}{3}
\]

(c) How many paths are there from \((0, 0)\) to \((10, 20)\) consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate)?

Solution. There is a bijection from 30-bit sequences with 10 zeros and 20 ones. The sequence \((b_1, \ldots, b_{30})\) maps to a path where the \( i \)-th step is right if \( b_i = 0 \) and up if \( b_i = 1 \). Therefore, the number of paths is equal to \( \binom{30}{10} \).

(d) An independent living group is hosting eight pre-frosh, affectionately known at \( P_1, \ldots, P_8 \) by the permanent residents. Each pre-frosh must be assigned a task: 2 must wash pots, 2 must clean the kitchen, 1 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can \( P_1, \ldots, P_8 \) be put to productive use?

Solution. There is a bijection from sequences containing two \( P \)'s, two \( K \)'s, a \( B \), a \( C \), and two \( D \)'s. In particular, the sequence \((t_1, \ldots, t_{10})\) corresponds to assigning \( P_i \) to washing pots if \( t_i = P \), to cleaning the kitchen if \( t_i = K \), to cleaning the bathrooms if \( t_i = B \), etc. Therefore, the number of possible assignments is:

\[
\frac{8!}{2! 2! 1! 1! 2!}
\]
(e) In how many ways can Mr. and Mrs. Grumperson distribute 13 indistinguishable pieces of coal to their two—no, three!—children for Christmas?

**Solution.** There is a bijection from 15-bit strings with two ones. In particular, the bit string $0^a10^b10^c$ maps to the assignment of $a$ coals to the first child, $b$ coals to the second, and $c$ coals to the third. Therefore, there are $\binom{15}{2}$ assignments.

(f) How many solutions over the natural numbers are there to the equation:

$$x_1 + x_2 + \ldots + x_{10} \leq 100$$

**Solution.** There is a bijection from 110-bit sequences with 10 ones to solutions to this equation. In particular, $x_i$ is the number of zeros before the $i$-th one but after the $(i - 1)$-st one (or the beginning of the sequence). Therefore, there are $\binom{110}{10}$ solutions.

(g) (Quiz 2, Fall ’03) Suppose that two identical 52-card decks of are mixed together. In how many ways can the cards in this double-size deck be arranged?

**Solution.** The number of sequences of the 104 cards containing 2 of each card is $104!/(2!)^{52}$. 