Notes for Recitation 10

\[
1 + z + z^2 + \ldots + z^{n-1} = \frac{1 - z^n}{1 - z} \quad (z \neq 1)
\]
\[
1 + z + x^2 + \ldots = \frac{1}{1 - z} \quad (|z| < 1)
\]
\[
1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}
\]
\[
1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(n + 1)}{3}
\]
\[
1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}
\]

**Theorem (Taylor’s theorem).** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is \( n + 1 \) times differentiable on the interval \([0, x]\). Then

\[
f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(z)x^{n+1}}{(n + 1)!}
\]

for some \( z \in [0, x] \).
1 Sums and Approximations

Problem 1. Evaluate the following sums.

(a) \[ 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \]

Solution. The formula for the sum of an infinite geometric series with ratio 1/2 gives:

\[ \frac{1}{1 - \frac{1}{2}} = 2 \]

(b) \[ 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots \]

Solution. The formula for the sum of an infinite geometric series with ratio \(-1/2\) gives:

\[ \frac{1}{1 - (-\frac{1}{2})} = \frac{2}{3} \]

(c) \[ 1 + 2 + 4 + 8 + \ldots + 2^{n-1} \]

Solution. The formula for the sum of a (finite) geometric series with ratio 2 gives:

\[ \frac{1 - 2^n}{1 - 2} = 2^n - 1 \]

(d) \[ \sum_{k=n}^{2n} k^2 \]

Solution.

\[ \sum_{k=n+1}^{2n} k^2 = \sum_{k=1}^{2n} k^2 - \sum_{k=1}^{n} k^2 \]

\[ = \frac{2n(2n + \frac{1}{2})(2n + 1)}{3} - \frac{n(n + \frac{1}{2})(n + 1)}{3} \]

(e) \[ \sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j} \]
Solution.

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j} = \sum_{i=0}^{n} \left( 3^i \cdot \sum_{j=0}^{m} 3^j \right)$$

$$= \left( \sum_{j=0}^{m} 3^j \right) \cdot \left( \sum_{i=0}^{n} 3^i \right)$$

$$= \left( \frac{3^{m+1} - 1}{2} \right) \cdot \left( \frac{3^{n+1} - 1}{2} \right)$$
Problem 2. You’ve seen this neat trick for evaluating a geometric sum:

\[
S = 1 + z + z^2 + \ldots + z^n \\
zS = z + z^2 + \ldots + z^n + z^{n+1} \\
S - zS = 1 - z^{n+1} \\
S = \frac{1 - z^{n+1}}{1 - z}
\]

Use the same approach to find a closed-form expression for this sum:

\[
T = 1z + 2z^2 + 3z^3 + \ldots + nz^n
\]

Solution.

\[
zT = 1z^2 + 2z^3 + 3z^4 + \ldots + nz^{n+1} \\
T - zT = z + z^2 + z^3 + \ldots + z^n - nz^{n+1} \\
= \frac{1 - z^{n+1}}{1 - z} - 1 - nz^{n+1} \\
T = \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{1 + nz^{n+1}}{1 - z}
\]
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Problem 3. Here is a nasty product:
\[
\left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2}{n^2}\right) \left(1 + \frac{3}{n^2}\right) \cdots \left(1 + \frac{n}{n^2}\right)
\]

Remarkably, an expression similar to this one comes up in analyzing the distribution of birthdays. Let’s make sense of it.

(a) Give a two-term Taylor approximation for \(e^x\). (Forget about the error term.)

Solution.
\[ e^x \approx 1 + x \]

(b) This is probably the most wide-used approximation in computer science. The fact that \(x\) appears at “ground level” in the approximation and in the exponent of \(e^x\) lets us translate sums into products and vice-versa. Rewrite the product using this approximation.

Solution.
\[ e^{1/n^2} \cdot e^{2/n^2} \cdot e^{3/n^2} \cdots e^{n/n^2} = e^{\frac{1+2+\ldots+n}{n^2}} \]

(c) Now use a standard summation formula to simplify the exponent.

Solution. The formula \(1 + 2 + 3 + \ldots + n = n(n + 1)/2\) gives:
\[ e^{n(n+1)/(2n^2)} = e^{1/2+1/2n} \]

(d) What constant does this approach for large \(n\)?

Solution. \(\sqrt{e}\)
Problem 4. Let’s use Taylor’s Theorem to find some approximations for the function \( \sqrt{1+x} \).

(a) Give a three-term Taylor approximation for \( \sqrt{1+x} \).

Solution. First, we compute two derivatives:

\[
\frac{d}{dx} \left( \sqrt{1+x} \right) = \frac{1}{2\sqrt{1+x}} \\
\frac{d^2}{dx^2} \left( \sqrt{1+x} \right) = \frac{1}{4(1+x)^{3/2}}
\]

Now we plug into Taylor’s theorem:

\[
f(x) \approx f(0) + xf'(0) + \frac{x^2}{2}f''(0) = 1 + \frac{x}{2} - \frac{x^2}{8}
\]

(b) Sketch the function \( \sqrt{1+x} \) and your approximation. How good is the approximation when \( x = 8 \)?

Solution. The approximation is pretty bad when \( x = 8 \). The actual value is 3, but the approximation is -3.

(c) Using this approximation and the fact that \( \sqrt{1+x} = \sqrt{x} \sqrt{1+1/x} \), give an approximation for \( \sqrt{1+x} \) that is accurate for large \( x \).

Solution.

\[
\sqrt{1+x} = \sqrt{x} \sqrt{1+1/x} \approx \sqrt{x} \left( 1 + \frac{1}{2x} + \frac{1}{8x^2} \right)
\]

(d) Estimate:

\[
\sqrt{1,000,001} \approx 1000 \cdot \left( 1 + \frac{1}{2 \cdot 10^6} + \frac{1}{8 \cdot 10^{12}} \right)
\]

Solution.

\[
\sqrt{1,000,001} \approx 1000.000500000125
\]