ERROR EXPONENT REGIONS FOR MULTI-USER CHANNELS

by

Lihua Weng

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Electrical Engineering: Systems)
in The University of Michigan
2005

Doctoral Committee:

Associate Professor Achilleas Anastasopoulos, Co-Chair
Assistant Professor Sandeep P. Sadanandarao, Co-Chair
Professor Wayne E. Stark
Associate Professor Serap Savari
Assistant Professor Dushyant Sharma

To my parents, my brother and my sisters.

ACKNOWLEDGEMENTS

I would like to thank my thesis co-advisors Achilleas Anastasopoulos and S. Sandeep Pradhan. Achilleas is humorous, brilliant, and creative. My thesis was inspired by his one question - what do the error exponents look like in a broadcast channel? After I plotted a few points on a piece of paper at home, I started to realize that there existed a new degree of freedom - a tradeoff among error exponents. I thank him for his valuable questions and suggestions which went into this thesis.

I am grateful to Sandeep for introducing me to multi-user information theory. Sandeep has broad knowledge in multi-user channels, distributed source coding, sensor networks, and other subjects important to me. His insight and perspective in these areas provided many valuable suggestions and guided me in the right direction. Furthermore, he was very patient regarding my progress and gave me a lot of encouragement while I experienced difficulty in my research. Thank you, Sandeep.

Finally, I would like to thank my parents, my brother and sisters for their love and support. My mother, who raised four children as a single parent twenty years ago, is a brave and strong woman. A thousand thanks for her endless love and understanding.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

LIST OF APPENDICES

APPENDIX

CHAPTER 1

Introduction

In modern communication systems, different applications have different requirements for quality of service (QoS). For example, the third-generation (3G) wireless system is designed to provide various services such as real-time voice service, video telephony, high-speed data transfer, full-motion video, high-quality audio, and so on [1, 2]. In the 3G system, the data transfer rates may vary from 32 kb/s in voice service to over 1 Mb/s in full-motion video, the delay requirements may vary from 1 ms in video telephony to a few seconds in web browsing, and the bit error rates may vary from as high as 10^{-2} in voice service to as low as 10^{-8} in video conferencing. The conventional internet protocols are designed solely for nonreal-time data services, and are inherently suboptimal for networks running heterogenous applications. One of the biggest challenges for modern communication system designers is to design a system which simultaneously supports several QoS requirements while still providing high-efficiency services. Although the complete design issues regarding QoS in communication networks are quite complex, in this work we focus on one key aspect of QoS - bit error rate. In particular, we are interested in achieving different bit error rates for different users in a multi-user system. Some practical solutions to this problem have been proposed in the literature, collectively known as unequal error protection (UEP). These techniques, which provide UEP to different users, can be divided roughly into two categories - time-division coded modulation (TDCM) and superposition coded modulation (SCM) [3, 4]. TDCM is a form of resource sharing in which different users transmit on disjoint time intervals. In SCM, both users transmit on the same time intervals using superposition of channel codes. For practical channel codes, there exist examples where TDCM, or a hybrid of TDCM and SCM, outperforms SCM [3, 4].

Although practical UEP schemes have been deployed in existing systems, there is currently no framework in information theory which deals with different bit error rates in a multi-user channel. A traditional approach concerning bit error rates in a point-to-point channel is the study of the reliability-rate tradeoff through the notion of error exponent [5, 6, 7, 8, 9, 10, 11], which is also known as the reliability function of a channel. A straightforward extension of this concept can be realized in a multi-user setting by defining the *probability of system error*. A system is considered to be in error if at least one user's codeword is decoded erroneously. For the study of the capacity region of a multi-user channel, it is sufficient to show that this single performance measure, probability of system error, approaches zero as the block length increases. This approach, however, does not solve the problem of assigning different error protections to different users in a multi-user channel, since there is only one error probability considered here, i.e., the probability of system error. Therefore, on the one hand, there are practical schemes to provide different error protections for different users, but on the other hand, the current information-theoretic analysis can not cope with the issues of QoS in a multi-user system. Hence our goal is to provide an information-theoretic framework which can address these issues by giving concrete design methodologies for such systems.

We ask the following question: is it possible to simultaneously provide an increased reliability for one user and a reduced reliability for the other user, while keeping their rates the same in a two-user channel? More generally, for a fixed pair of data rates for the two users, is it possible to provide a set of choices of individual reliabilities for these two users? The main contribution of this work is to provide a positive answer to these questions by formalizing these ideas in the context of information theory, studying the fundamental limits of such tradeoffs of individual reliabilities among the users for a fixed vector of data rates, and developing efficient transmission strategies that approach these limits. This is done by defining individual error probabilities for each user and studying the tradeoff of the corresponding error exponents. This tradeoff is quantified by introducing the concept of *error exponent region* (EER) for a multi-user channel. Although the idea proposed in this work is very general, we present it in the context of broadcast and multiple access channels [12, 13, 14, 15].

1.1 Thesis Outline

We first review the concept of error exponent in Chapter 2, and provide an overview of the reliability-rate tradeoff for single-user and multi-user channels as studied in the literature. We then review the diversity-multiplexing tradeoff in multiple-input multiple-output (MIMO) fading channels obtained by Zheng and Tse [16]. The multiplexing gain and the diversity gain defined in [16] can be regarded as the rate and the error exponent at high signal-to-noise ratio (SNR), respectively.

In Chapter 3, we introduce the notion of error exponent regions for multi-user channels. Our approach hinges on the following two observations. First, one can define a separate probability of error for each user. Therefore, there can be multiple error exponents, one for each user. Second, in contrast to a single-user channel where the error exponent is fixed for a given rate, in a multi-user channel one can tradeoff the error exponents among different users even for a fixed vector of users' rates. We consider only two kinds of multi-user channels in this work, either a two-user broadcast channel or a two-user multiple access channel. Thus the term "multi-user channel" in this work is referred to either one of these two kinds of multi-user channels.

In Chapter 4, we consider the EERs for Gaussian broadcast channels. We derive inner and outer bounds for the EERs. The inner bound is derived based on the random codebook method. In contrast to the standard approach, we use *two* different probability distributions to construct each user's random codebook. It turns out that the achievable region is enlarged by this approach. The outer bound is derived by transforming the original broadcast channel into a superior single-user channel, then applying the error exponent upper bounds for a single-user channel. One of the main goals in this work is to show that one can tradeoff error exponents among the users even for a fixed vector of transmission rates. From the derived inner and outer bounds, we show that this tradeoff indeed exists.

In Chapter 5, we consider the EERs for Gaussian multiple access channels. Inner and outer bounds for the EERs are derived based on similar techniques used in broadcast channels. In contrast to Gaussian broadcast channels, the EER inner and outer bounds for Gaussian multiple access channels are tight in some cases. We also show that a Gaussian multiple access channel is equivalent to two independent Gaussian single-user channels in the sense that each user does not suffer the side-interference from the other user at low rates.

In Chapter 6, we provide improved outer bounds for the EERs for Gaussian broadcast and multiple access channels, which explicitly incorporate the fact that two users are simultaneously communicating with one transmitter (in a broadcast channel) or receiver (in a multiple access channel). In particular, we extend the concept of minimum distance bound to a multi-user setting with multiple message sets. The proofs of these new outer bounds are based on a geometric conjecture, which deals with packing codewords on a high-dimension sphere under an area constraint. Thus the final results depend on the correctness of this geometric conjecture.

In Chapter 7, we turn our attention to MIMO fading channels and consider the diversity gain regions (DGR). The DGR can be regarded as the EER at high SNR. In contrast to a Gaussian broadcast channel, we show that it is possible for either one of the two users to achieve the optimum single-user diversity gain without suffering the side-interference from the other user. We also define the multiplexing gain region (MGR), which is the counterpart of the channel capacity region, and derive inner and outer bounds for the MGR.

In Chapter 8, we consider the DGRs for MIMO fading multiple access channels. Due to lack of an effective bounding technique for our decoding strategy, the DGR inner bound derived here is achieved by two simpler encoding strategies. Similar to Gaussian multiple access channels, the DGR inner and outer bounds are tight in some cases.

In Chapter 9, we discuss some future directions and consider an on-going work - the EERs for discrete memoryless multi-user channels. In contrast to Gaussian or MIMO fading multi-user channels, there is no one single "optimum" input distribution in discrete memoryless multi-user channels, whereas in Gaussian or MIMO fading multi-user channels the optimum input distribution is (shelled) Gaussian. We derive an EER outer bound for a discrete memoryless multiple access channel, where the boundary of the EER outer bound is achieved by different probability distributions.

1.2 Notation

The following notation is used throughout this work. $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ denote the sets of real numbers, complex numbers and integers, respectively. We write $a \triangleq b$ to mean "a is defined as b". $\mathbb{R}_+^n$ is the set of real n-vectors with nonnegative elements, and $(x)^+$ is defined as $\max(x, 0)$, i.e., $(x)^+ \triangleq \max(x, 0)$. We use boldface letters to denote random variables (e.g. $\mathbf{X}$), and lightface letters to denote their realizations (e.g. X). The calligraphic letters $\mathcal{A}$, $\mathcal{B}$, etc., denote general sets or probability events. The abbreviation "i.i.d" stands for independent and identically distributed. The zero-mean, unit-variance, real Gaussian distribution is denoted by $\mathcal{N}(0, 1)$, and the zero-mean, unit-variance, circular symmetric, complex Gaussian distribution is denoted by $\mathcal{CN}(0, 1)$. We don't distinguish between a scalar and a matrix in our notation.

CHAPTER 2

Background: Error Exponent and Diversity Gain

It is well-known that the error exponent for a single-user channel provides the rate of exponential decay of the average probability of error as a function of the block length of the codebooks [5, 6, 7, 8, 9, 10, 11]. The concept of the error exponent was extended to a Gaussian multiple access channel in [17, 18], where an upper bound on the *probability of system error* (i.e., the probability that any user is in error) was derived for random codes. Zheng and Tse considered error exponents under a high signal-to-noise ratio (SNR) approximation, called diversity gains, for MIMO fading single-user channels [16], and for MIMO fading multiple access channels [19]. In this chapter, we briefly review some basic results regarding error exponents and diversity gains for single-user and multiple access channels.

2.1 Error Exponent

Consider a discrete-time memoryless stationary single-user channel. Let $P_e(N, R)$ denote the smallest average probability of block decoding error, i.e., codeword error, of any code of block length N and rate R for this channel. The error exponent at rate R is defined as

$$E(R) \triangleq \lim_{N\to\infty} -\frac{\log P_e(N,R)}{N}, \tag{2.1}$$

where the limit in (2.1) (and throughout this work) should be interpreted as lim sup or lim inf from the context whenever the limit does not exist. Define $f(N) \cong e^{Nb}$ if

$$\lim_{N\to\infty} \frac{\log f(N)}{N} = b, \tag{2.2}$$

and $\gtrsim$, $\lesssim$ are defined similarly. Thus, the probability of error $P_e(N,R)$ can be written as $P_e(N,R) \cong e^{-NE(R)}$.

Error exponents have been studied in detail for discrete memoryless channels and additive white Gaussian noise (AWGN) channels [5, 6, 7, 8, 9, 10, 11]. Lower and upper bounds are known for the error exponent $E(R)$ for these channels. A lower bound, known as random coding exponent $E_r(R)$, was developed by Fano [8]. The random coding exponent was tightened at low rates by Gallager to yield expurgated exponent $E_{ex}(R)$ [10]. Two upper bounds, known as sphere packing exponent $E_{sp}(R)$ and minimum distance exponent $E_{md}(R)$, were developed by Shannon, Gallager, and Berlekamp [11]. A straight line connecting any two points of $E_{sp}(R)$ and $E_{md}(R)$ was shown to be an error exponent upper bound, which is known as straight line exponent $E_{st}(R)$ [11]. The random coding exponent and the sphere packing exponent agree at rates $R \geq R_{crit}$, where R_{crit} is called the critical rate. These five bounds are shown in Fig. 2.1. The lower solid curve and the upper solid curve are the expurgated exponent $E_{ex}(R)$ and the straight line exponent $E_{st}(R)$, respectively. The dashed curve is the random coding exponent $E_r(R)$. The dash-dotted curve is the sphere packing exponent $E_{sp}(R)$, which agrees with $E_r(R)$ for $R \geq R_{crit}$. The dotted curve is the minimum distance exponent $E_{md}(R)$. For $R < R_{crit}$, the error exponent lies inside the shaded region.

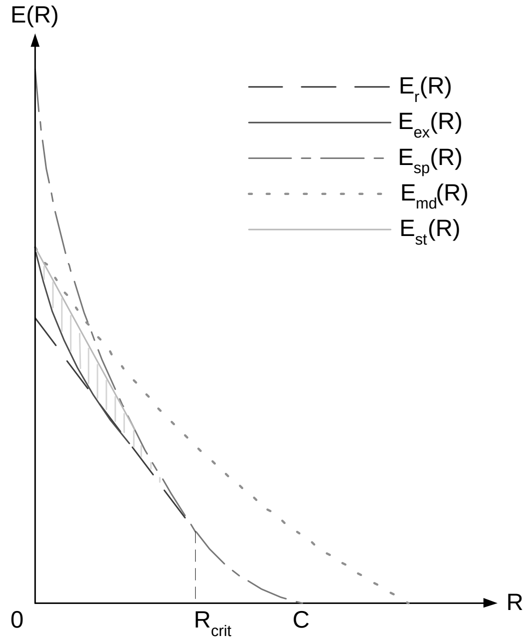


Figure 2.1: Error exponent lower and upper bounds.

Error exponents have also been studied for multiple access channels [17, 18]. For a given multiple access channel, let $P_{e,sys}(N, R_1, R_2)$ denote the smallest average probability of block decoding system error of any code of block length N and rates R_1, R_2 for user 1, user 2, respectively. The error exponent for a multiple access channel is defined as

$$E_{sys}(R_1, R_2) \triangleq \lim_{N\to\infty} -\frac{\log P_{e,sys}(N, R_1, R_2)}{N}. \tag{2.3}$$

In the following, we summarize the basic technique used by Gallager to provide an upper bound on the probability of system error in a multiple access channel [17]. A variation of this method will be used later to provide similar upper bounds. Consider a codebook $CB_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,M_1}\}$ for user 1, where $C_{1,i}$ is the i^{th} codeword with length N $(1 \le i \le M_1)$ and M_1 is the number of the codewords in the codebook CB_1. Similarly, $CB_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,M_2}\}$ is a codebook for user 2. Gallager [17] derived the random coding exponent using *joint* maximum likelihood (ML) decoding, i.e., decoding users' messages based on the pair (i, j) maximizing $P(Y^N|C_{1,i}, C_{2,j})$, where Y^N is the received sequence of length N. Let $(\hat{i}, \hat{j})$ denote the indexes of the decoded codewords for user 1 and user 2. The probability of system error can be written as

$$\begin{aligned} P_{e,sys} &= P(\hat{i} \neq i \text{ or } \hat{j} \neq j) \\ &= P(\hat{i} \neq i \text{ and } \hat{j} = j) + P(\hat{i} = i \text{ and } \hat{j} \neq j) + P(\hat{i} \neq i \text{ and } \hat{j} \neq j) \\ &= P_{e,t1} + P_{e,t2} + P_{e,t3}, \end{aligned} \tag{2.4}$$

where we define

$$\begin{aligned} P_{e,t1} &\triangleq P(\hat{i} \neq i \text{ and } \hat{j} = j) \\ P_{e,t2} &\triangleq P(\hat{i} = i \text{ and } \hat{j} \neq j) \\ P_{e,t3} &\triangleq P(\hat{i} \neq i \text{ and } \hat{j} \neq j). \end{aligned} \tag{2.5}$$

Thus there are three types of error events. Type 1 error occurs when user 1's codeword is decoded erroneously, but user 2's codeword is decoded correctly. Type 2 error occurs when user 2's codeword is decoded erroneously, but user 1's codeword is decoded correctly. Type 3 error occurs when both users' codewords are decoded as wrong codewords. Applying the random coding argument, it was shown in [17] that there exist codebooks CB_1 and CB_2 such that $P_{e,ti}$ can be upper bounded by

$$\begin{aligned} P_{e,t1} &\leq e^{-NE_{t1}(R_1)} \\ P_{e,t2} &\leq e^{-NE_{t2}(R_2)} \\ P_{e,t3} &\leq e^{-NE_{t3}(R_1+R_2)}, \end{aligned} \tag{2.6}$$

where E_{ti}, $1 \leq i \leq 3$, is an exponent which accounts for type i error. The probability of system error can be upper bounded by

$$\begin{aligned} P_{e,sys} &= P_{e,t1} + P_{e,t2} + P_{e,t3} \\ &\leq e^{-NE_{t1}(R_1)} + e^{-NE_{t2}(R_2)} + e^{-NE_{t3}(R_1+R_2)} \\ &\leq 3e^{-N\min\{E_{t1}(R_1),E_{t2}(R_2),E_{t3}(R_1+R_2)\}}, \end{aligned} \tag{2.7}$$

and the system error exponent can be lower bounded by

$$E_{sys}(R_1, R_2) \geq \min\{E_{t1}(R_1), E_{t2}(R_2), E_{t3}(R_1 + R_2)\}. \tag{2.8}$$

2.2 Diversity Gain

Error exponents have also been studied for MIMO fading single-user channels in the high SNR regime [16]. Consider a MIMO fading single-user channel with m transmit antennas and n receive antennas. The channel model is

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}}\mathbf{H}\mathbf{X} + \mathbf{Z}. \tag{2.9}$$

The channel fading matrix between the transmitter and the receiver is represented by an $n \times m$ matrix $\mathbf{H}$. We assume that $\mathbf{H}$ remains constant over a block with length l, and changes to a new independent realization in the next block. $\mathbf{H}$ has i.i.d. entries distributed as $\mathcal{CN}(0, 1)$. We assume that the fading matrix $\mathbf{H}$ is known by the receiver but not known by the transmitter. $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average SNR at each receive antenna is SNR. The noise $\mathbf{Z}$ is an $n \times l$ matrix with i.i.d. entries $\mathcal{CN}(0, 1)$. The channel output $\mathbf{Y}$ is an $n \times l$ matrix.

In contrast to a single-antenna system, multiple antennas provide spatial diversity, which can be used to support higher data rate than a single-antenna system. The channel capacity of a MIMO fading channel can be written as

$$C(SNR) = \min\{m, n\} \log SNR + O(1) \tag{2.10}$$

at high SNR [20]. In addition to supporting higher data rate, multi-antenna channels can also improve the reliability of the link. As an example, the probability of error at high SNR for uncoded binary phase-shift keying (PSK) signals over a single-antenna fading channel ($m = n = l = 1$) is approximately $\frac{1}{4} SNR^{-1}$, but the probability of error for a receiver equipped with two antennas is approximately $\frac{3}{16} SNR^{-2}$ at high SNR [21]. Note that in both cases the probabilities of error go to zero as SNR goes to infinity and this implies that the error exponent goes to infinity as SNR goes to infinity. Nevertheless, we can define a "normalized" error exponent, with respect to SNR, for a multi-antenna channel. To be specific, let the channel be operated at a rate $R = R(SNR)$ which is a fraction of the channel capacity at high SNR, i.e.,

$$\lim_{SNR \to \infty} \frac{R(SNR)}{\log SNR} = r. \tag{2.11}$$

The normalized error exponent $e(r)$ (with respect to SNR) is thus defined as

$$e(r) \triangleq \lim_{SNR \to \infty} \frac{E(R(SNR))}{\log SNR} = \lim_{SNR \to \infty} \left\{ \lim_{N \to \infty} -\frac{\log P_e(N, R(SNR))}{N \log SNR} \right\}. \tag{2.12}$$

If we define $d(N, r)$ as

$$d(N, r) \triangleq \lim_{SNR \to \infty} -\frac{\log P_e(N, R(SNR))}{N \log SNR}, \tag{2.13}$$

and if we can exchange the order of $\lim_{SNR \to \infty}$ and $\lim_{N \to \infty}$ in (2.12), then the normalized error exponent can be expressed as $e(r) = \lim_{N\to\infty} d(N, r)$. [1] Define $f \doteq SNR^b$ if

$$\lim_{SNR\to\infty} \frac{\log f}{\log SNR} = b, \tag{2.14}$$

and $\dot{\geq}, \dot{\leq}$ are defined similarly. Thus (2.13) can be written as $P_e(N, R(SNR)) \doteq SNR^{-Nd(N,r)}$.

It was shown in [16] that in a MIMO fading single-user channel with m transmit antennas, n receive antennas, and block length l, both the random coding diversity gain $d_{m,n,l}(r)$ and the expurgated diversity gain $d^{ex}_{m,n,l}(r)$ are lower bounds of $d(r) \triangleq d(1, r)$, where $d(r)$ is used as a shorthand notation for $d(1, r)$, and r and $d(r)$ were referred to as the multiplexing gain and the diversity gain, respectively, in [16]. The random coding diversity gain $d_{m,n,l}(r)$ is defined as

$$d_{m,n,l}(r) = \min_{\underline{\alpha}\in\mathbb{R}_+^{\min(m,n)}\setminus\mathcal{B}} \Big\{ \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i + l\big[\sum_{i=1}^{\min(m,n)} (1-\alpha_i)^+ - r\big] \Big\}, \tag{2.15}$$

with

$$\mathcal{B} = \Big\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{\min(m,n)} \geq 0; \sum_{i=1}^{\min(m,n)} (1-\alpha_i)^+ < r \Big\}, \tag{2.16}$$

where $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{\min(m,n)})$. The expurgated diversity gain $d^{ex}_{m,n,l}(r)$ is defined

[1] In general, the order of $\lim_{SNR\to\infty}$ and $\lim_{N\to\infty}$ might not be exchangeable, i.e. $\lim_{x\to\infty}\lim_{y\to\infty} f(x,y)$ might not be equal to $\lim_{y\to\infty}\lim_{x\to\infty} f(x,y)$ in general, where $f(x,y)$ is an arbitrary function with variables x and y. Moreover, we don't know how to evaluate $\lim_{N\to\infty} -\frac{\log P_e(N,R(SNR))}{N\log SNR}$, but an upper bound and a lower bound for $\lim_{SNR\to\infty} -\frac{\log P_e(N,R(SNR))}{N\log SNR}$ were derived in [16].

as

$$d^{ex}_{m,n,l}(r) = n\ d^{-1}_{m,l,n}(lr), \tag{2.17}$$

where $d^{-1}_{m,l,n}$ is the inverse function of $d_{m,l,n}(r)$. In addition, it was shown in [16] that $d(N, r)$ is upper bounded by the outage diversity gain $d^{out}_{m,n}(r)$, which is the piecewise linear function connecting the points $(k, d^{out}_{m,n}(k)) = (k, (m-k)(n-k)), k \in \{0, 1, \ldots, \min(m, n)\}$. Finally, it was also shown in [16] that $d_{m,n,l}(r)$ and $d^{out}_{m,n}(r)$ coincide for $l \geq m + n - 1$. As an illustration in Fig. 2.2, the solid curve is the random coding diversity gain $d_{m,n,l}(r)$, the dashed curve is the expurgated diversity gain $d^{ex}_{m,n,l}(r)$, and the dash-dotted curve is the outage diversity gain $d^{out}_{m,n}(r)$.

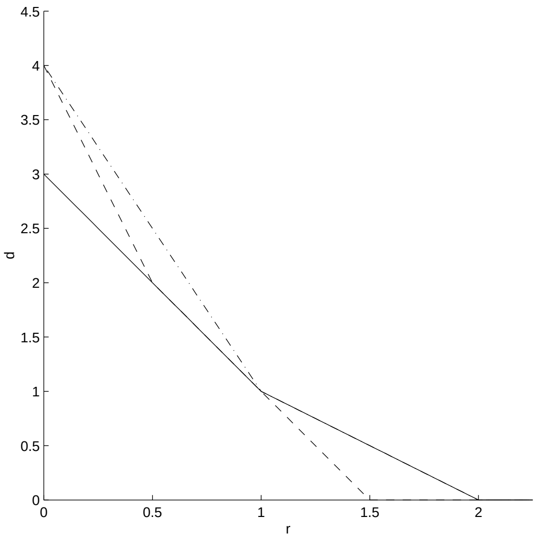


Figure 2.2: Random coding diversity gain (solid), expurgated diversity gain (dashed) and outage diversity gain (dash-dotted) for $m = 2, n = 2, l = 2$.

In [19], these concepts were extended to MIMO fading multiple access channels. In particular, the channel was considered to operate at a rate pair $R_1 = R_1(SNR)$,

$R_2 = R_2(SNR)$, such that

$$\lim_{SNR \to \infty} \frac{R_1(SNR)}{\log SNR} = r_1, \ \lim_{SNR \to \infty} \frac{R_2(SNR)}{\log SNR} = r_2, \tag{2.18}$$

and $d_{sys}(N, r_1, r_2)$ was defined as

$$d_{sys}(N, r_1, r_2) \triangleq \lim_{SNR \to \infty} -\frac{\log P_{e,sys}(N, R_1(SNR), R_2(SNR))}{N \log SNR}. \tag{2.19}$$

For a MIMO fading multiple access channel with m transmit antennas for user 1 and user 2, n receive antennas, and block length $l \geq 2m+n-1$, it was shown in [19] that $d_{sys}(N, r_1, r_2) = d_{sys}(1, r_1, r_2) = d_{sys}(r_1, r_2) = \min\{d^{out}_{m,n}(r_1), d^{out}_{m,n}(r_2), d^{out}_{2m,n}(r_1 + r_2)\}$, where $d_{sys}(r_1, r_2)$ is a shorthand notation for $d_{sys}(1, r_1, r_2)$.

CHAPTER 3

Error Exponent Region

In this chapter, we introduce the notion of error exponent region (EER) for a multi-user channel. Recall that for a multi-user channel, the probability of system error (or equivalently, the corresponding system error exponent) is not sufficient to capture the different reliability requirements of the users. Our approach to addressing this issue hinges on the following two observations. First, one can define a separate probability of error for each user. Therefore, there can be multiple error exponents, one for each user. Some earlier results in this area are the work by Marton and Sgarro [22] which considers a broadcast channel with degraded message sets, and the work by Diggavi *et al.* [23, 24], which considers a single-user channel with two different messages, i.e., a high- and a low-reliability message.

Second, in contrast to a single-user channel where the error exponent is fixed for a given rate, in a multi-user channel one can tradeoff the error exponents among different users even for a fixed vector of users' rates. To illustrate this point, consider the capacity region of a multi-user channel as shown in Fig. 3.1(a). As expected, the error exponents for the two users are functions of both the operating point A and the channel capacity region. However, unlike the case in a single-user channel where the channel capacity boundary is a single point, in a multi-user channel we have multiple points on the capacity boundary (e.g. B, D in Fig. 3.1(a)). Thus one can expect to get different error exponents (and thus a tradeoff among them) depending on which target point on the capacity boundary is considered. For instance, consider the operating point A (corresponding to a rate pair (R_1, R_2)) obtained by backing off from a target point B on the capacity boundary in Fig. 3.1(a). It is expected that the error exponent for user 1 is smaller than that for user 2, since user 1 operates at rate R_1 which is very close to the corresponding capacity (determined by B, see Fig. 3.1(b)), while user 2 backs off significantly from the corresponding capacity (determined again by B, see Fig. 3.1(b)). On the other hand, if we consider point A as if it is obtained by backing off from a target point D on the capacity boundary in Fig. 3.1(a), we then expect the error exponent for user 1 to be larger than that for user 2 (see Fig. 3.1(c)). Therefore, a tradeoff of error exponents between users might be possible by considering different points on the capacity boundary.

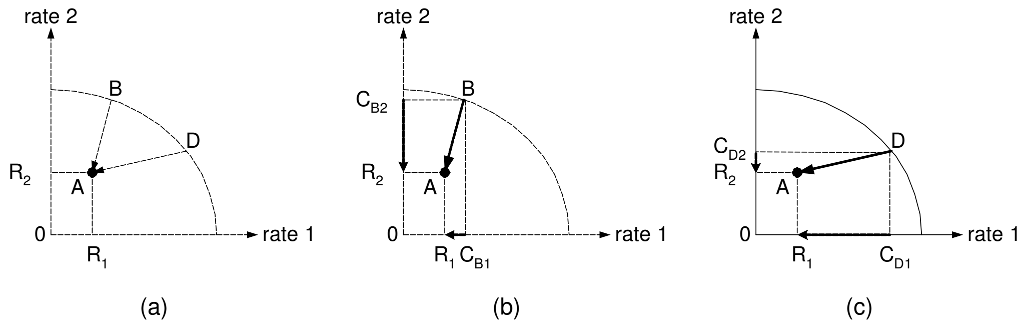


Figure 3.1: Channel capacity region: (a) multiple points on the capacity boundary; (b) users back off from point B to point A; (c) users back off from point D to point A.

This leads us to the notion of error exponent region (EER) for a multi-user channel. For a given operating point characterized by the rate pair (R_1, R_2), the error exponent region consists of all achievable error exponent pairs for the two users. For example, the error exponent region for a channel operated at point A in Fig. 3.1 is a two-dimensional region which depends on rates R_1 and R_2 (see Fig. 3.2). Note that the concepts of EER and channel capacity region (CCR) are fundamentally different. For a given channel, there is only one CCR. One the other hand, an EER depends on the channel operating point (R_1, R_2). Thus, for a given channel, there is one EER for every operating point inside the CCR.

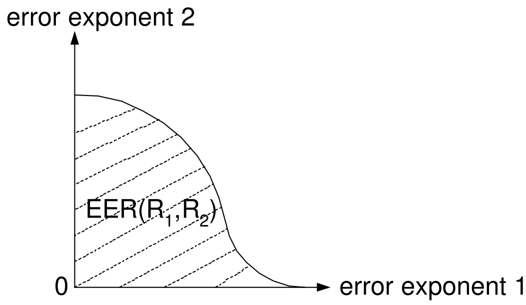


Figure 3.2: Error exponent region for a rate pair (R_1, R_2).

CHAPTER 4

Error Exponent Regions for Gaussian Broadcast Channels

A discrete memoryless stationary broadcast channel with two receivers is a tuple $\{\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P(Y_1, Y_2|X)\}$ of input alphabet $\mathcal{X}$, output alphabets $\mathcal{Y}_i$ for $i = 1, 2$, and a conditional probability distribution $P(Y_1, Y_2|X)$. We formally define the EER for a broadcast channel in the following.

Definition 4.1 *An $(N, M_1, M_2, P_{e1}, P_{e2})$ code for a broadcast channel consists of an encoder*

$$e : \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\} \to \mathcal{X}^N, \tag{4.1}$$

a pair of decoders

$$d_i : \mathcal{Y}_i^N \to \{1, 2, \ldots, M_i\} \tag{4.2}$$

for $i = 1, 2$, and a pair of error probabilities

$$P_{e1} = \frac{1}{M_1 M_2} \sum_{k=1}^{M_1} \sum_{l=1}^{M_2} P[d_1(\mathbf{Y}_1^N) \neq k | X^N = e(k,l)]$$
$$P_{e2} = \frac{1}{M_1 M_2} \sum_{k=1}^{M_1} \sum_{l=1}^{M_2} P[d_2(\mathbf{Y}_2^N) \neq l | X^N = e(k,l)]. \quad (4.3)$$

Definition 4.2 *Given a pair of transmission rates (R_1, R_2), a pair of error exponents (E_1, E_2) is said to be achievable for a broadcast channel if for all $\delta > 0$, there exists a sequence of $(N, M_1, M_2, P_{e1}, P_{e2})$ codes such that*

$$\frac{1}{N} \log M_1 > R_1 - \delta, \quad -\frac{1}{N} \log P_{e1} > E_1 - \delta$$
$$\frac{1}{N} \log M_2 > R_2 - \delta, \quad -\frac{1}{N} \log P_{e2} > E_2 - \delta \quad (4.4)$$

for all sufficiently large N.

Definition 4.3 *Given a pair of transmission rates (R_1, R_2), the error exponent region is the set of all achievable error exponent pairs.*

Now, let us consider a scalar Gaussian broadcast channel [12, 13]

$$\mathbf{Y}_1 = \mathbf{X} + \mathbf{Z}_1$$
$$\mathbf{Y}_2 = \mathbf{X} + \mathbf{Z}_2, \quad (4.5)$$

where $\mathbf{X}$ is the channel input with average power constraint P, and $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are the channel outputs for user 1 and user 2. Assume that the noise power for $\mathbf{Z}_1$ is σ_1^2 and for $\mathbf{Z}_2$ is σ_2^2. We derive inner and outer bounds for the EER in the following sections.

4.1 Inner Bound for Error Exponent Region

Define the *shelled* Gaussian distribution $\mathcal{N}^{sh}(N,P)$ [25, Chap. 7] as the following.

Definition 4.4 *The probability density function $Q(X^N)$ of an N-dimensional shelled Gaussian random vector $\mathbf{X}^N = (\mathbf{X}_1, \ldots, \mathbf{X}_N)$ with variance (power) P is given by*

$$Q(X^N) = \mu^{-1}\phi(X^N)\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi P}}\ e^{-\frac{X_k^2}{2P}}\ , \tag{4.6}$$

where

$$\phi(X^N) = \begin{cases} 1, & \textit{for } NP-\delta < \sum_{k=1}^{N} X_k^2 \le NP \\ 0, & \textit{otherwise} \end{cases}\ , \tag{4.7}$$

and δ is an arbitrary positive number and μ is a normalizing constant to make $Q(X^N)$ integrate to 1.

We write $\mathcal{N}^{sh}(N,P)$ as $\mathcal{N}^{sh}(P)$ when the dimension N is clear from the context.

We now derive an EER inner bound using two encoding strategies - single-code encoding and superposition encoding. In single-code encoding, we construct a random codebook $\mathbf{CB} = \{\mathbf{C}_{i,j} |\ 1 \le i \le M_1, 1 \le j \le M_2\}$ of size $M_3 \triangleq M_1 M_2$. Each random vector $\mathbf{C}_{i,j}$ is i.i.d. with $\mathcal{N}^{sh}(N,P)$. In the receivers, user 1 decodes the message based on the pair (i,j) maximizing $P(\mathbf{Y}_1^N|\mathbf{C}_{i,j})$, and user 2 decodes the message based on the pair (i,j) maximizing $P(\mathbf{Y}_2^N|\mathbf{C}_{i,j})$.

In superposition encoding, we construct two independent random codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$ of size M_1 and M_2, respectively (see Fig. 4.1). Let $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$ denote the i^{th} and the j^{th} codewords in the codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$, respectively. The channel input $\mathbf{X}^N$ is equal to $\mathbf{C}_{1,i} + \mathbf{C}_{2,j}$. Further, let $\mathbf{C}_{1,i}(k)$ and $\mathbf{C}_{2,j}(k)$ denote the k^{th} elements in the codewords $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$, respectively. The random vectors $(\mathbf{C}_{1,i}(1), \ldots, \mathbf{C}_{1,i}(\alpha N))$ and $(\mathbf{C}_{1,i}(\alpha N+1), \ldots, \mathbf{C}_{1,i}(N))$ are independent with distributions $\mathcal{N}^{sh}(\alpha N, P_{11})$ and $\mathcal{N}^{sh}((1-\alpha)N, P_{12})$, respectively, where $\alpha = \frac{a}{N}$ for some $a \in \{0, 1, \ldots, N\}$. Similarly, the random vectors $(\mathbf{C}_{2,j}(1), \ldots, \mathbf{C}_{2,j}(\alpha N))$ and $(\mathbf{C}_{2,j}(\alpha N+1), \ldots, \mathbf{C}_{2,j}(N))$ are independent with distributions $\mathcal{N}^{sh}(\alpha N, P_{21})$ and $\mathcal{N}^{sh}((1-\alpha)N, P_{22})$, respectively. Due to the power constraint P, we have the following equality

$$\alpha(P_{11} + P_{21}) + (1-\alpha)(P_{12} + P_{22}) = P. \tag{4.8}$$

Note that superposition includes two special and important encoding schemes, namely "uniform" superposition and "on-off" superposition. In uniform superposition, the parameter α in Fig. 4.1 is chosen to be zero or one, so the random codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$ have uniform entries. In on-off superposition, the parameters P_{12} and P_{21} in Fig. 4.1 are chosen to be zero, so the transmitter switches between user 1 and user 2 (on-off) during the transmission. On-off superposition is more commonly referred to as time-sharing in the literature.

In the receivers, the optimum decoding strategy is *individual* ML decoding, which minimizes the probabilities of error for user 1 and user 2. In particular, decoding user 1's message is based on the index i maximizing $P(\mathbf{Y}_1^N|\mathbf{C}_{1,i}) = \sum_{j=1}^{M_2} P(\mathbf{Y}_1^N|\mathbf{C}_{1,i} + \mathbf{C}_{2,j})P(\mathbf{C}_{2,j})$ and decoding user 2's message is based on the index j maximizing $P(\mathbf{Y}_2^N|\mathbf{C}_{2,j}) = \sum_{i=1}^{M_1} P(\mathbf{Y}_2^N|\mathbf{C}_{1,i} + \mathbf{C}_{2,j})P(\mathbf{C}_{1,i})$, where $\mathbf{Y}_1^N$ and $\mathbf{Y}_2^N$ are the received channel outputs (with length N) for user 1 and user 2, respectively. However, it turns out that it is difficult to derive analytical, single-letter expressions for error exponents using individual ML decoding, so we use *joint* ML decoding to analyze the performance instead. In joint ML decoding, user 1's message is decoded based on the pair (i, j) maximizing $P(\mathbf{Y}_1^N|\mathbf{C}_{1,i} + \mathbf{C}_{2,j})$, and user 2's message is decoded

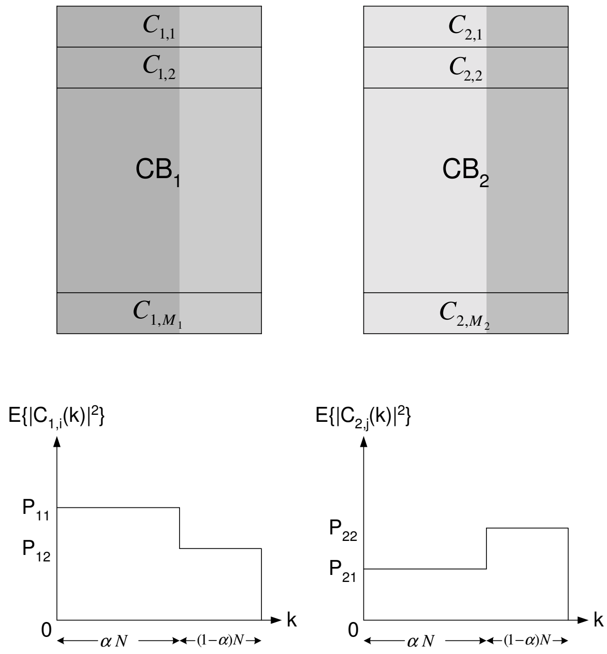


Figure 4.1: Random codebooks for user 1 and user 2 using superposition encoding.

based on the pair (i, j) maximizing $P(\mathbf{Y}_2^N|\mathbf{C}_{1,i} + \mathbf{C}_{2,j})$. Note that we can substitute the optimal decoder with the joint ML decoder or any other decoding scheme and still provide valid inner bounds for the EER. Furthermore, as it will become evident in the subsequent analysis, the performance bounds based on joint ML decoding can be tightened by considering another decoding strategy, namely the *naive* single-user decoding. In naive single-user decoding, user 1 simply regards user 2 as noise, and similarly, user 2 regards user 1 as noise.

Before summarizing the EER inner bound in the following theorem, we define a few error exponent functions. Let $E_r(R, SNR)$ and $E_{ex}(R, SNR)$ denote the random coding exponent and the expurgated exponent for a scalar Gaussian channel with rate R and signal-to-noise ratio SNR. Define the nonuniform-power random coding exponent $E_r^{np}(R, SNR_1, SNR_2, \alpha)$ as

$$\begin{aligned}
E_r^{np}(R, SNR_1, SNR_2, \alpha) &\triangleq \max_{\rho,\theta_1,\theta_2} \{E_{r,0}^{np}(\rho, \theta_1, \theta_2, \alpha) - \rho R\} \\
E_{r,0}^{np}(\rho, \theta_1, \theta_2, \alpha) &\triangleq \alpha \left[\frac{1+\rho}{2} \ln\left(\frac{e\theta_1}{1+\rho}\right) - \frac{\theta_1}{2} + \frac{\rho}{2} \ln\left(1 + \frac{SNR_1}{\theta_1}\right)\right] + \\
&\quad (1-\alpha)\left[\frac{1+\rho}{2} \ln\left(\frac{e\theta_2}{1+\rho}\right) - \frac{\theta_2}{2} + \frac{\rho}{2} \ln\left(1 + \frac{SNR_2}{\theta_2}\right)\right],
\end{aligned} \tag{4.9}$$

where the maximization is over $0 \le \rho \le 1$ and $0 < \theta_1, \theta_2 \le 1 + \rho$.

Define the nonuniform-power expurgated exponent $E_{ex}^{np}(R, SNR_1, SNR_2, \alpha)$ as

$$\begin{aligned}
E_{ex}^{np}(R, SNR_1, SNR_2, \alpha) &\triangleq \max_{\rho,\theta_1,\theta_2} \{E_{ex,0}^{np}(\rho, \theta_1, \theta_2, \alpha) - \rho R\} \\
E_{ex,0}^{np}(\rho, \theta_1, \theta_2, \alpha) &\triangleq \alpha \left[\rho \ln\left(\frac{e\theta_1}{2\rho}\right) - \frac{\theta_1}{2} + \frac{\rho}{2} \ln\left(1 + \frac{SNR_1}{\theta_1}\right)\right] + \\
&\quad (1-\alpha)\left[\rho \ln\left(\frac{e\theta_2}{2\rho}\right) - \frac{\theta_2}{2} + \frac{\rho}{2} \ln\left(1 + \frac{SNR_2}{\theta_2}\right)\right],
\end{aligned} \tag{4.10}$$

where the maximization is over $\rho \ge 1$ and $0 < \theta_1, \theta_2 \le 2\rho$.

Define $E_{t3}^{np}(R, SNR_{11}, SNR_{12}, SNR_{21}, SNR_{22}, \alpha)$ as

$$\begin{aligned}
&E_{t3}^{np}(R, SNR_{11}, SNR_{12}, SNR_{21}, SNR_{22}, \alpha) \triangleq \\
&\qquad \max_{\rho,\theta_{11},\theta_{12},\theta_{21},\theta_{22}} \{E_{t3,0}^{np}(\rho, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \alpha) - \rho R\} \\
&E_{t3,0}^{np}(\rho, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \alpha) \triangleq \\
&\quad \alpha \left[(1+\rho) \ln\left(\frac{e\sqrt{\theta_{11}\theta_{21}}}{1+\rho}\right) - \frac{\theta_{11}+\theta_{21}}{2} + \frac{\rho}{2} \ln\left(1 + \frac{SNR_{11}}{\theta_{11}} + \frac{SNR_{21}}{\theta_{21}}\right)\right] + \\
&\quad (1-\alpha)\left[(1+\rho) \ln\left(\frac{e\sqrt{\theta_{12}\theta_{22}}}{1+\rho}\right) - \frac{\theta_{12}+\theta_{22}}{2} + \frac{\rho}{2} \ln\left(1 + \frac{SNR_{12}}{\theta_{12}} + \frac{SNR_{22}}{\theta_{22}}\right)\right],
\end{aligned} \tag{4.11}$$

where the maximization is over $0 \leq \rho \leq 1$ and $0 < \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} \leq 1+\rho$. The function $E_{t3}^{np}(\cdot)$ accounts for type 3 error in a scalar Gaussian multiple access channel [17] when the random codebooks for the two users are given in Fig. 4.1.

Finally, let $E^{np}(R, SNR_1, SNR_2, \alpha)$ denote the maximum of the nonuniform-power random coding exponent $E_r^{np}(R, SNR_1, SNR_2, \alpha)$ and the nonuniform-power expurgated exponent $E_{ex}^{np}(R, SNR_1, SNR_2, \alpha)$. We now summarize the EER inner bound based on single-code and superposition encoding in the following theorem.

Theorem 4.1 *For a Gaussian broadcast channel with power constraint P and noise power σ_1^2 and σ_2^2 for user 1 and user 2, respectively, an inner bound for EER is $EER_{sc}(R_1, R_2) \cup EER_{sp}(R_1, R_2)$, where $EER_{sc}(R_1, R_2)$ and $EER_{sp}(R_1, R_2)$ are given by*

$$
\begin{aligned}
EER_{sc}(R_1, R_2) = \big\{(E_1, E_2) : E_1 &\leq \max\{E_r(R_1 + R_2, \frac{P}{\sigma_1^2}), E_{ex}(R_1 + R_2, \frac{P}{\sigma_1^2})\} \\
E_2 &\leq \max\{E_r(R_1 + R_2, \frac{P}{\sigma_2^2}), E_{ex}(R_1 + R_2, \frac{P}{\sigma_2^2})\}\big\} \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
EER_{sp}(R_1, R_2) = \Big\{&(E_1, E_2) : 0 \leq \alpha \leq 1,\ \alpha(P_{11} + P_{21}) + (1-\alpha)(P_{12} + P_{22}) = P, \\
E_1 \leq \max\Big\{ &\min\big\{E^{np}(R_1, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \alpha), E_{t3}^{np}(R_1 + R_2, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \frac{P_{21}}{\sigma_1^2}, \frac{P_{22}}{\sigma_1^2}, \alpha)\big\}, \\
&E^{np}(R_1, \frac{P_{11}}{\sigma_1^2 + P_{21}}, \frac{P_{12}}{\sigma_1^2 + P_{22}}, \alpha)\Big\} \\
E_2 \leq \max\Big\{ &\min\big\{E^{np}(R_2, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha), E_{t3}^{np}(R_1 + R_2, \frac{P_{11}}{\sigma_2^2}, \frac{P_{12}}{\sigma_2^2}, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)\big\}, \\
&E^{np}(R_2, \frac{P_{21}}{\sigma_2^2 + P_{11}}, \frac{P_{22}}{\sigma_2^2 + P_{12}}, \alpha)\Big\}\Big\}, \qquad (4.13)
\end{aligned}
$$

and the subscript "sc" of $EER_{sc}(R_1, R_2)$ and the subscript "sp" of $EER_{sp}(R_1, R_2)$ denote single-code and superposition, respectively.

Proof: The probabilities of error for user 1 and user 2 using single-code encoding can be upper bounded by

$$P_{e1} = P(i \neq i_1) \leq P(i \neq i_1 \text{ or } j \neq j_1) \leq e^{-N \max\{E_r(R_1+R_2, \frac{P}{\sigma_1^2}), E_{ex}(R_1+R_2, \frac{P}{\sigma_1^2})\}} \quad (4.14a)$$

$$P_{e2} = P(j \neq j_2) \leq P(i \neq i_2 \text{ or } j \neq j_2) \leq e^{-N \max\{E_r(R_1+R_2, \frac{P}{\sigma_2^2}), E_{ex}(R_1+R_2, \frac{P}{\sigma_2^2})\}}, \quad (4.14b)$$

where user 1 decodes (i, j) as (i_1, j_1) and user 2 decoded (i, j) as (i_2, j_2). The last inequalities in (4.14a) and (4.14b) are derived based on the achievable error exponents for Gaussian single-user channels. Thus the achievable error exponents using single-code encoding are

$$\begin{aligned} E_1^{sc} &= \max\{E_r(R_1 + R_2, \frac{P}{\sigma_1^2}), E_{ex}(R_1 + R_2, \frac{P}{\sigma_1^2})\} \\ E_2^{sc} &= \max\{E_r(R_1 + R_2, \frac{P}{\sigma_2^2}), E_{ex}(R_1 + R_2, \frac{P}{\sigma_2^2})\}. \end{aligned} \quad (4.15)$$

We next consider the superposition encoding shown in Fig. 4.1. The proof is given in three steps. The inner bound (4.13) is derived based on joint ML decoding and naive single-user decoding. The achievable error exponents based on joint ML decoding are derived in step 1 and step 2, and the achievable error exponents based on naive single-user decoding are derived in step 3. In particular, in step 1, we show that there exist a pair of *random* codebooks achieving the error exponents given in (4.13) (based on joint ML decoding), and in step 2, we show that there exist a pair of *deterministic* codebooks achieving the error exponents given in (4.13) (based on joint ML decoding).

Step 1 Let P_{e11} denote type 11 error probability, the probability that user 1 decodes (i, j) as $(\hat{i}, j)$, and let P_{e13} denote type 13 error probability, the probability that user 1 decodes (i, j) as $(\hat{i}, \hat{j})$, where $i \neq \hat{i}$ and $j \neq \hat{j}$. Similarly, let P_{e22} denote type 22 error probability, the probability that user 2 decodes (i, j) as $(i, \hat{j})$, and let P_{e23} denote type 23 error probability, the probability that user 2 decodes (i, j) as $(\hat{i}, \hat{j})$. Applying the random coding argument used in [17], it can be shown that there exist codebooks for user 1 and user 2 using joint ML decoding such that

$$\begin{aligned}
P_{e11} &\leq e^{-NE^{np}(R_1, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \alpha)} \\
P_{e22} &\leq e^{-NE^{np}(R_2, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)} \\
P_{e13} &\leq e^{-NE_{t3}^{np}(R_1+R_2, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \frac{P_{21}}{\sigma_1^2}, \frac{P_{22}}{\sigma_1^2}, \alpha)} \\
P_{e23} &\leq e^{-NE_{t3}^{np}(R_1+R_2, \frac{P_{11}}{\sigma_2^2}, \frac{P_{12}}{\sigma_2^2}, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)}.
\end{aligned} \tag{4.16}$$

The probabilities of error for user 1 and user 2 using joint ML decoding can be upper bounded by

$$\begin{aligned}
P_{e1} = P_{e11} + P_{e13} &\leq e^{-NE^{np}(R_1, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \alpha)} + e^{-NE_{t3}^{np}(R_1+R_2, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \frac{P_{21}}{\sigma_1^2}, \frac{P_{22}}{\sigma_1^2}, \alpha)} \\
&\leq 2e^{-N\min\{E^{np}(R_1, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \alpha), E_{t3}^{np}(R_1+R_2, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \frac{P_{21}}{\sigma_1^2}, \frac{P_{22}}{\sigma_1^2}, \alpha)\}} \\
P_{e2} = P_{e22} + P_{e23} &\leq e^{-NE^{np}(R_2, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)} + e^{-NE_{t3}^{np}(R_1+R_2, \frac{P_{11}}{\sigma_2^2}, \frac{P_{12}}{\sigma_2^2}, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)} \\
&\leq 2e^{-N\min\{E^{np}(R_2, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha), E_{t3}^{np}(R_1+R_2, \frac{P_{11}}{\sigma_2^2}, \frac{P_{12}}{\sigma_2^2}, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)\}}.
\end{aligned} \tag{4.17}$$

Thus the error exponents obtained using joint ML decoding are upper bounded by

$$\begin{aligned}
E_1^{sp,jm} &= \min\{E^{np}(R_1, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \alpha), E_{t3}^{np}(R_1 + R_2, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \frac{P_{21}}{\sigma_1^2}, \frac{P_{22}}{\sigma_1^2}, \alpha)\} \\
E_2^{sp,jm} &= \min\{E^{np}(R_2, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha), E_{t3}^{np}(R_1 + R_2, \frac{P_{11}}{\sigma_2^2}, \frac{P_{12}}{\sigma_2^2}, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)\},
\end{aligned} \tag{4.18}$$

where the superscript "sp,jm" denotes superposition and joint ML.

Step 2 In the previous discussion, we have showed that, averaged over the ensemble of the *random* codebooks, $(\mathbf{CB}_1, \mathbf{CB}_2)$, the error probabilities satisfy $P_{e1} \leq$

$e^{-NE_1^{sp,jm}}$ and $P_{e2} \le e^{-NE_2^{sp,jm}}$, where $E_1^{sp,jm}$ and $E_2^{sp,jm}$ are given in (4.18). This implies that there exist a pair of *deterministic* codebooks (CB_1', CB_2') with user 1's error probability P_{e1}' satisfying $P_{e1}' \le e^{-NE_1^{sp,jm}}$, and there exist another pair of *deterministic* codebooks (CB_1'', CB_2'') with user 2's error probability P_{e2}'' satisfying $P_{e2}'' \le e^{-NE_2^{sp,jm}}$. However, this does not mean that there exist a pair of deterministic codebooks (CB_1^*, CB_2^*) with a pair of error probabilities (P_{e1}^*, P_{e2}^*) satisfying $P_{e1}^* \le e^{-NE_1^{sp,jm}}$ and $P_{e2}^* \le e^{-NE_2^{sp,jm}}$ *simultaneously.* [1] To prove the existence of deterministic codebooks (CB_1^*, CB_2^*), we can apply Markov inequality to get

$$\begin{aligned}
P(\mathbf{P}_{e1} > \beta P_{e1}) &\le \frac{1}{\beta} \\
P(\mathbf{P}_{e2} > \beta P_{e2}) &\le \frac{1}{\beta}
\end{aligned} \tag{4.19}$$

for any $\beta > 0$, where $\mathbf{P}_{e1}$ and $\mathbf{P}_{e2}$ are the (random) probabilities of error for user 1 and user 2, respectively, based on random codebooks $(\mathbf{CB}_1, \mathbf{CB}_2)$, and P_{e1} and P_{e2} are the ensemble averages of $\mathbf{P}_{e1}$ and $\mathbf{P}_{e2}$, respectively. Thus

$$\begin{aligned}
P(\{\mathbf{P}_{e1} \le \beta P_{e1}\} \cap \{\mathbf{P}_{e2} \le \beta P_{e2}\}) &= 1 - P(\{\mathbf{P}_{e1} > \beta P_{e1}\} \cup \{\mathbf{P}_{e2} > \beta P_{e2}\}) \\
&\ge 1 - P(\mathbf{P}_{e1} > \beta P_{e1}) - P(\mathbf{P}_{e2} > \beta P_{e2}) \\
&\ge 1 - \frac{2}{\beta} \\
&> 0
\end{aligned} \tag{4.20}$$

[1] This difficulty does not arise in a multi-user channel when there is only one error probability criterion. For example, in the case of the system error probability for a multiple access channel considered in [17], the existence of a pair of random codebooks $(\mathbf{CB}_1, \mathbf{CB}_2)$ satisfying $P_{e,sys} \le e^{-NE}$ implies directly the existence of a pair of deterministic codebooks (CB_1^*, CB_2^*) satisfying $P_{e,sys}^* \le e^{-NE}$.

by choosing an appropriate β, say $\beta = 10$. This implies that there exist at least a pair of deterministic codebooks (CB_1^*, CB_2^*) with

$$\begin{aligned} P_{e1}^* &\leq \beta e^{-NE_1^{sp,jm}} \\ P_{e2}^* &\leq \beta e^{-NE_2^{sp,jm}}, \end{aligned} \tag{4.21}$$

where the factor β has no effect on the error exponents.

Step 3 When naive single-user decoding is utilized, the probability of error for user 1 can be upper bounded by

$$P_{e1} \leq e^{-NE^{np}(R_1, \frac{P_{11}}{\sigma_1^2+P_{21}}, \frac{P_{12}}{\sigma_1^2+P_{22}}, \alpha)}. \tag{4.22}$$

Strictly speaking, the side-interference $\mathbf{X}_2^N$ is shelled Gaussian distributed, so the noise $\mathbf{X}_2^N + \mathbf{Z}^N$ seen by user 1 is not exactly Gaussian, whereas (4.22) is written based on the assumption that the noise $\mathbf{X}_2^N + \mathbf{Z}^N$ is Gaussian. Nevertheless, the shelled Gaussian distribution $\mathcal{N}(N, P)$ given in Definition 4.4 can be upper bounded by

$$Q(X^N) = \mu^{-1}\phi(X^N)\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi P}}\ e^{-\frac{X_k^2}{2P}} \leq \mu^{-1}\prod_{k=1}^{N}\frac{1}{\sqrt{2\pi P}}\ e^{-\frac{X_k^2}{2P}}, \tag{4.23}$$

where the right hand side of the last inequality is a Gaussian distribution except for the factor μ^{-1}. We can use the upper bound in (4.23) to derive an upper bound for P_{e1}, and the final result is

$$P_{e1} \leq \mu^{-2} e^{-NE^{np}(R_1, \frac{P_{11}}{\sigma_1^2+P_{21}}, \frac{P_{12}}{\sigma_1^2+P_{22}}, \alpha)}, \tag{4.24}$$

where the factor $\mu^{-2} = \mu^{-1} \cdot \mu^{-1}$ is due to using two shelled Gaussian distributions in each random codebook, and each shelled Gaussian distribution results in one μ^{-1}

in (4.24). Thus (4.22) is a valid upper bound except for the factor μ^{-2}, which has no effect on the error exponent and is omitted for simplicity.

Similarly, the probability of error for user 2 can be upper bounded by

$$P_{e2} \leq e^{-NE^{np}(R_2, \frac{P_{21}}{\sigma_2^2+P_{11}}, \frac{P_{22}}{\sigma_2^2+P_{12}}, \alpha)}. \tag{4.25}$$

Therefore, the achievable error exponents using naive single-user decoding are

$$\begin{aligned} E_1^{sp,ns} &= E^{np}(R_1, \frac{P_{11}}{\sigma_1^2 + P_{21}}, \frac{P_{12}}{\sigma_1^2 + P_{22}}, \alpha) \\ E_2^{sp,ns} &= E^{np}(R_2, \frac{P_{21}}{\sigma_2^2 + P_{11}}, \frac{P_{22}}{\sigma_2^2 + P_{12}}, \alpha), \end{aligned} \tag{4.26}$$

where the superscript "sp,ns" denotes superposition and naive single-user.

Since both users can choose either joint ML decoding or naive single-user decoding, the maximum of the corresponding error exponents are achievable, i.e.,

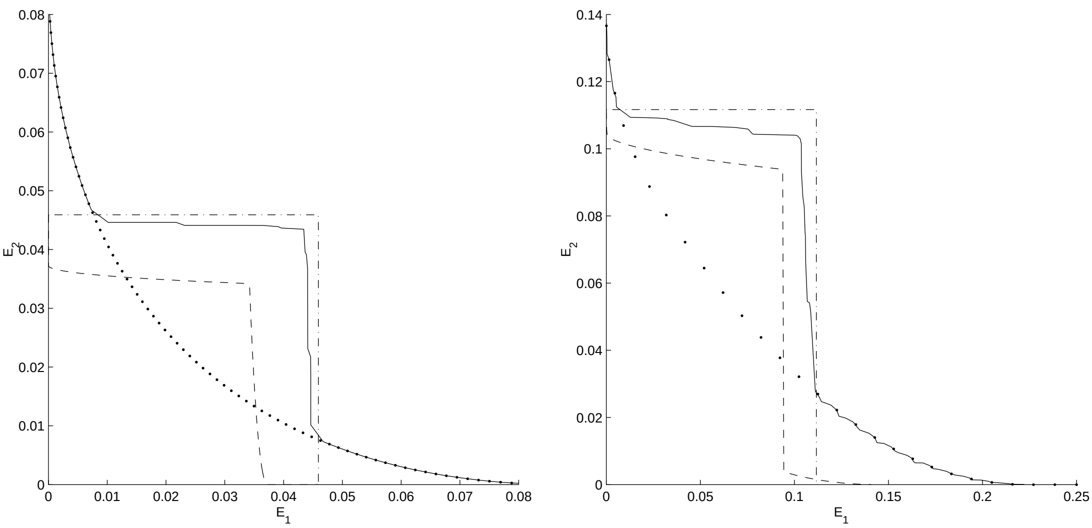
$$\begin{aligned} E_1^{sp} = \max\Big\{ & \min\big\{E^{np}(R_1, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \alpha), E_{t3}^{np}(R_1 + R_2, \frac{P_{11}}{\sigma_1^2}, \frac{P_{12}}{\sigma_1^2}, \frac{P_{21}}{\sigma_1^2}, \frac{P_{22}}{\sigma_1^2}, \alpha)\big\}, \\ & E^{np}(R_1, \frac{P_{11}}{\sigma_1^2 + P_{21}}, \frac{P_{12}}{\sigma_1^2 + P_{22}}, \alpha)\Big\} \\ E_2^{sp} = \max\Big\{ & \min\big\{E^{np}(R_2, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha), E_{t3}^{np}(R_1 + R_2, \frac{P_{11}}{\sigma_2^2}, \frac{P_{12}}{\sigma_2^2}, \frac{P_{21}}{\sigma_2^2}, \frac{P_{22}}{\sigma_2^2}, \alpha)\big\}, \\ & E^{np}(R_2, \frac{P_{21}}{\sigma_2^2 + P_{11}}, \frac{P_{22}}{\sigma_2^2 + P_{12}}, \alpha)\Big\}. \end{aligned} \tag{4.27}$$

This completes the proof. ∎

Several comments are in order at this point.

- It may seem surprising that we use two different probability distributions $\mathcal{N}^{sh}(P_{11})$ and $\mathcal{N}^{sh}(P_{12})$ to construct the random codebook $\mathbf{CB}_1$ (and similarly for $\mathbf{CB}_2$). This requires some explanation. Consider two special cases of superposition encoding - uniform superposition and on-off superposition.

In Fig. 4.2(a), the achievable EERs obtained by these two special cases are illustrated. The dashed curve is the boundary of the achievable EER using uniform superposition, and the dotted curve is the boundary of the achievable EER using on-off superposition (the dotted curve merging with the solid curve at $(E_1, E_2) = (0.046, 0.008)$ and $(E_1, E_2) = (0.008, 0.046)$). In Fig. 4.2(b), the achievable EERs for the same Gaussian channel but with unequal rates for user 1 and user 2 are illustrated. Based on these two encoding schemes, it is now clear that superposition encoding includes these two special cases (uniform and on-off) and also serves as a smooth transition between these two encoding schemes. One may ask if it is possible to improve the EER by using three, four, or even more probability distributions to construct each random codebook. Our numerical results indicated that going beyond two distributions provides only marginal improvements. However, multiple distributions might be beneficial for a broadcast channel with more than two users.

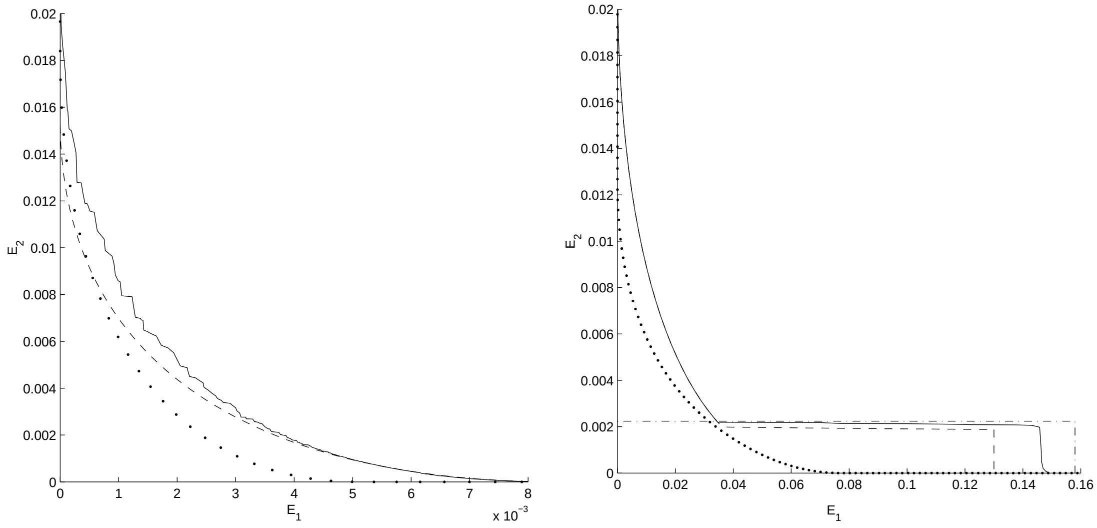


(a) (b)

Figure 4.2: EER inner bound using on-off superposition (dotted), uniform superposition (dashed), superposition (solid) and single-code (dash-dotted) for (a) $R_1 = 0.5, R_2 = 0.5, \frac{P}{\sigma_1^2} = 10, \frac{P}{\sigma_2^2} = 10$; (b) $R_1 = 0.2, R_2 = 0.7, \frac{P}{\sigma_1^2} = 10, \frac{P}{\sigma_2^2} = 10$.

- In Fig. 4.2(a), the maximum equal error exponent pair achieved by superposition encoding is $(E_1, E_2) = (0.044, 0.044)$, which is slightly smaller than the error exponent pair $(E_1, E_2) = (0.046, 0.046)$ achieved by single-code encoding. The broadcast channel in Fig. 4.2(a) is symmetric and it is easy to show that single-code encoding is optimum (in the sense of equal error exponents) for symmetric broadcast channels. It also happens that in Fig. 4.2(a) the use of (nonuniform) superposition does not enlarge the achievable EER beyond what is obtained by uniform superposition, on-off superposition, and single-code encoding. This is not true in general. We point out that the $EER_{sp}(R_1, R_2)$ achieved by superposition in (4.13) is non-vanishing for any point (R_1, R_2) inside the capacity region, but the $EER_{sc}(R_1, R_2)$ achieved by single-code encoding in (4.12) is empty when $R_1 + R_2 > \frac{1}{2}\log(1 + \frac{P}{\sigma_2^2})$ (assuming $\sigma_2^2 > \sigma_1^2$). As illustrated in Fig. 4.3(a), the $EER_{sc}(R_1, R_2)$ achieved by single-code encoding is empty, and (nonuniform) superposition indeed enlarges the region achieved by using only uniform and on-off superposition. In Fig. 4.3(b), it happens that the achievable EER using on-off superposition is completely inside the achievable EER using uniform superposition (the dashed curve merging with the solid curve at $(E_1, E_2) = (0.038, 0.002)$). Note that on-off superposition is not a capacity-achieving strategy. On the other hand, it is easy to verify that the achievable EER using uniform superposition is non-vanishing for any point (R_1, R_2) inside the capacity region. Hence, it is possible that the achievable EER using on-off superposition is included in the achievable EER using uniform superposition for some operating points (R_1, R_2)'s.

- In Fig. 4.2(a), the maximum achievable equal error exponent pair using uniform superposition is $E_1 = E_2 = 0.0319$, which is smaller than the maximum achievable equal error exponent pair $E_1 = E_2 = 0.044$ using (nonuniform) su-



(a) (b)

Figure 4.3: EER inner bound using on-off superposition (dotted), uniform superposition (dashed), superposition (solid) and single-code (dash-dotted) for (a) $R_1 = 1, R_2 = 0.1, \frac{P}{\sigma_1^2} = 10, \frac{P}{\sigma_2^2} = 5$; (b) $R_1 = 0.2, R_2 = 0.65, \frac{P}{\sigma_1^2} = 10, \frac{P}{\sigma_2^2} = 5$.

perposition. Given that the broadcast channel is symmetric and is operated at equal rates $R_1 = R_2$, why does uniform superposition ($P_{11} = P_{21} = \frac{P}{2}$) not achieve the maximum equal error exponent pair? Recall that in joint ML decoding, there are four types of error events - type 11, type 13, type 22, and type 23 errors. The point $(E_1, E_2) = (0.0319, 0.0319)$ in Fig. 4.2(a) can be achieved using uniform superposition, and the corresponding error exponent lower bound for type 11 error is $E_{t11} = 0.2103$, which is much larger than the error exponent lower bound for type 13 error $E_{t13} = 0.0319$. Further, if we plot E_{t11} and E_{t13} as a function of P_{11} given by

$$E_{t11} = E^{np}(R_1, P_{11}, P - P_{11}, \alpha)$$
$$E_{t13} = E_{t3}^{np}(R_1 + R_2, P_{11}, P - P_{11}, P - P_{11}, P_{11}, \alpha) \tag{4.28}$$

while keeping $R_1 = R_2 = 0.5$, $P = 10$, $\alpha = \frac{1}{2}$ fixed, then E_{t11} decreases as P_{11} increases from $\frac{P}{2}$ to P, but E_{t13} increases as P_{11} increases from $\frac{P}{2}$ to P (see Fig. 4.4). Since $E_1 = \min\{E_{t11}, E_{t13}\}$, the error exponent for user 1 (and the error exponent for user 2) increases when we use superposition. Thus superposition (compared to uniform superposition) provides one more degree of freedom to tradeoff between type 11 and type 13 errors, which increases the maximum achievable equal error exponent pair when the dominant error event is type 13 error.

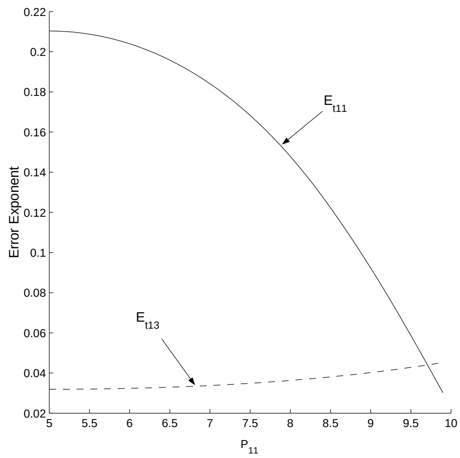


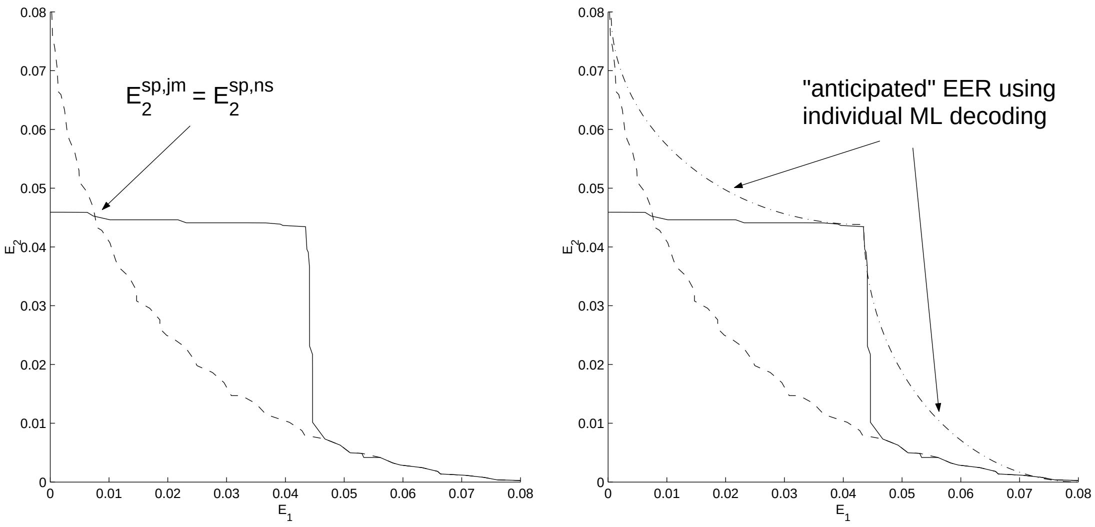
Figure 4.4: E_{t11} and E_{t13} plotted as a function of P_{11}.

- The result that the performance bound based on joint ML decoding can be improved by naive single-user decoding might not have been anticipated. To illustrate this, let's consider a broadcast channel operated at $(R_1, R_2) = (0.4, 1)$ with $P = 50$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 5$. The sum rate $R_1 + R_2 = 1.4 > 1.2 = \frac{1}{2}\log(1+\frac{P}{\sigma_2^2})$, so $E_2^{sc} = 0$ and $E_2^{sp,jm} = 0$, because it can be verified (numerically) that for any R, SNR_{11}, SNR_{12}, SNR_{21}, SNR_{22} and α, we have

$$E_{t3}^{np}(R, SNR_{11}, SNR_{12}, SNR_{21}, SNR_{22}, \alpha) \leq$$
$$E_r(R, \alpha(SNR_{11} + SNR_{12}) + (1-\alpha)(SNR_{21}, SNR_{22})), \qquad (4.29)$$

so $E_{t3}^{np} = 0$ for user 2 in the case $(R_1, R_2) = (0.4, 1)$. On the other hand, if we use uniform superposition ($\alpha = 1$) with $P_{11} = 10$ and $P_{21} = 40$, then even user 2 simply regards the side-interference $\mathbf{X}_1^N$ as noise, the achievable error exponent for user 2 is $E_2^{sp,ns} = E_r(R_2, \frac{P_{21}}{P_{11}+\sigma_2^2}) = 0.084$ (and $E_1^{sp,jm} = 0.328$). This example illustrates that the achievable error exponents derived using joint ML decoding might be much worse than the actual performance using individual ML decoding. There are two possible explanations for this, though we can not verify which one is the main reason: either joint ML decoding is significantly inferior to individual ML decoding, or the bound derived for joint ML decoding is loose. Nevertheless, naive single-user decoding serves as an assisted decoding scheme to partially close the performance gap between the optimum individual ML decoding and the suboptimum joint ML decoding.

- In Fig. 4.2(a), it seems that there are abrupt changes in the achievable EER using superposition encoding around $(E_1, E_2) = (0.008, 0.046)$ and $(E_1, E_2) = (0.046, 0.008)$. This is due to the switch between the joint ML and naive single decoding at the receivers. To illustrate this point, we plot two curves in Fig. 4.5(a), where in the solid curve user 2 uses only joint ML decoding and in the dashed curve user 2 uses only naive single-user decoding (user 1 uses a mixture of joint ML and naive single-user decoding in both curves). Although $E_2^{sp,jm}$ increases slowly as E_1 decreases from 0.044 to 0, $E_2^{sp,ns}$ increases much more rapidly as E_1 decreases. $E_2^{sp,ns}$ is equal to $E_2^{sp,jm}$ around $(E_1, E_2) = (0.008, 0.046)$, and this is why there is an abrupt change over here. We believe that the abrupt change of the achievable EER using superposition in Fig. 4.2(a) is an artificial effect due to the switch between joint ML and naive single-user decoding, and we anticipate that the actual achievable EER using the optimum individual ML decoding would be much more smooth (see the dash-dotted curve in Fig. 4.5(b)). However, we can not verify this speculation.



(a) (b)

Figure 4.5: EER inner bound for $R_1 = 0.5, R_2 = 0.5, \frac{P}{\sigma_1^2} = 10, \frac{P}{\sigma_2^2} = 10$ (a) user 2 using joint ML decoding (solid) and naive single-user decoding (dashed); (b) anticipated EER using individual ML decoding (dash-dotted).

4.2 Outer Bound for Error Exponent Region

We now derive an EER outer bound and summarize the result in the following theorem.

Theorem 4.2 *For a Gaussian broadcast channel with power constraint P and noise power σ_1^2 and σ_2^2 for user 1 and user 2, respectively, an outer bound for EER is*

$$\begin{aligned} E_1 &\leq E_{su}(R_1, \frac{P}{\sigma_1^2}) \\ E_2 &\leq E_{su}(R_2, \frac{P}{\sigma_2^2}) \\ \min\{E_1, E_2\} &\leq \max\big\{E_{su}(R_1 + R_2, \frac{P}{\sigma_1^2}), E_{su}(R_1 + R_2, \frac{P}{\sigma_2^2})\big\}, \end{aligned} \tag{4.30}$$

where $E_{su}(\cdot)$ is any error exponent upper bound for a scalar Gaussian channel and the subscript "su" denotes single-user upper (bound).

Proof: For any broadcast channel, the probability of decoding error for user i can always be lower bounded by the probability of decoding error for user i operating over a point-to-point channel defined by the marginal distribution $P(Y_i|X)$, where $i = 1$ or 2. This implies that

$$\begin{aligned} E_1 &\leq E_{su}(R_1, \frac{P}{\sigma_1^2}) \\ E_2 &\leq E_{su}(R_2, \frac{P}{\sigma_2^2}). \end{aligned} \tag{4.31}$$

Given any encoding and decoding schemes, it is true that

$$P_{e,sys} \leq P_{e1} + P_{e2} \leq 2\max\{P_{e1}, P_{e2}\}, \tag{4.32}$$

where the first inequality follows from the union bound. The broadcast channel considered in (4.5) is *stochastically* degraded [26, Chap. 14]. Since the performance of a broadcast channel depends only on the marginal distributions, we may further assume that the broadcast channel considered in (4.5) is *physically* degraded, i.e., $P(Y_1, Y_2|X) = P(Y_1|X)P(Y_2|Y_1)$ if $\sigma_2^2 \geq \sigma_1^2$. If we now allow the two receivers to cooperate, we have a single-user channel, whose probability of error, P'_e, should be less than or equal to the probability of system error $P_{e,sys}$ in the original broadcast channel [27]. The probability of error P'_e of the new single-user channel can be lower bounded by

$$e^{-N\max\{E_{su}(R_1+R_2, \frac{P}{\sigma_1^2}), E_{su}(R_1+R_2, \frac{P}{\sigma_2^2})\}} \leq P'_e, \tag{4.33}$$

since the original broadcast channel is physically degraded. Combining (4.32) and (4.33), we have

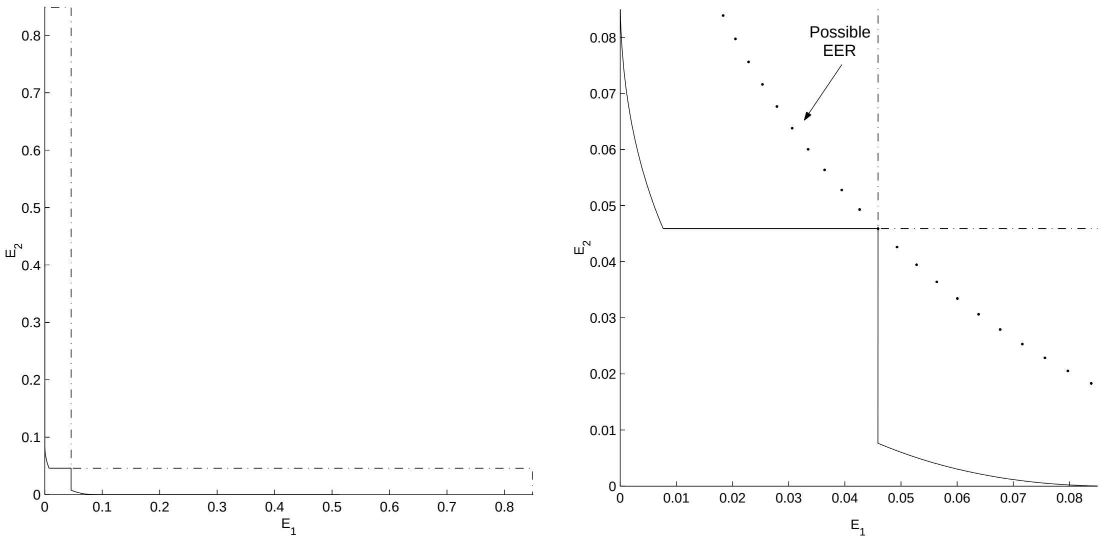
$$e^{-N\max\{E_{su}(R_1+R_2,\frac{P}{\sigma_1^2}),E_{su}(R_1+R_2,\frac{P}{\sigma_2^2})\}} \le P'_e \le P_{e,sys} \le 2\max\{P_{e1}, P_{e2}\}, \tag{4.34}$$

which implies that

$$\min\{E_1, E_2\} \le \max\{E_{su}(R_1+R_2, \frac{P}{\sigma_1^2}), E_{su}(R_1+R_2, \frac{P}{\sigma_2^2})\}. \tag{4.35}$$

This completes the proof. ■

This outer bound is illustrated in Fig. 4.6(a), where the solid curve is the EER inner bound, and the dash-dotted curve is the EER outer bound. One of the main goals in this work is to show that one can tradeoff the error exponents among the users even for a fixed vector of transmission rates in a multi-user channel. This is equivalent to saying that the EER is not a rectangle. A possible boundary of the EER (dotted curve) is shown in Fig. 4.6(b), where Fig. 4.6(b) is a zoom-in version of Fig. 4.6(a). It is clear from Fig. 4.6(b) that there is indeed a tradeoff between user 1's and user 2's error exponents, i.e., the EER is not a rectangle, when the channel is operated at $(R_1, R_2) = (0.5, 0.5)$. Note that the EER inner and outer bounds are tight at the equal error exponents $(E_1, E_2) = (0.046, 0.046)$. This follows from the fact that the broadcast channel in Fig. 4.6 is symmetric and is operated at high rates, so the random coding exponent and the sphere packing exponent are tight at the sum rate $R_1 + R_2$.



(a) (b)

Figure 4.6: EER inner bound (solid) and outer bound (dash-dotted) for $R_1 = 0.5, R_2 = 0.5, \frac{P}{\sigma_1^2} = 10, \frac{P}{\sigma_2^2} = 10$.

CHAPTER 5

Error Exponent Regions for Gaussian Multiple Access Channels

Consider a discrete-time memoryless stationary scalar Gaussian multiple access channel

$$\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{Z}, \tag{5.1}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are the channel inputs for user 1 and user 2 with average power constraints P_1 and P_2, and $\mathbf{Y}$ is the channel output. Assume that the noise power for $\mathbf{Z}$ is σ^2. We derive inner and outer bounds for the EER in the following sections.

5.1 Inner Bound for Error Exponent Region

In the transmitters, we use superposition encoding and construct two independent random codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$ of size M_1 and M_2, respectively. Let $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$ denote the i^{th} and the j^{th} codewords in the codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$, respectively. Let $\mathbf{C}_{1,i}(k)$ and $\mathbf{C}_{2,j}(k)$ denote the k^{th} elements in the codewords $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$. The random vectors $(\mathbf{C}_{1,i}(1), \ldots, \mathbf{C}_{1,i}(\alpha N))$ and $(\mathbf{C}_{1,i}(\alpha N+1), \ldots, \mathbf{C}_{1,i}(N))$

are independent with distributions $\mathcal{N}^{sh}(\alpha N, P_{11})$ and $\mathcal{N}^{sh}((1-\alpha)N, P_{12})$, respectively, where $\alpha = \frac{a}{N}$ for some $a \in \{0, 1, \ldots, N\}$. Similarly, the random vectors $(\mathbf{C}_{2,j}(1), \ldots, \mathbf{C}_{2,j}(\alpha N))$ and $(\mathbf{C}_{2,j}(\alpha N+1), \ldots, \mathbf{C}_{2,j}(N))$ are independent with distributions $\mathcal{N}^{sh}(\alpha N, P_{21})$ and $\mathcal{N}^{sh}((1-\alpha)N, P_{22})$, respectively. Due to the power constraint P, we have the following equalities

$$\begin{aligned} \alpha P_{11} + (1-\alpha)P_{12} &= P_1 \\ \alpha P_{21} + (1-\alpha)P_{22} &= P_2. \end{aligned} \tag{5.2}$$

In the receivers, we use a mixture of joint ML decoding and naive single-user decoding. We summarize the result in the following theorem.

Theorem 5.1 *For a Gaussian multiple access channel with power constraints P_1 and P_2 for user 1 and user 2 and noise power σ^2, an inner bound for EER is*

$$\begin{aligned} EER(R_1, R_2) = \Big\{&(E_1, E_2) : 0 \le \alpha \le 1, \alpha P_{11} + (1-\alpha)P_{12} = P_1,\ \alpha P_{21} + (1-\alpha)P_{22} = P_2, \\ &E_1 \le \max\Big\{\min\{E^{np}(R_1, \frac{P_{11}}{\sigma^2}, \frac{P_{12}}{\sigma^2}, \alpha), E^{np}_{t3}(R_1+R_2, \frac{P_{11}}{\sigma^2}, \frac{P_{12}}{\sigma^2}, \frac{P_{21}}{\sigma^2}, \frac{P_{22}}{\sigma^2}, \alpha)\}, \\ &\qquad E^{np}(R_1, \frac{P_{11}}{\sigma^2+P_{21}}, \frac{P_{12}}{\sigma^2+P_{22}}, \alpha)\Big\} \\ &E_2 \le \max\Big\{\min\{E^{np}(R_2, \frac{P_{21}}{\sigma^2}, \frac{P_{22}}{\sigma^2}, \alpha), E^{np}_{t3}(R_1+R_2, \frac{P_{11}}{\sigma^2}, \frac{P_{12}}{\sigma^2}, \frac{P_{21}}{\sigma^2}, \frac{P_{22}}{\sigma^2}, \alpha)\}, \\ &\qquad E^{np}(R_2, \frac{P_{21}}{\sigma^2+P_{11}}, \frac{P_{22}}{\sigma^2+P_{12}}, \alpha)\Big\}\Big\}, \end{aligned} \tag{5.3}$$

where $E^{np}(\cdot)$ is the maximum of the nonuniform-power random coding exponent $E^{np}_r(\cdot)$ and the nonuniform-power expurgated exponent $E^{np}_{ex}(\cdot)$, and $E^{np}_{t3}(\cdot)$ is the function which accounts for type 3 error in a scalar Gaussian multiple access channel.

Proof: Following [17], we define three types of error events, type 1, type 2, and type 3, using joint ML decoding. It can be shown by using random coding arguments that there exist codebooks for user 1 and user 2 using joint ML decoding such that

$$\begin{aligned}
P_{et1} &\leq e^{-NE^{np}(R_1,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\alpha)} \\
P_{et2} &\leq e^{-NE^{np}(R_2,\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)} \\
P_{et3} &\leq e^{-NE_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)}.
\end{aligned} \tag{5.4}$$

The probabilities of error for user 1 and user 2 using joint ML decoding can be upper bounded by

$$\begin{aligned}
P_{e1} = P_{et1} + P_{et3} &\leq e^{-NE^{np}(R_1,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\alpha)} + e^{-NE_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)} \\
&\leq 2e^{-N\min\{E^{np}(R_1,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\alpha),E_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)\}} \\
P_{e2} = P_{et2} + P_{et3} &\leq e^{-NE^{np}(R_2,\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)} + e^{-NE_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)} \\
&\leq 2e^{-N\min\{E^{np}(R_2,\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha),E_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)\}}.
\end{aligned} \tag{5.5}$$

Thus the achievable error exponents using joint ML decoding are

$$\begin{aligned}
E_1^{sp,jm} &= \min\big\{E^{np}(R_1,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\alpha), E_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)\big\} \\
E_2^{sp,jm} &= \min\big\{E^{np}(R_2,\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha), E_{t3}^{np}(R_1+R_2,\frac{P_{11}}{\sigma^2},\frac{P_{12}}{\sigma^2},\frac{P_{21}}{\sigma^2},\frac{P_{22}}{\sigma^2},\alpha)\big\}.
\end{aligned} \tag{5.6}$$

So far we have shown that there exist a pair of *random* codebooks satisfying (5.6). The proof for the existence of a pair of *deterministic* random codebooks satisfying (5.6) is the same as given in the Gaussian broadcast channel and is omitted here.

When naive single-user decoding is utilized, the achievable error exponents are

$$
\begin{aligned}
E_1^{sp,ns} &= E^{np}(R_1, \frac{P_{11}}{\sigma^2 + P_{21}}, \frac{P_{12}}{\sigma^2 + P_{22}}, \alpha) \\
E_2^{sp,ns} &= E^{np}(R_2, \frac{P_{21}}{\sigma^2 + P_{11}}, \frac{P_{22}}{\sigma^2 + P_{12}}, \alpha).
\end{aligned} \tag{5.7}
$$

Since both users can choose either joint ML or naive single-user decoding, the maximum of the corresponding error exponents are achievable. This completes the proof.

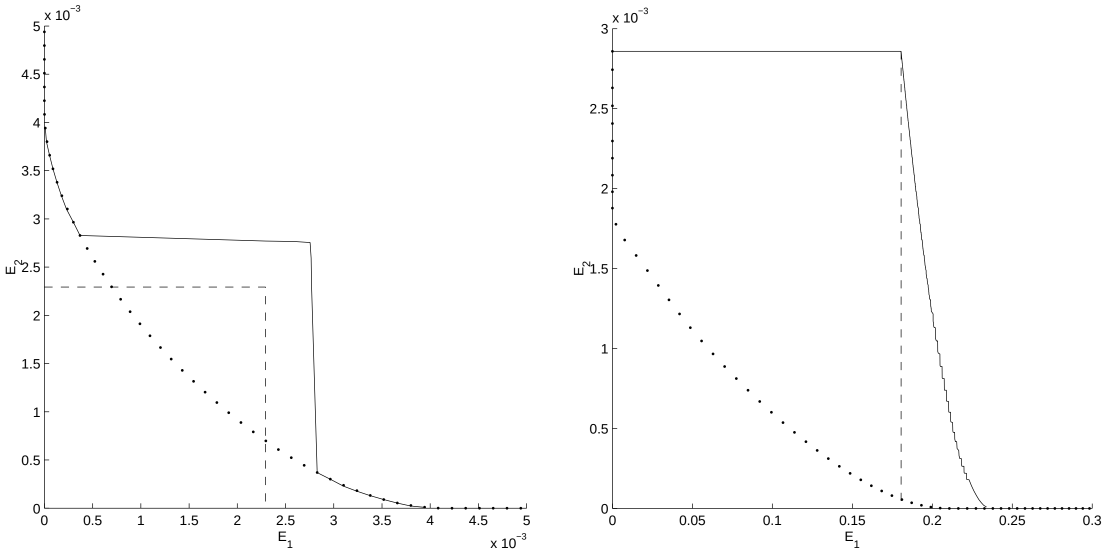
■

In Fig. 5.1(a), we illustrate this inner bound using an example. The dotted curve is the boundary of the achievable region obtained by on-off superposition, which merges with the solid curve (obtained by superposition) at $(E_1, E_2) = (0.0028, 0.0004)$ and $(E_1, E_2) = (0.0028, 0.0004)$. The maximum equal error exponent pair achieved by uniform superposition (dashed curve) is $(E_1, E_2) = (0.0023, 0.0023)$, which is less than the maximum equal error exponent pair $(E_1, E_2) = (0.0028, 0.0028)$ achieved by superposition (solid curve). In Fig. 5.1(b), we illustrate the inner bound using an example when the power constraints for user 1 and user 2 are different.

Before we continue for the outer bound, we make a remark here. Although we have shown that (nonuniform) superposition encoding provides an improvement over uniform superposition in terms of the error exponent region, it can also be shown that the former continues to perform better than the latter when the performance measure is system error exponent. The system error exponent is equal to $\min\{E_1, E_2\}$, which can be derived easily from the EER (the maximum value along the $E_1 = E_2$ line inside the EER). It is mentioned in [17] that

"This means that we can define a region R_α of rate pairs as the convex hull of all pairs $R_1 > 0$, $R_2 > 0$ for which $E_r(R_1, R_2) > \alpha$. ... First, the random coding ensemble itself could use different probability assignments Q_1Q_2 on different letters of the block. ... No examples have been found where this approach enlarges the region R_α defined above;"

The example of Fig. 5.1(a) indeed shows such an improvement is possible.



(a) (b)

Figure 5.1: EER inner bound using on-off superposition (dotted), uniform superposition (dashed) and superposition (solid) for (a) $R_1 = 0.25, R_2 = 0.25, \frac{P_1}{\sigma^2} = 1, \frac{P_2}{\sigma^2} = 1$; (b) $R_1 = 0.1, R_2 = 0.5, \frac{P_1}{\sigma^2} = 4, \frac{P_2}{\sigma^2} = 2$.

5.2 Outer Bound for Error Exponent Region

We now derive an EER outer bound and summarize the result in the following theorem.

Theorem 5.2 *For a Gaussian multiple access channel with power constraints* P_1

and P_2 for user 1 and user 2 and noise power σ^2, an outer bound for EER is

$$\begin{aligned} E_1 &\le E_{su}(R_1, \frac{P_1}{\sigma^2}) \\ E_2 &\le E_{su}(R_2, \frac{P_2}{\sigma^2}) \\ \min\{E_1, E_2\} &\le E_{su}(R_1 + R_2, \frac{P_1 + P_2}{\sigma^2}), \end{aligned} \tag{5.8}$$

where $E_{su}(\cdot)$ is any error exponent upper bound for a scalar Gaussian channel.

Proof: For a Gaussian multiple access channel with power constraints P_1 and P_2 for user 1 and user 2, respectively, the probabilities of decoding error for user 1 and user 2 can always be lower bounded by the probabilities of decoding error for user 1 and user 2 operating over the point-to-point channel $\mathbf{Y} = \mathbf{X}_i + \mathbf{Z}$ with power constraint P_i, for $i = 1, 2$. This implies that

$$\begin{aligned} E_1 &\le E_{su}(R_1, \frac{P_1}{\sigma^2}) \\ E_2 &\le E_{su}(R_2, \frac{P_2}{\sigma^2}). \end{aligned} \tag{5.9}$$

Given any two codebooks $CB_1 = \{C_{1,1}, \ldots, C_{1,M_1}\}$ and $CB_2 = \{C_{2,1}, \ldots, C_{2,M_2}\}$ for the Gaussian multiple access channel satisfying the power constraints P_1 and P_2, we have

$$\frac{1}{M_1 M_2} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} (C_{1,i} + C_{2,j})^2 = \frac{1}{M_1} \sum_{i=1}^{M_1} C_{1,i}^2 + \frac{1}{M_2} \sum_{j=1}^{M_2} C_{2,j}^2 \le P_1 + P_2. \tag{5.10}$$

Here we assume that the codebooks CB_1 and CB_2 are zero-mean, i.e.,

$$\sum_{i=1}^{M_1} C_{1,i} = \sum_{j=1}^{M_2} C_{2,j} = 0, \tag{5.11}$$

since any nonzero-mean codebook can be modified to zero-mean with the same performance and using less power. Let $D_{i,j}$ denote the decision region associated with the codewords $C_{1,i}$ and $C_{2,j}$. Now construct a codebook $CB = \{C_1, \ldots, C_{M_3}\}$ with codewords $C_{(i-1)M_2+j} = C_{1,i} + C_{2,j}$ and decision regions $D_{(i-1)M_2+j} = D_{i,j}$, where $M_3 = M_1 M_2$, then the probability of system error is lower bounded by

$$\begin{aligned} P_{e,sys} &= \frac{1}{M_1 M_2} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} P(\mathbf{Y}^N \notin D_{i,j} | C_{1,i} + C_{2,j}) \\ &= \frac{1}{M_3} \sum_{k=1}^{M_3} P(\mathbf{Y}^N \notin D_k | C_k) \\ &\geq \min_{\frac{1}{M_3}\sum_{k=1}^{M_3} C_k'^2 \leq P_1 + P_2} \frac{1}{M_3} \sum_{k=1}^{M_3} P(\mathbf{Y}^N \notin D_k' | C_k') \\ &\geq e^{-NE_{su}(R_1+R_2, \frac{P_1+P_2}{\sigma^2})}, \end{aligned} \tag{5.12}$$

where $CB' = \{C_1', \ldots, C_{M_3}'\}$ is any codebook with M_3 codewords and D_k' is the optimum decision region associated with the codewords C_k'. This implies that

$$\min\{E_1, E_2\} \leq E_{su}(R_1 + R_2, \frac{P_1 + P_2}{\sigma^2}). \tag{5.13}$$

This completes the proof. ■

In Fig. 5.2, we illustrate this outer bound using an example. The solid curve is the achievable EER and the dash-dotted curve is the outer bound for the EER. It is also clear from Fig. 5.2 that for the rate pair $(R_1, R_2) = (0.25, 0.25)$, there is a tradeoff between user 1's and user 2's error exponents.

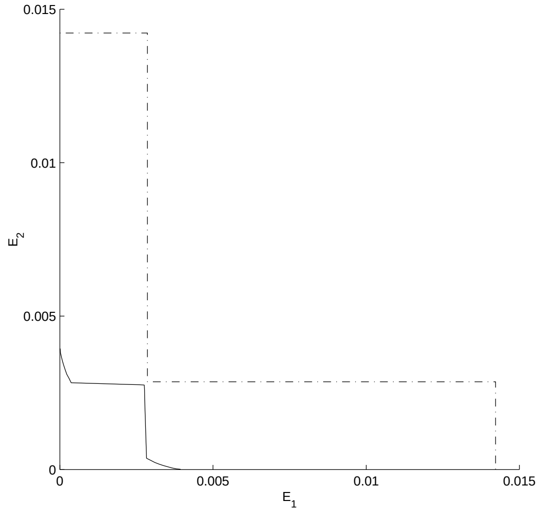


Figure 5.2: EER inner bound (solid) and outer bound (dash-dotted) for $R_1 = 0.25, R_2 = 0.25, \frac{P_1}{\sigma^2} = 1, \frac{P_2}{\sigma^2} = 1$.

5.3 Operating Points with Tight Inner and Outer Bounds

From Theorem 5.1 and Theorem 5.2, we can show that the EER inner and outer bounds are tight for certain operating points (R_1, R_2). It is known that for a single-user channel the random coding exponent $E_r(R, SNR)$ and the sphere packing exponent $E_{sp}(R, SNR)$ are tight for rates $R \geq R_{crit}$, where R_{crit} is the critical rate [7]. From Theorem 5.1, the achievable error exponents using uniform superposition and joint ML decoding are

$$\begin{aligned} E_1^{us,jm} &= \min\left\{E(R_1, \frac{P_1}{\sigma^2}), E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2})\right\} \\ E_2^{us,jm} &= \min\left\{E(R_2, \frac{P_2}{\sigma^2}), E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2})\right\}, \end{aligned} \tag{5.14}$$

where the superscript "us,jm" denotes uniform superposition and joint ML, and

$$E(R, SNR) \triangleq \max\{E_r(R, SNR), E_{ex}(R, SNR)\}$$
$$E_{t3}(R, SNR_1, SNR_2) \triangleq E_{t3}^{np}(R, SNR_1, SNR_1, SNR_2, SNR_2, \alpha). \tag{5.15}$$

Thus the CCR of the Gaussian multiple access channel can be divided into four regions $\mathcal{R}_{12}$, $\mathcal{R}_{13}$, $\mathcal{R}_{23}$, and $\mathcal{R}_3$ as the following

$$\begin{aligned}
\mathcal{R}_{12} &\triangleq \{(R_1, R_2) : E(R_1, \frac{P_1}{\sigma^2}) \leq E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2}), E(R_2, \frac{P_2}{\sigma^2}) \leq E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2})\} \\
\mathcal{R}_{13} &\triangleq \{(R_1, R_2) : E(R_1, \frac{P_1}{\sigma^2}) \leq E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2}) \leq E(R_2, \frac{P_2}{\sigma^2})\} \\
\mathcal{R}_{23} &\triangleq \{(R_1, R_2) : E(R_2, \frac{P_2}{\sigma^2}) \leq E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2}) \leq E(R_1, \frac{P_1}{\sigma^2})\} \\
\mathcal{R}_3 &\triangleq \{(R_1, R_2) : E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2}) \leq E(R_1, \frac{P_1}{\sigma^2}), E_{t3}(R_1 + R_2, \frac{P_1}{\sigma^2}, \frac{P_2}{\sigma^2}) \leq E(R_2, \frac{P_2}{\sigma^2})\}
\end{aligned} \tag{5.16}$$

depending on whether the bound for type 1 error, type 2 error, or type 3 error dominates when using uniform superposition and joint ML decoding (see Fig. 5.3). In region $\mathcal{R}_{12}$, each user attains the maximal achievable single-user error exponent. In region $\mathcal{R}_{13}$, the first user achieves the maximal single-user error exponent, while the second user's error probability is dominated by type 3 error. A similar statement also holds for region $\mathcal{R}_{23}$. In region $\mathcal{R}_3$, type 3 error is dominant over both type 1 and type 2 errors. If $(R_1, R_2) \in \mathcal{R}_{12}$ with $R_1 \geq R_{1,crit}$ and $R_2 \geq R_{2,crit}$, where $R_{1,crit}$ and $R_{2,crit}$ are the critical rates for a Gaussian single-user channel with SNR equal to $\frac{P_1}{\sigma^2}$ and $\frac{P_2}{\sigma^2}$, respectively, then

$$\begin{aligned}
E_1^{us,jm} &= E_r(R_1, \frac{P_1}{\sigma^2}) = E_{sp}(R_1, \frac{P_1}{\sigma^2}) \\
E_2^{us,jm} &= E_r(R_2, \frac{P_2}{\sigma^2}) = E_{sp}(R_2, \frac{P_2}{\sigma^2}),
\end{aligned} \tag{5.17}$$

i.e., the EER inner and outer bounds are tight. In Fig 5.3, the dotted region is the rate region with tight EER inner and outer bounds.

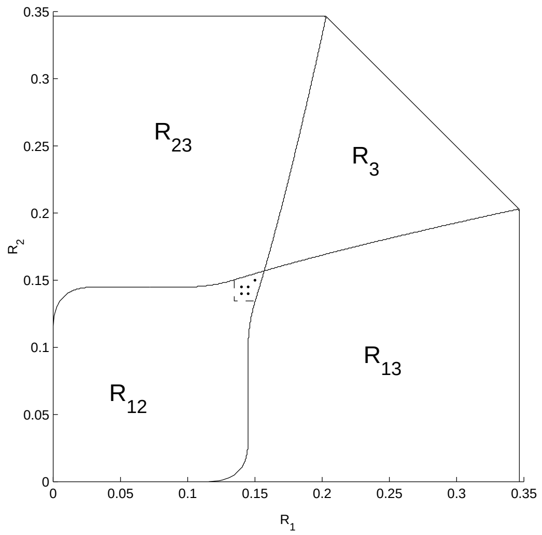


Figure 5.3: Channel capacity region for $\frac{P_1}{\sigma^2} = 1, \frac{P_2}{\sigma^2} = 1$.

CHAPTER 6

Conjectured EER Outer Bound for Gaussian Multi-User Channels

The EER outer bounds derived in the previous chapters are essentially based on the upper bounds for the error exponent for a single-user channel. In particular, the performance of the system with two independent messages of the two users is bounded by that of a system involving only one message of one user. In this chapter our objective is to provide improved outer bounds for the EERs which explicitly incorporate the fact that two users are simultaneously communicating with one transmitter or receiver. In particular, we extend the concept of minimum distance bound to a multi-user setting with multiple message sets. To do this, we first consider a single-user communication system where a transmitter wishes to communicate two messages to a receiver, where these messages require different reliabilities. This leads to the notion of EER for a single-user channel with two message sets. The basic idea behind this outer bound for EER can be understood in the following. The output space is divided into several regions (dashed curve in Fig. 6.1), where each region contains codewords with the same value of the first index (for message 1). The set of the codewords inside each region, i.e., the set of codewords with the same value of the first index, is called one subcode. When E_1 and E_2 are roughly the same, the codewords are distributed uniformly inside the regions (see Fig. 6.2(a)). On the other hand, when E_1 and E_2 are extremely asymmetric, e.g., $E_1 >> E_2$, the distance between the subcodes, e.g. the distance d in Fig. 6.1, increases and the minimum distance of the subcode, e.g. the distance d' in Fig. 6.1, reduces. Thus the codewords concentrate at the center of the regions and the minimum distance of the codebook is reduced (see Fig. 6.2(b)). This induces a tradeoff between E_1 and E_2. The proof of this new outer bound is based on a geometric conjecture (Conjecture 6.1), so the final result depends on the correctness of this geometric conjecture. This single-user two-message EER outer bound can then be applied to Gaussian broadcast and multiple access channels (Theorem 6.2, Theorem 6.3).

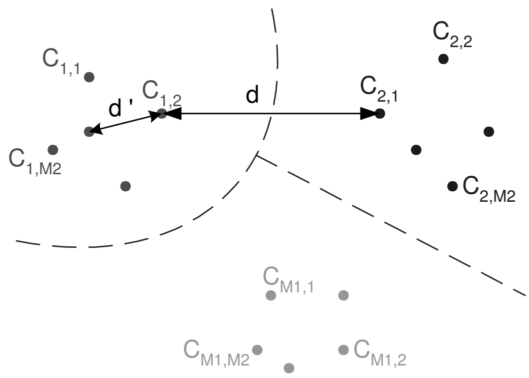


Figure 6.1: Partition codewords into subcodes.

We first review spherical codes and the minimum distance bound [7, 28, 29]. We state our main result in Theorem 6.1, which is based on Conjecture 6.1 given in Section 6.3.

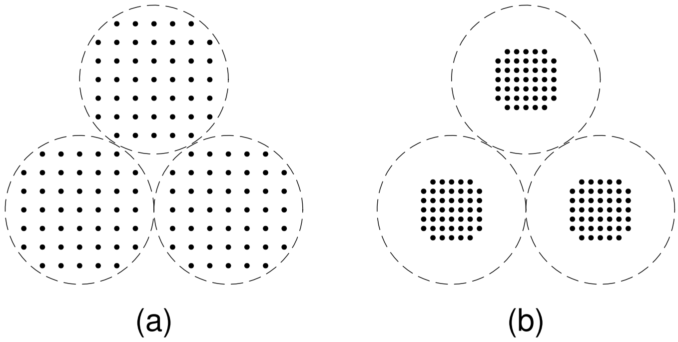


Figure 6.2: Codeword distributions when (a) E_1 and E_2 are roughly the same; (b) E_1 and E_2 are extremely asymmetric.

6.1 Preliminaries: Spherical Code and Minimum Distance Bound

In the following, we review spherical codes and the minimum distance bound for Gaussian single-user channels (with one message set) [7, 28]. A brief summary of spherical codes and the minimum distance bound for Gaussian channels can be found in [29].

For a scalar Gaussian channel with average power constraint P, it was shown in [7] that the optimum codebook (with sufficiently large codeword length N) can be constructed with each codeword having power exactly P. Therefore, it is sufficient to consider only spherical codes when transmitting over a Gaussian channel, i.e., a code with codewords on the unit sphere S^{N-1} in $\mathbb{R}^N$. The noise variance is normalized correspondingly to satisfy the required SNR.

In [7], the following notation is used.

- $\Omega(N, \theta)$: solid angle in $\mathbb{R}^N$ of a circular cone of half-angle θ, i.e., the area of unit sphere S^{N-1} cut out by the cone.

As pointed out in [30], for any spherical code (with positive rate) the angle between any two codewords is less than $\frac{\pi}{2}$, so it can be assumed that any angle θ

appeared here is less than $\frac{\pi}{4}$. It is shown in [7] that

$$\frac{\Omega(N,\theta)}{\Omega(N,\pi)} \cong \sin^N \theta. \tag{6.1}$$

Now consider the minimum distance bound for Gaussian channels. Consider a spherical code $\mathcal{C} = \{C_1, \ldots, C_M\} \subset S^{N-1}$ with $M = e^{NR}$ codewords. Define

- $d_{min}(\mathcal{C})$: minimum (Euclidean) distance of spherical code $\mathcal{C}$. $d_{min}(\mathcal{C}) \triangleq \min_{k \neq k'} d(C_k, C_{k'})$, where $d(x,y)$ is the distance between any two points x and y in $\mathbb{R}^N$.

- $d_{min}(N,R)$: $d_{min}(N,R) \triangleq \max_{\mathcal{C} \subset S^{N-1}} d_{min}(\mathcal{C})$, where the maximization is over all spherical codes $\mathcal{C} \subset S^{N-1}$ with $M = e^{NR}$ codewords.

- $d_{min}(R)$: $d_{min}(R) \triangleq \limsup_{N\to\infty} d_{min}(N,R)$.

The minimum distance exponent $E_{md}(R, SNR)$ is defined as

$$E_{md}(R, SNR) \triangleq \frac{SNR}{8} d_{min}^2(R). \tag{6.2}$$

It can be shown that the minimum distance exponent is an error exponent upper bound for Gaussian channels by evaluating the pairwise error probability of the two codewords with the minimum distance in the codebook. The minimum distance $d_{min}(R)$ is unknown, but there are upper and lower bounds. The best known upper bound is given in [28] as

$$d_{min}(R) \leq \begin{cases} \frac{\sqrt{2}(\sqrt{1+\rho}-\sqrt{\rho})}{\sqrt{1+2\rho}} & 0 \leq R \leq 0.234 \\ \sqrt{2}e^{-R-0.0686} & 0.234 < R \end{cases}, \tag{6.3}$$

where the real numbers 0.0686 and 0.234 here are approximate, and ρ is the root of the equation

$$R = (1+\rho)H(\frac{\rho}{1+\rho}). \tag{6.4}$$

6.2 Outer Bound for Error Exponent Region

In the following, we derive an EER outer bound for a Gaussian single-user channel with two message sets based on Conjecture 6.1 (to be stated in Section 6.3), then apply this EER outer bound (with slight modifications) to Gaussian broadcast and multiple access channels. Roughly speaking, Conjecture 6.1 concerns the problem of packing codewords on S^{N-1} such that the surface area "occupied" by the codewords is minimal. However, Conjecture 6.1 is lengthy and technical in nature, so, for better readability, we present it after the EER outer bounds.

Consider transmission of two messages over a discrete-time memoryless stationary scalar Gaussian single-user channel

$$\mathbf{Y} = \mathbf{X} + \mathbf{Z}, \tag{6.5}$$

where $\mathbf{X}$ is the channel input with average power constraint P, $\mathbf{Y}$ is the channel output, and $\mathbf{Z}$ is the additive noise with variance σ^2. The codebook $\mathcal{C} = \{C_{i,j} \mid 1 \leq i \leq M_1, 1 \leq j \leq M_2\}$ consists of $M_3 \triangleq M_1 M_2$ codewords with codeword length N. Denote (i,j) as the index of the transmitted codeword and $(\hat{i},\hat{j})$ as the index of the decoded codeword. The first index i is used for message 1 and the second index j is used for message 2. Message 1 is decoded in error if $\hat{i} \neq i$ and message 2 is decoded in error if $\hat{j} \neq j$. Define message 1's rate $R_1 \triangleq \frac{\log M_1}{N}$, message 2's rate $R_2 \triangleq \frac{\log M_2}{N}$, and sum rate $R_3 \triangleq \frac{\log M_3}{N} = R_1 + R_2$. The EER for a single-user channel with two message sets is defined in the following.

Definition 6.1 *Consider a single-user channel operated at rates R_1 and R_2 for message 1 and message 2, respectively. The EER is defined as the set of all achievable error exponent pairs $(E_1(R_1,R_2), E_2(R_1,R_2))$, where $E_1(R_1,R_2)$ and $E_2(R_1,R_2)$ are defined as*

$$\begin{aligned} E_1(R_1,R_2) &\triangleq \lim_{N\to\infty} -\frac{\log P_{e1}(N,R_1,R_2)}{N} \\ E_2(R_1,R_2) &\triangleq \lim_{N\to\infty} -\frac{\log P_{e2}(N,R_1,R_2)}{N}, \end{aligned} \tag{6.6}$$

and $P_{e1}(N,R_1,R_2)$ and $P_{e2}(N,R_1,R_2)$ are the probabilities of error for message 1 and message 2, respectively.

An EER outer bound for the Gaussian channel with two message sets is summarized in the following theorem.

Theorem 6.1 *Based on Conjecture 6.1, for a Gaussian single-user channel with two message sets under power constraint P and noise power σ^2, an outer bound for EER is*

$$\begin{aligned} E_1 &\leq \left[\sin\eta(R_2,R_1,E_2,\frac{P}{\sigma^2})\right]^2\ E_{md}(R_1,\frac{P}{\sigma^2}) \\ E_2 &\leq \left[\sin\eta(R_1,R_2,E_1,\frac{P}{\sigma^2})\right]^2\ E_{md}(R_2,\frac{P}{\sigma^2}), \end{aligned} \tag{6.7}$$

where

$$\eta(R_1,R_2,E,SNR) \triangleq \sin^{-1}\left(\frac{e^{-2R_1}}{\sin\left(\sin^{-1}(e^{-R_1}) + \delta\big(R_2, \sin^{-1}(e^{-R_1}), \sin^{-1}(\sqrt{\frac{2E}{SNR}}\,)\big)\right)}\right) \tag{6.8a}$$

$$\delta(R,\theta_r',\theta_e') \triangleq \psi(R,\theta_r',\theta_e') - \theta_r' \tag{6.8b}$$

and

$$\psi(R,\theta'_r,\theta'_e) = \sin^{-1}\big(\frac{\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\big) \tag{6.9a}$$

$$\alpha = \sin^2\theta'_r\cos^2\phi + \cos^2\theta'_r \tag{6.9b}$$

$$\beta = (2-4\sin^2\frac{\theta'_e}{2})\sin\theta'_r\cos\phi \tag{6.9c}$$

$$\gamma = 1-4\sin^2\frac{\theta'_e}{2}+4\sin^4\frac{\theta'_e}{2}-\cos^2\theta'_r \tag{6.9d}$$

$$\phi = \cos^{-1}\big(1-\frac{1}{2}\,d^2_{min}(R)\big) \tag{6.9e}$$

with the assumption that $2\sin\frac{\theta'_e}{2} > \sin\theta'_r\ d_{min}(R)$ *for the arguments of* $\psi(R,\theta'_r,\theta'_e)$. [1]

Proof: The proof is given in Appendix A. The function $d_{min}(\cdot)$ in (6.9e) can be substituted for any upper bound for $d_{min}(\cdot)$. ■

The EER outer bound given in Theorem 6.1 can be easily modified and applied to Gaussian broadcast and multiple access channels. We summarize the result in the following theorems.

Theorem 6.2 *Based on Conjecture 6.1, for a Gaussian broadcast channel with power constraint P and noise power σ_1^2 and σ_2^2 for user 1 and user 2, respectively, an outer bound for EER is*

$$\begin{aligned} E_1 &\le \big[\sin\eta(R_2,R_1,E_2,\frac{P}{\sigma_2^2})\big]^2\ E_{md}(R_1,\frac{P}{\sigma_1^2}) \\ E_2 &\le \big[\sin\eta(R_1,R_2,E_1,\frac{P}{\sigma_1^2})\big]^2\ E_{md}(R_2,\frac{P}{\sigma_2^2}). \end{aligned} \tag{6.10}$$

Proof: The proof for this theorem is the same as the proof given for Theorem 6.1 with some minor changes and is omitted here. ■

[1]The outer bound is not valid when this condition is not satisfied.

Theorem 6.3 *Based on Conjecture 6.1, for a Gaussian multiple access channel with power constraints P_1 and P_2 for user 1 and user 2 and noise power σ^2, an outer bound for EER is*

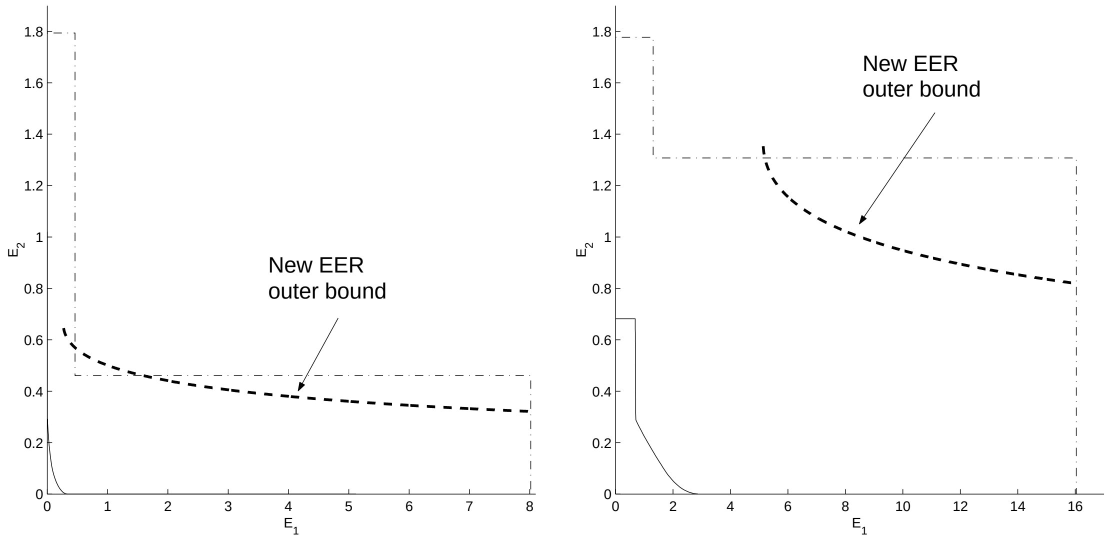
$$E_1 \leq \big[\sin\eta(R_2, R_1, E_2, \frac{P_1+P_2}{\sigma^2})\big]^2 \; E_{md}(R_1, \frac{P_1+P_2}{\sigma^2})$$
$$E_2 \leq \big[\sin\eta(R_1, R_2, E_1, \frac{P_1+P_2}{\sigma^2})\big]^2 \; E_{md}(R_2, \frac{P_1+P_2}{\sigma^2}). \tag{6.11}$$

Proof: The proof follows directly from Theorem 6.1 by comparing the performance achieved by the codebook CB in a single-user channel under power constraint P_1+P_2 with the performance achieved by the codebooks CB_1 and CB_2 in the multiple access channel under power constraints P_1 and P_2. ■

We illustrate these outer bounds in Fig. 6.3. In Fig. 6.3(a) the solid curve is the boundary of the EER inner bound, the dashed-dotted curve is the boundary of the EER outer bound given in Theorem 4.2, and the dashed curve is the boundary of the EER outer bound given in Theorem 6.2. In Fig. 6.3(b), the solid curve is the boundary of the EER inner bound, the dashed-dotted curve is the boundary of the EER outer bound given in Theorem 5.2, and the dashed curve is the boundary of the EER outer bound given in Theorem 6.3. Note that in Theorem 6.1 we require the arguments of $\psi(R, \theta'_r, \theta'_e)$ to satisfy $2\sin\frac{\theta'_e}{2} > \sin\theta'_r \; d_{min}(R)$. This is true for $E_1 \geq 0.25$ in Fig. 6.3(a) and for $E_1 \geq 5$ in Fig. 6.3(b). For $E_1 < 0.25$ in Fig. 6.3(a) and $E_1 < 5$ in Fig. 6.3(b), the EER outer bounds given in Theorem 6.2 and Theorem 6.3 are no longer valid.

6.3 Conjectured Minimum-Area Spherical Code

In this section, we state a geometric conjecture about spherical codes which is used in the derivation of the two-message EER outer bound given in Theorem 6.1.



(a) (b)

Figure 6.3: EER inner bound (solid) and outer bound (dashed-dotted, dashed) for (a) broadcast channel with $R_1 = 0.5, R_2 = 2.4, \frac{P}{\sigma_1^2} = 100, \frac{P}{\sigma_2^2} = 1000$; (b) multiple access channel with $R_1 = 0.5, R_2 = 1.6, \frac{P_1}{\sigma^2} = 200, \frac{P_2}{\sigma^2} = 200$.

As mentioned earlier, the EER outer bound is derived by partitioning the output space into several regions, where the area of each region is upper bounded by some quantity depending only on the rates. When E_1 and E_2 are extremely asymmetric, the codewords inside each region are far away from the region boundary and this reduces the minimum distance of the codewords due to the area constraint for each region. In order to estimate the reduction of the minimum distance, we consider an equivalent problem and its conjectured solution - Conjecture 6.1.

Define the following quantities.

- $Cone(N, \theta, W)$: circular cone of half angle θ with vertex at the origin O and axis $\overline{OW}$, where W is a point on S^{N-1} (see Fig. 6.4).

- $Cone^s(N, \theta, W)$: surface cap. $Cone^s(N, \theta, W) \triangleq Cone(N, \theta, W) \cap S^{N-1}$.

- $\partial Cone^s(N, \theta, W)$: boundary of surface cap $Cone^s(N, \theta, W)$.

- $A(\cdot)$: measure of surface areas on S^{N-1}.

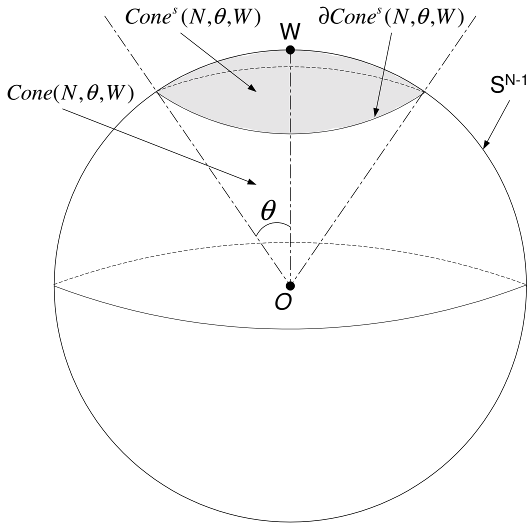


Figure 6.4: Circular cone $Cone(N, \theta, W)$, surface cap $Cone^s(N, \theta, W)$ and its boundary $\partial Cone^s(N, \theta, W)$.

We write $Cone(N, \theta, W)$ as $Cone(\theta, W)$ when the dimension N is clear from the context. Consider the following problem:

"Assuming $d < 2 \sin \Phi$, what is the minimum of $A\big(\bigcup_{k=1}^{M} Cone^s(N, \Phi, C_k)\big)$ under the distance constraint $d_{min}(\mathcal{C}) \geq d$, where the minimization is over all spherical codes $\mathcal{C} = \{C_1, \ldots, C_M\} \subset S^{N-1}$ with $M = e^{NR}$ codewords?"

We are only interested in this problem in high dimensions ($N \to \infty$). In the following, we give a motivation of our conjecture.

Motivation of the conjecture: Under the assumption that $d < 2 \sin \Phi$ given in the problem, any two different surface caps $Cone^s(N, \Phi, C_k)$ and $Cone^s(N, \Phi, C_{k'})$ can overlap and still satisfy the distance constraint $d(C_k, C_{k'}) \geq d$. Intuitively, the optimum code should pack the codewords on S^{N-1} as densely as possible (without violating the distance constraint d), because the more the surface caps $Cone^s(N, \Phi, C_k)$ overlap, the less the total area $A\big(\bigcup_{k=1}^{M} Cone^s(N, \Phi, C_k)\big)$. It seems to us that the most efficient way to pack the codewords C_k's in S^{N-1} is to pack them in some surface cap $Cone^s(N, \theta', W)$, where θ' is the smallest angle such that the spherical code $\mathcal{C}$ still satisfies the distance constraint $d_{min}(\mathcal{C}) \geq d$ (see Fig. 6.5). The area around the boundary of $Cone^s(N, \theta', W)$ is dominant in high dimensions, and this can be seen by the following

$$\frac{A\big(Cone^s(N, \theta' - \epsilon, W)\big)}{A\big(Cone^s(N, \theta', W)\big)} = \frac{\Omega(N, \theta' - \epsilon)}{\Omega(N, \theta')} \cong \frac{\sin^N(\theta' - \epsilon)}{\sin^N \theta'} \cong 0 \tag{6.12}$$

for any $0 < \epsilon < \theta'$. Therefore, we may further assume that all the codewords are in the boundary $\partial Cone^s(N, \theta', W)$.

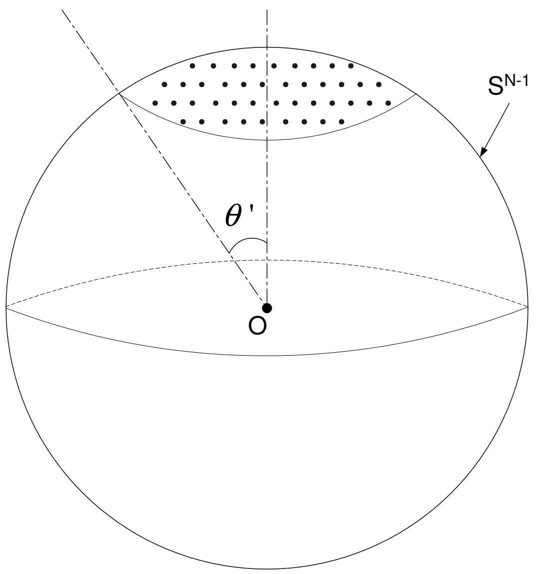


Figure 6.5: Spherical code in surface cap.

We now formally provide a description of a code that we conjecture to be one of the solutions to the above optimization problem.

Description: Define an angle $\Upsilon \triangleq \sin^{-1} \frac{d}{d_{min}(N-1, \frac{N}{N-1}R)}$. Note that $\partial Cone^s(N, \Upsilon, W)$ is an $(N-2)$-dimensional sphere with radius $\sin \Upsilon$. From the definition of $d_{min}(N, R)$, there exists a code $\mathcal{D} = \{D_1, D_2, \ldots, D_M\} \subset \partial Cone^s(N, \Upsilon, W)$ with M codewords and $d_{min}(\mathcal{D}) = d$. Next, we augment the code $\mathcal{D}$ to become the "densest" code $\mathcal{D}' \subset \partial Cone^s(N, \Upsilon, W)$ in the following way:

(a) Start with $\mathcal{D}' = \mathcal{D}$.

(b) If there is a point $U \in \partial Cone^s(N, \Upsilon, W)$ such that the interior of the surface cap $Cone^s(N, 2\sin^{-1}\frac{d}{2}, U)$ contains no codewords of $\mathcal{D}'$, then add this point U to $\mathcal{D}'$ as a new codeword.

(c) Continue this procedure until we can't add any more codeword to $\mathcal{D}'$.

From the construction of $\mathcal{D}'$, any surface cap $Cone^s(N, 2\sin^{-1}\frac{d}{2}, V)$ must include at least one codeword of $\mathcal{D}'$, where V is an arbitrary point in $\partial Cone^s(N, \Upsilon, W)$. Hence we refer to $\mathcal{D}'$ as the densest code due to this property. This densest code property of $\mathcal{D}'$ is used in the proof of Theorem 6.1.

Define $M' \triangleq |\mathcal{D}'|$, and we have $M' \cong M$ due to the assumption that $\mathcal{D}$ is the code achieves $d_{min}(N-1, \frac{N}{N-1}R)$. Let the k^{th} codeword of $\mathcal{D}'$ be denoted as D'_k. We make the following conjecture.

Conjecture 6.1 *The area corresponding to $\mathcal{D}'$, i.e. $A\big(\bigcup_{k=1}^{M'} Cone^s(N, \Phi, D'_k)\big)$, is asymptotically equal to the area corresponding to the optimum code for the above problem in the following sense:*

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{\min_{\mathcal{C}\subset S^{N-1}} A\big(\bigcup_{k=1}^{M} Cone^s(N, \Phi, C_k)\big)}{A\big(\bigcup_{k=1}^{M'} Cone^s(N, \Phi, D'_k)\big)} = 0. \tag{6.13}$$

■

From Conjecture 6.1, we also have

$$\lim_{N\to\infty}\frac{1}{N}\log\frac{\min_{\mathcal{C}\subset S^{N-1}} A\big(\bigcup_{k=1}^{M} Cone^s(N,\Phi,C_k)\big)}{A\big(\bigcup_{k=1}^{M} Cone^s(N,\Phi,D_k)\big)} = 0, \tag{6.14}$$

since

$$\min_{\mathcal{C}\subset S^{N-1}} A\big(\bigcup_{k=1}^{M} Cone^s(N,\Phi,C_k)\big) \le A\big(\bigcup_{k=1}^{M} Cone^s(N,\Phi,D_k)\big) \le A\big(\bigcup_{k=1}^{M'} Cone^s(N,\Phi,D'_k)\big). \tag{6.15}$$

Therefore, the area corresponding to the code $\mathcal{D}$ is also asymptotically minimum. Since the code rates of $\mathcal{D}'$ and $\mathcal{D}$ are asymptotically equal, i.e. $M' \cong M$, we will simply regard these two codes $\mathcal{D}'$ and $\mathcal{D}$ the same and say that the code $\mathcal{D}$ (with M codewords) is asymptotically optimum (in the sense of the corresponding area) and is also the densest code. Here the densest code means that any surface cap $Cone^s(N, 2\sin^{-1}\frac{d}{2}, V)$ must include at least one codeword of $\mathcal{D}$ for any $V \in \partial Cone^s(N,\Upsilon,W)$.

CHAPTER 7

Diversity Gain Regions for MIMO Fading Broadcast Channels

7.1 Diversity Gain Region

Consider a MIMO fading broadcast channel with m transmit antennas and n_1 and n_2 receive antennas for user 1 and user 2, respectively. The channel model is

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{H}_1\mathbf{X} + \mathbf{Z}_1$$
$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}}\mathbf{H}_2\mathbf{X} + \mathbf{Z}_2. \tag{7.1}$$

The channel fading matrices between the transmitter and the receiver 1 and the receiver 2 are represented by an $n_1 \times m$ matrix $\mathbf{H}_1$ and an $n_2 \times m$ matrix $\mathbf{H}_2$, respectively. We assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ remain constant over a block with length l, and change to a new independent realization in the next block. $\mathbf{H}_1$ and $\mathbf{H}_2$ have i.i.d. entries and each entry is distributed as $\mathcal{CN}(0, 1)$. We assume that the fading matrices are known to the receivers but unknown to the transmitter. The channel input $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average power at each transmit antenna is 1, which means that the average SNR at each receive antenna is SNR. The noise $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $n_1 \times l$ and $n_2 \times l$ matrices with i.i.d. entries distributed as $\mathcal{CN}(0,1)$. The channel outputs $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are $n_1 \times l$ and $n_2 \times l$ matrices, respectively.

We define the diversity gain region (DGR) for a MIMO fading broadcast channel in the following.

Definition 7.1 *Consider a MIMO fading broadcast channel operated at a rate pair $R_1 = R_1(SNR)$, $R_2 = R_2(SNR)$ over N blocks such that*

$$\begin{aligned} \lim_{SNR\to\infty} \frac{R_1(SNR)}{\log SNR} &= r_1 \\ \lim_{SNR\to\infty} \frac{R_2(SNR)}{\log SNR} &= r_2, \end{aligned} \tag{7.2}$$

where r_1 and r_2 are referred to as the multiplexing gains for user 1 and user 2, respectively. Given a multiplexing gain pair (r_1, r_2) of the two users, we define the DGR (corresponding to coding over N blocks) as the set of all achievable diversity gain pairs $(d_1(N, r_1, r_2), d_2(N, r_1, r_2))$ for all encoding schemes, where $d_1(N, r_1, r_2)$ and $d_2(N, r_1, r_2)$ are defined as

$$\begin{aligned} d_1(N, r_1, r_2) &\triangleq \lim_{SNR\to\infty} -\frac{\log P_{e1}(N, R_1(SNR), R_2(SNR))}{N \log SNR} \\ d_2(N, r_1, r_2) &\triangleq \lim_{SNR\to\infty} -\frac{\log P_{e2}(N, R_1(SNR), R_2(SNR))}{N \log SNR}, \end{aligned} \tag{7.3}$$

where $P_{e1}(N, R_1(SNR), R_2(SNR))$, $P_{e2}(N, R_1(SNR), R_2(SNR))$ are the probabilities of error for user 1 and user 2, respectively.

When $N = 1$, we use $d_1(r_1, r_2)$ and $d_2(r_1, r_2)$ as shorthand notations for $d_1(1, r_1, r_2)$ and $d_2(1, r_1, r_2)$, respectively. Note that if (d_1^*, d_2^*) is achieved by an encoding scheme over one single block, i.e., $(d_1^*, d_2^*) = (d_1(r_1, r_2), d_2(r_1, r_2))$ for some encoding scheme, then for any $N > 1$ there exists an encoding scheme over N blocks such that $d_1^* \leq d_1(N, r_1, r_2)$ and $d_2^* \leq d_2(N, r_1, r_2)$, since encoding over multiple blocks can only improve the diversity gains of the users. Therefore, a DGR inner bound achieved by an encoding scheme over one single block is also a valid DGR inner bound for an encoding scheme over N blocks ($N > 1$). Similarly, if a DGR outer bound is valid for all encoding schemes over sufficiently large N ($N \to \infty$), this DGR outer bound is also valid for any encoding scheme over a finite number of blocks.

7.2 Inner Bound for Diversity Gain Region

The inner bound derived in this section is based on encoding over one block, i.e., $N = 1$, so it is also a valid inner bound for encoding over multiple blocks $N > 1$.

We derive a DGR inner bound using superposition encoding, and a mixture of joint ML and naive single-user decoding. In superposition encoding, we construct two independent random codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$, of size M_1 and M_2, respectively (see Fig. 7.1). Let $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$ denote the i^{th} and the j^{th} codewords in the codebooks $\mathbf{CB}_1$ and $\mathbf{CB}_2$, respectively. Note that $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$ are $m \times l$ random matrices. The channel input $\mathbf{X}$ is equal to $\mathbf{C}_{1,i} + \mathbf{C}_{2,j}$. Let $\mathbf{C}_{1,i}(q,k)$ and $\mathbf{C}_{2,j}(q,k)$ denote the k^{th} elements in the q^{th} transmit antenna in the codewords $\mathbf{C}_{1,i}$ and $\mathbf{C}_{2,j}$, respectively. Each random variable $\mathbf{C}_{1,i}(q,k)$ is i.i.d. with distribution $\mathcal{CN}(0,1)$ for $1 \leq k \leq \beta l$, and is i.i.d. with distribution $\mathcal{CN}(0, SNR^{-(1-p_1)})$ for $\beta l + 1 \leq k \leq l$, where $\beta = \frac{a}{l}$, for some $a \in \{0, 1, \ldots, l\}$, and $0 \leq p_1 \leq 1$. Similarly, each random variable $\mathbf{C}_{2,j}(q,k)$ is i.i.d. with distribution $\mathcal{CN}(0, SNR^{-(1-p_2)})$ for $1 \leq k \leq \beta l$ and is i.i.d. with distribution $\mathcal{CN}(0,1)$ for $\beta l + 1 \leq k \leq l$, where $0 \leq p_2 \leq 1$. Note that the average power per transmit antenna in superposition encoding is $1 + SNR^{-(1-p_2)}$ for the first βl transmissions and is $1 + SNR^{-(1-p_1)}$ for the remaining $(1-\beta)l$ transmissions, which are slightly more than the power constraint 1. However, we are only interested in the high SNR approximation, so a power constraint of 1 or 2 makes no difference on the diversity gains.

For comparison, we also consider two special cases of superposition encoding - uniform superposition and on-off superposition. In uniform superposition, the parameter β in Fig. 7.1 is chosen to be zero or one, so the random codebook $\mathbf{CB}_1$ or $\mathbf{CB}_2$ has i.i.d. entries. In on-off superposition, the parameters p_1 and p_2 in Fig. 7.1 are chosen to be zero, so the transmitter switches between user 1 and user 2 during the transmission.

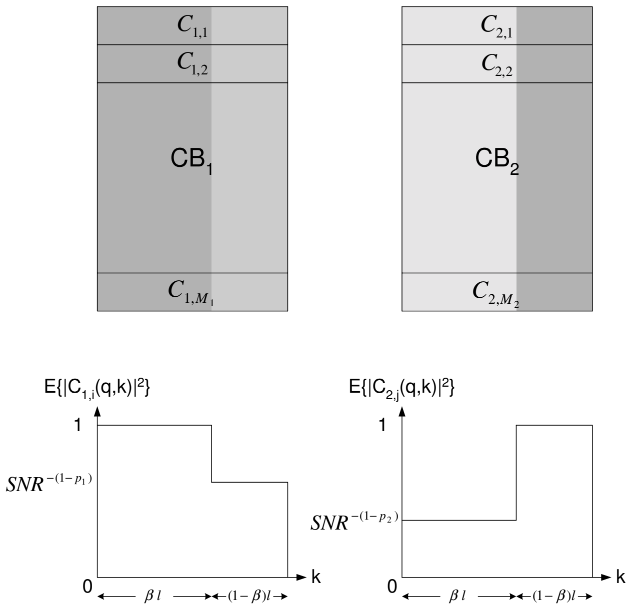


Figure 7.1: Random codebooks for user 1 and user 2 using superposition encoding.

Before deriving the DGR inner bound, we first derive two intermediate results. The first result is a diversity gain $d^{ns}_{m,n,l,p}(r)$ for a MIMO fading broadcast channel assuming naive single-user decoding. The second result is a nonuniform-power random coding diversity gain $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ for a MIMO fading single-user channel.

7.2.1 Naive Single-User Diversity Gain

For the naive single-user diversity gain, consider the broadcast channel using uniform superposition encoding ($\beta = 1$). If we decode user 1's message using naive single-user decoding, i.e., user 1 simply regards user 2 as noise, we can derive an achievable diversity gain for user 1. We summarize the result in the following lemma.

Lemma 7.1 *For a MIMO fading broadcast channel operated at a multiplexing gain pair* (r_1, r_2) *with* m *transmit antennas,* n_1*,* n_2 *receive antennas and block length* l*, the optimal probability of detection error for user 1 using uniform superposition encoding* ($\beta = 1$) *and naive single-user decoding is upper-bounded by*

$$P_{e1} \dot{\leq} SNR^{-d^{ns}_{m,n_1,l,p_2}(r_1)}, \tag{7.4}$$

where

$$d^{ns}_{m,n_1,l,p_2}(r_1) = \min_{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n_1)} \setminus \mathcal{B}} \Big\{ \sum_{i=1}^{\min(m,n_1)} (2i - 1 + |m - n_1|)\alpha_i + l\Big[\sum_{i=1}^{\min(m,n_1)} (1 - \alpha_i - (p_2 - \alpha_i)^+)^+ - r_1 \Big] \Big\}, \tag{7.5}$$

and

$$\mathcal{B} = \Big\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n_1)} \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n_1)} \geq 0; \sum_{i=1}^{\min(m,n_1)} (1 - \alpha_i - (p_2 - \alpha_i)^+)^+ < r_1 \Big\}. \tag{7.6}$$

Proof: If we use uniform superposition encoding with $\beta = 1$, the channel output for user 1 is

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{H}_1(\mathbf{X}_1 + \mathbf{X}_2) + \mathbf{Z}_1, \tag{7.7}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent with i.i.d elements $\mathcal{CN}(0,1)$ and $\mathcal{CN}(0, SNR^{-(1-p_2)})$, respectively. If we decode user 1's message using naive single-user decoding, we can consider the following MIMO fading side-interference single-user channel

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}}\mathbf{H}(\mathbf{X} + \mathbf{S}) + \mathbf{Z}, \tag{7.8}$$

where $\mathbf{H}$ is an $n \times m$ matrix with i.i.d. entries $\mathcal{CN}(0,1)$, and $\mathbf{Z}$ and $\mathbf{S}$ are $n \times l$ noise and side-interference matrices with i.i.d. entries $\mathcal{CN}(0,1)$ and $\mathcal{CN}(0, SNR^{-(1-p)})$ respectively. The channel input $\mathbf{X}$ is an $m \times l$ matrix and is normalized such that the average power at each transmit antenna is 1. We want to show that the probability of detection error in this side-interference channel is upper bounded by $SNR^{-d^{ns}_{m,n,l,p}(r)}$. Note that $d^{ns}_{m,n,l,p}(r) = 0$ for $r \geq \min(m,n)(1-p)$, so $SNR^{-d^{ns}_{m,n,l,p}(r)}$ is a valid (trivial) upper bound for $r \geq \min(m,n)(1-p)$.

At high SNR, we can ignore the integral of the probability of error over the range $\mathbf{H} \notin \mathbb{R}_+^{\min(m,n)}$. The probability of error can be upper bounded by

$$P_e \leq P(\mathbf{H} \in \mathcal{B}) + P(\text{error}, \mathbf{H} \notin \mathcal{B}), \tag{7.9}$$

where

$$\mathcal{B} = \Big\{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0; \sum_{i=1}^{\min(m,n)} (1 - \alpha_i - (p - \alpha_i)^+)^+ < r\Big\}. \tag{7.10}$$

Define

$$\underline{\alpha}^* \triangleq \arg\min_{\underline{\alpha}\in\overline{\mathcal{B}}} \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i, \tag{7.11}$$

where $\overline{\mathcal{B}}$ is the closure of $\mathcal{B}$, i.e.,

$$\overline{\mathcal{B}} = \Big\{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0; \sum_{i=1}^{\min(m,n)} (1-\alpha_i-(p-\alpha_i)^+)^+ \leq r\Big\}. \tag{7.12}$$

It is easy to verify that

$$\sum_{i=1}^{\min(m,n)} (1-\alpha_i^*-(p-\alpha_i^*)^+)^+ = r \tag{7.13}$$

for $0 \leq r \leq \min(m,n)(1-p)$, so

$$\begin{aligned}
\min_{\underline{\alpha}\in\overline{\mathcal{B}}} \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i &= \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i^* \\
&= \sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i^* + l\big[\sum_{i=1}^{\min(m,n)} (1-\alpha_i^*-(p-\alpha_i^*)^+)^+ - r\big] \\
&\geq \min_{\underline{\alpha}\in\mathbb{R}_+^{\min(m,n)}\setminus\mathcal{B}} \Big\{\sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i + l\big[\sum_{i=1}^{\min(m,n)} (1-\alpha_i-(p-\alpha_i)^+)^+ - r\big]\Big\} \\
&= d_{m,n,l,p}^{ns}(r), && (7.14)
\end{aligned}$$

where the above inequality is due to $\underline{\alpha}^* \in \mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}$. At high SNR, $P(\underline{\alpha})$ can be approximated by $\prod_{i=1}^{\min(m,n)} SNR^{-(2i-1+|m-n|)\alpha_i}$ [16]. Hence

$$P(\mathbf{H} \in \mathcal{B}) = \int_{\mathcal{B}} P(\underline{\alpha}) d\underline{\alpha} \doteq SNR^{-\min_{\underline{\alpha}\in\overline{\mathcal{B}}} \sum_{i=1}^{\min(m,n)}(2i-1+|m-n|)\alpha_i} \leq SNR^{-d^{ns}_{m,n,l,p}(r)}, \tag{7.15}$$

so it remains to prove that $P(\text{error}, \mathbf{H} \notin \mathcal{B})$ is also upper bounded by $SNR^{-d^{ns}_{m,n,l,p}(r)}$.

Conditioned on a channel realization $\mathbf{H} = H$, we can write the channel as

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}} H\mathbf{X} + (\sqrt{\frac{SNR}{m}} H\mathbf{S} + \mathbf{Z}). \tag{7.16}$$

Since H is known at the receiver, we can whiten the noise $\sqrt{\frac{SNR}{m}} H\mathbf{S}+\mathbf{Z}$ by multiplying $\big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}}$ at the channel output $\mathbf{Y}$, where H' is the conjugate transpose of H. Thus we have the following equivalent channel

$$\mathbf{Y} = \sqrt{\frac{SNR}{m}} \big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}} H\mathbf{X} + \mathbf{Z}. \tag{7.17}$$

Assume $X(0)$, $X(1)$ are two possible transmitted codewords and $\Delta X = X(1) - X(0)$. Suppose $X(0)$ is transmitted, then the probability that a receiver will make a detection error in favor of $X(1)$ is

$$\begin{aligned} &P(\mathbf{X}(0) = X(0) \rightarrow \mathbf{X}(1) = X(1) \mid \mathbf{H} = H) = \\ &\qquad P\Big(\frac{SNR}{m} \Big\| \frac{1}{2} \big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}} H\Delta X \Big\|_F^2 \leq ||\mathbf{w}||^2\Big), \end{aligned} \tag{7.18}$$

where I is an identity matrix, $\|\cdot\|_F$ is the Frobenius norm, and $\mathbf{w}$ is the additive noise with variance 1/2 on the direction of $\big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}} H\Delta X$. With the standard approximation of the Gaussian tail function: $Q(t) \leq 1/2 \exp(-t^2/2)$, we have

$$P(\mathbf{X}(0) = X(0) \to \mathbf{X}(1) = X(1) \mid \mathbf{H} = H) \leq \exp\Big\{ -\frac{SNR}{4m} \Big\| \big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}} H \Delta X \Big\|_F^2 \Big\}. \tag{7.19}$$

Averaging over the ensemble of random codes, we have the average pairwise error probability (PEP) given the channel realization H

$$\begin{aligned} P(\mathbf{X}(0) \to \mathbf{X}(1) \mid \mathbf{H} = H) &\leq \Big| I + \frac{SNR}{2m} \big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}} HH' \big(\frac{SNR^p}{m} HH' + I\big)^{-\frac{1}{2}} \Big|^{-l} \\ &= \Big\{ \prod_{i=1}^{\min(m,n)} \Big(1 + \frac{\frac{SNR}{2m}\lambda_i}{1 + \frac{SNR^p}{m}\lambda_i}\Big) \Big\}^{-l} \\ &\doteq \prod_{i=1}^{\min(m,n)} SNR^{-l(1-\alpha_i-(p-\alpha_i)^+)^+}, \end{aligned} \tag{7.20}$$

where $\lambda_i = SNR^{-\alpha_i}$ ($\alpha_1 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0$) and λ_i's are the nonzero eigenvalues of HH'.

Applying the union bound, we have

$$\begin{aligned} P(\text{error} \mid \underline{\alpha}) &\dot{\leq} SNR^{lr} \prod_{i=1}^{\min(m,n)} SNR^{-l(1-\alpha_i-(p-\alpha_i)^+)^+} \\ &= SNR^{-l[\sum_{i=1}^{\min(m,n)} (1-\alpha_i-(p-\alpha_i)^+)^+ - r]}. \end{aligned} \tag{7.21}$$

Therefore,

$$\begin{aligned} P(\text{error}, \mathbf{H} \notin \mathcal{B}) &\doteq \int_{\mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} P(\underline{\alpha}) P(\text{error} \mid \underline{\alpha}) d\underline{\alpha} \\ &\dot{\leq} \int_{\mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} SNR^{-\sum_{i=1}^{\min(m,n)} (2i-1+|m-n|)\alpha_i} SNR^{-l[\sum_{i=1}^{\min(m,n)} (1-\alpha_i-(p-\alpha_i)^+)^+ - r]} d\underline{\alpha} \\ &\doteq SNR^{-d^{ns}_{m,n,l,p}(r)}. \end{aligned} \tag{7.22}$$

This completes the proof. $\blacksquare$

We now give an explicit expression for $d^{ns}_{m,n,l,p}(r)$ in the form of a piecewise linear function of the multiplexing gain r. It is easy to see that $d^{ns}_{m,n,l,p}(r) = 0$ for $r \geq \min(m,n)(1-p)$. For $r \leq \min(m,n)(1-p)$, $d^{ns}_{m,n,l,p}(r)$ can be classified into the following three cases.

Case 1: $p \leq 1 - \frac{m+n-1}{l}$

$d^{ns}_{m,n,l,p}(k(1-p)) = (m-k)(n-k)$ for $k \in \{0, \ldots, \min(m,n)\}$. For $(k-1)(1-p) < r < k(1-p)$, $d^{ns}_{m,n,l,p}(r)$ consists of two line segments with slopes $m+n-2k+1$ and l, i.e. $d^{ns}_{m,n,l,p}(r)$ is a function connecting the following three points $(r,d) = ((k-1)(1-p),(m-k+1)(n-k+1))$, $(k(1-p) - \frac{sp}{1-s}, (m-k)(n-k) + \frac{sp}{1-s}l)$, and $(k(1-p),(m-k)(n-k))$, where $s = \frac{m+n-2k+1}{l}$ (see Fig. 7.2).

Case 2: $1 - \frac{m+n-1}{l} < p < 1 - \frac{|m-n|+1}{l}$

Define $k' = \max\{a \in \mathbb{Z} \mid (1-p)l \leq m+n-2a+1\}$. For $r \leq k'(1-p)$, $d^{ns}_{m,n,l,p}(r) = (m-k')(n-k') + (k'(1-p)-r)l$. For $r > k'(1-p)$ and $(k-1)(1-p) < r < k(1-p)$, $d^{ns}_{m,n,l,p}(r)$ consists of two line segments with slopes $m+n-2k+1$ and l, i.e. $d^{ns}_{m,n,l,p}(r)$ is a function connecting the following three points $(r,d) = ((k-1)(1-p),(m-k+1)(n-k+1))$, $(k(1-p) - \frac{sp}{1-s}, (m-k)(n-k) + \frac{sp}{1-s}l)$, and $(k(1-p),(m-k)(n-k))$, where $s = \frac{m+n-2k+1}{l}$ (see Fig. 7.3).

Case 3: $p \geq 1 - \frac{|m-n|+1}{l}$

$d^{ns}_{m,n,l,p}(r)$ is a linear function with slope l, i.e. $d^{ns}_{m,n,l,p}(r) = (\min(m,n)(1-p) - r)l$ (see Fig. 7.4).

Naive single-user diversity gains for case 1, 2, and 3 are illustrated in Fig. 7.5.

In case 1, $d^{ns}_{m,n,l,p}(r) = d_{m,n,l}(r)$ for $r \leq \frac{(1-p)l-(m+n-1)}{l-(m+n-1)}$. This means that one can still achieve the (single-user) random coding diversity gain $d_{m,n,l}(r)$ even with side

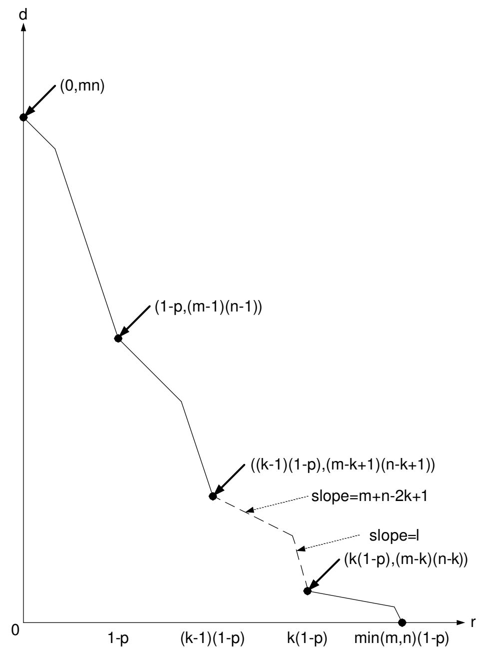


Figure 7.2: Naive single-user diversity gain: $p \leq 1 - \frac{m+n-1}{l}$.

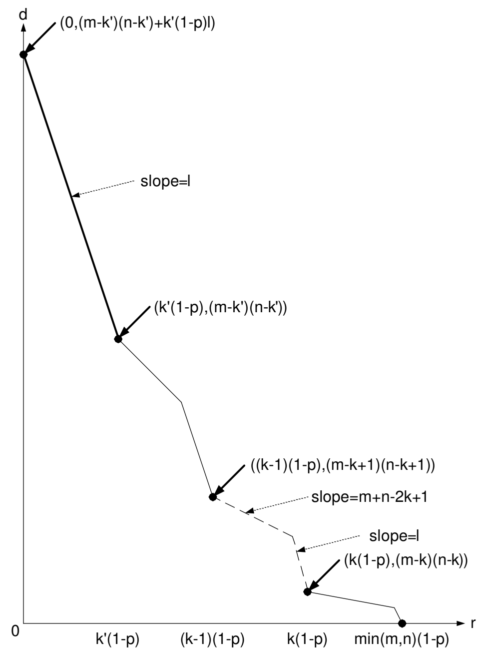


Figure 7.3: Naive single-user diversity gain: $1 - \frac{m+n-1}{l} < p < 1 - \frac{|m-n|+1}{l}$.

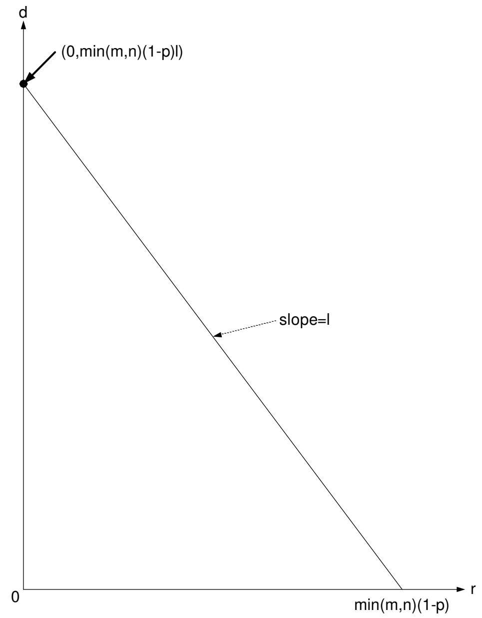
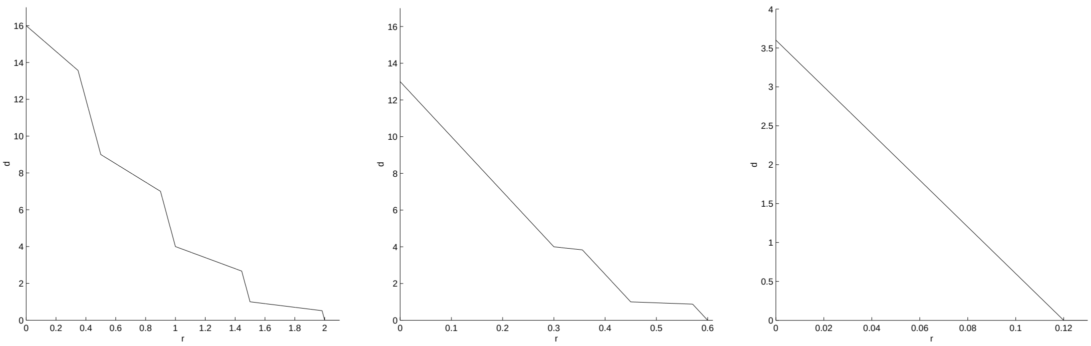


Figure 7.4: Naive single-user diversity gain: $p \geq 1 - \frac{|m-n|+1}{l}$.



(a) (b) (c)

Figure 7.5: Naive single-user diversity gain for $m = n = 4, l = 30$ (a) $p = 0.5$; (b) $p = 0.85$; (c) $p = 0.97$.

interference. This is a quite interesting and important phenomenon, so we give some intuition behind this. Consider the following two conditions

$$p < 1 - \frac{m+n-1}{l} \tag{7.23a}$$

$$r \leq \frac{(1-p)l - (m+n-1)}{l-(m+n-1)}. \tag{7.23b}$$

Note that the power of the side-interference at the receiver is $SNR \cdot SNR^{-(1-p)} = SNR^p$, so roughly speaking the first condition $p < 1 - \frac{m+n-1}{l}$ says that the power of the side-interference is small and the second condition $r \leq \frac{(1-p)l-(m+n-1)}{l-(m+n-1)}$ says that the channel is lightly used (small multiplexing gain). Since $\frac{(1-p)l-(m+n-1)}{l-(m+n-1)}$ is always less than $1-p$, we can never achieve full (single-user) random coding diversity gain for $r \geq 1-p$. To understand the implications of these two conditions in more detail, we can consider a single-input single-output (SISO) fading side-interference single-user channel ($m = n = 1$). The optimum probability of detection error for a SISO fading single-user channel can be upper bounded by

$$P_e(SNR) \dot{\leq} P(\mathbf{H} \in \mathcal{B}) + P(\text{error}, \mathbf{H} \notin \mathcal{B}), \tag{7.24}$$

where

$$\mathcal{B} = \Big\{ \alpha \geq 0 \mid (1 - \alpha - (p-\alpha)^+)^+ < r \Big\}. \tag{7.25}$$

We can upper bound $P(\mathbf{H} \in \mathcal{B})$ by

$$P(\mathbf{H} \in \mathcal{B}) \dot{\leq} SNR^{-\inf_{\alpha \in \mathcal{B}} \alpha}, \tag{7.26}$$

and upper bound $P(\text{error}, \mathbf{H} \notin \mathcal{B})$ by

$$P(\text{error}, \mathbf{H} \notin \mathcal{B}) \dot{\leq} SNR^{-d^{ns}_{1,1,l,p}(r)} = SNR^{-\min_{\alpha \in \mathbb{R}_+ \setminus \mathcal{B}} \{\alpha + l[(1-\alpha-(p-\alpha)^+)^+ - r]\}}. \qquad (7.27)$$

Under conditions (7.23a), (7.23b), the minimization in (7.26), (7.27) is achieved by $\alpha = 1 - r$, so the dominant error event happens when the power of the side-interference at the channel output

$$SNR^p \cdot SNR^{-\alpha} = SNR^{-(1-r-p)} \dot{<} 1 \qquad (7.28)$$

is less than the power of the Gaussian noise $\mathbf{Z}$, i.e., the side interference is negligible compared to additive Gaussian noise. This explains why one can still achieve the single-user diversity gain even with side interference. In general, the values of the minimization in (7.26), (7.27) might be different such that there is a gap between the naive single-user diversity gain $d^{ns}_{m,n,l,p}(r)$ and the random coding diversity gain $d_{m,n,l}(r)$.

7.2.2 Nonuniform-Power Random Coding Diversity Gain

For the nonuniform-power random coding diversity gain, consider a random codebook $\mathbf{CB}$ with M codewords (see Fig. 7.6). Denote $\mathbf{C}_i$ as the i^{th} codeword in the codebook $\mathbf{CB}$. Note that $\mathbf{C}_i$ is a $m \times l$ random matrix. Denote $\mathbf{C}_i(q,k)$ the k^{th} element in the q^{th} transmit antenna in the codeword $\mathbf{C}_i$. Each random variable $\mathbf{C}_i(q,k)$ is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_1)})$ for $1 \leq k \leq \beta l$, and is i.i.d. with $\mathcal{CN}(0, SNR^{-(1-p_2)})$ for $\beta l + 1 \leq k \leq l$, where $\beta = \frac{a}{l}$, for some $a \in \{0, 1, \ldots, l\}$, and $0 \leq p_1, p_2 \leq 1$.

Extending the derivation of the random coding diversity gain $d_{m,n,l}(r)$ in [16], we can derive a nonuniform-power random coding diversity gain $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ for a nonuniform-power random codebook $\mathbf{CB}$. The result is summarized in the following

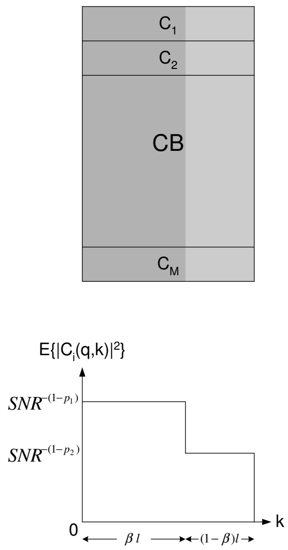


Figure 7.6: Nonuniform-power random codebook.

lemma.

Lemma 7.2 *For a MIMO fading single-user channel operated at a multiplexing gain r with m transmit antennas, n receive antennas and block length l, the optimal probability of detection error is upper-bounded by*

$$P_e \dot{\leq} SNR^{-d^{np}_{m,n,l,p_1,p_2,\beta}(r)}, \tag{7.29}$$

where

$$d^{np}_{m,n,l,p_1,p_2,\beta}(r) = \min_{\underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} \Big\{ \sum_{i=1}^{\min(m,n)} (2i - 1 + |m-n|)\alpha_i \;\; + \\ l\big[\beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+ - r\big]\Big\}, \tag{7.30}$$

and

$$\mathcal{B} = \Big\{ \underline{\alpha} \in \mathbb{R}_+^{\min(m,n)} \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)} \geq 0;$$
$$\beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+ < r \Big\}. \quad (7.31)$$

Proof: The derivation for $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ is a straight forward extension of the derivation for the random coding diversity gain $d_{m,n,l}(r)$ in [16]. The probability of error can be upper bounded by

$$P_e \leq P(\mathbf{H} \in \mathcal{B}) + P(\text{error}, \mathbf{H} \notin \mathcal{B}). \quad (7.32)$$

It is easy to verify that $P(\mathbf{H} \in \mathcal{B})$ is upper bounded by $SNR^{-d^{np}_{m,n,l,p_1,p_2,\beta}(r)}$, so we only need to prove that $P(\text{error}, \mathbf{H} \notin \mathcal{B})$ is also upper bounded by $SNR^{-d^{np}_{m,n,l,p_1,p_2,\beta}(r)}$.

The pairwise error probability given the channel realization H is upper bounded by

$$\begin{aligned} P(\mathbf{X}(0) \to \mathbf{X}(1) \mid \mathbf{H} = H) &\leq \left| \frac{SNR^{p_1}}{2m} HH' + I \right|^{-\beta l} \cdot \left| \frac{SNR^{p_2}}{2m} HH' + I \right|^{-(1-\beta)l} \\ &\doteq SNR^{-l[\beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+]}. \end{aligned} \quad (7.33)$$

Applying the union bound, we have

$$\begin{aligned} P(\text{error} \mid \underline{\alpha}) &\dot{\leq} SNR^{lr} \cdot SNR^{-l[\beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+]} \\ &\doteq SNR^{-l[\beta \sum_{i=1}^{\min(m,n)} (p_1 - \alpha_i)^+ + (1-\beta) \sum_{i=1}^{\min(m,n)} (p_2 - \alpha_i)^+ - r]}. \end{aligned} \quad (7.34)$$

Therefore,

$$P(\text{error}, \mathbf{H} \notin \mathcal{B}) \doteq \int_{\mathbb{R}_+^{\min(m,n)} \setminus \mathcal{B}} P(\underline{\alpha}) P(\text{error} | \ \underline{\alpha}) d\underline{\alpha} \ \dot{\leq} \ SNR^{-d^{np}_{m,n,l,p_1,p_2,\beta}(r)}. \qquad (7.35)$$

This completes the proof. ∎

We now give an explicit expression for $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ in the form of a piecewise linear function of the multiplexing gain r. It is easy to verify that $d^{np}_{m,n,l,p_1,p_2,\beta}(r) = p_2 \ d_{m,n,l}(\frac{r}{p_2})$ for $\beta = 0$, and $d^{np}_{m,n,l,p_1,p_2,\beta}(r) = p_1 \ d_{m,n,l}(\frac{r}{p_1})$ for $\beta = 1$. For $0 < \beta < 1$, we may assume $p_2 \leq p_1$ without loss of generality. $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ is a piecewise linear function and it is easy to see that $d^{np}_{m,n,l,p_1,p_2,\beta}(r) = 0$ for $r \geq \min(m,n)(\beta p_1 + (1-\beta)p_2)$. For $(k-1)(\beta p_1 + (1-\beta)p_2) < r < k(\beta p_1 + (1-\beta)p_2)$ and $k \in \{1, 2, \ldots, \min(m,n)\}$, $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ consists of two line segments with slopes $\min(\frac{m+n-2k+1}{\beta}, l)$ and $\min(m+n-2k+1, l)$, respectively, i.e., $d^{np}_{m,n,l,p_1,p_2,\beta}(r)$ is a function connecting the following three points $(r, d) =$

$$\Big((k-1)(\beta p_1 + (1-\beta)p_2), \sum_{q=k}^{\min(m,n)} \{p_1 \min(s_q, \beta l) + p_2[\min(s_q, l) - \min(s_q, \beta l)]\}\Big),$$

$$\Big(k(\beta p_1 + (1-\beta)p_2) - p_2, \sum_{q=k+1}^{\min(m,n)} \{p_1 \min(s_q, \beta l) + p_2[\min(s_q, l) - \min(s_q, \beta l)]\} + p_2 \min(s_k, l)\Big),$$

$$\Big(k(\beta p_1 + (1-\beta)p_2), \sum_{q=k+1}^{\min(m,n)} \{p_1 \min(s_q, \beta l) + p_2[\min(s_q, l) - \min(s_q, \beta l)]\}\Big),$$

where $s_q = m + n - 2q + 1$ (see Fig. 7.7).

7.2.3 Inner Bound for Diversity Gain Region

We are now ready to derive a DGR inner bound. We summarize the result in the following theorem.

Theorem 7.1 *For a MIMO fading broadcast channel operated at a multiplexing gain pair (r_1, r_2) with m transmit antennas, n_1, n_2 receive antennas and block length l,*

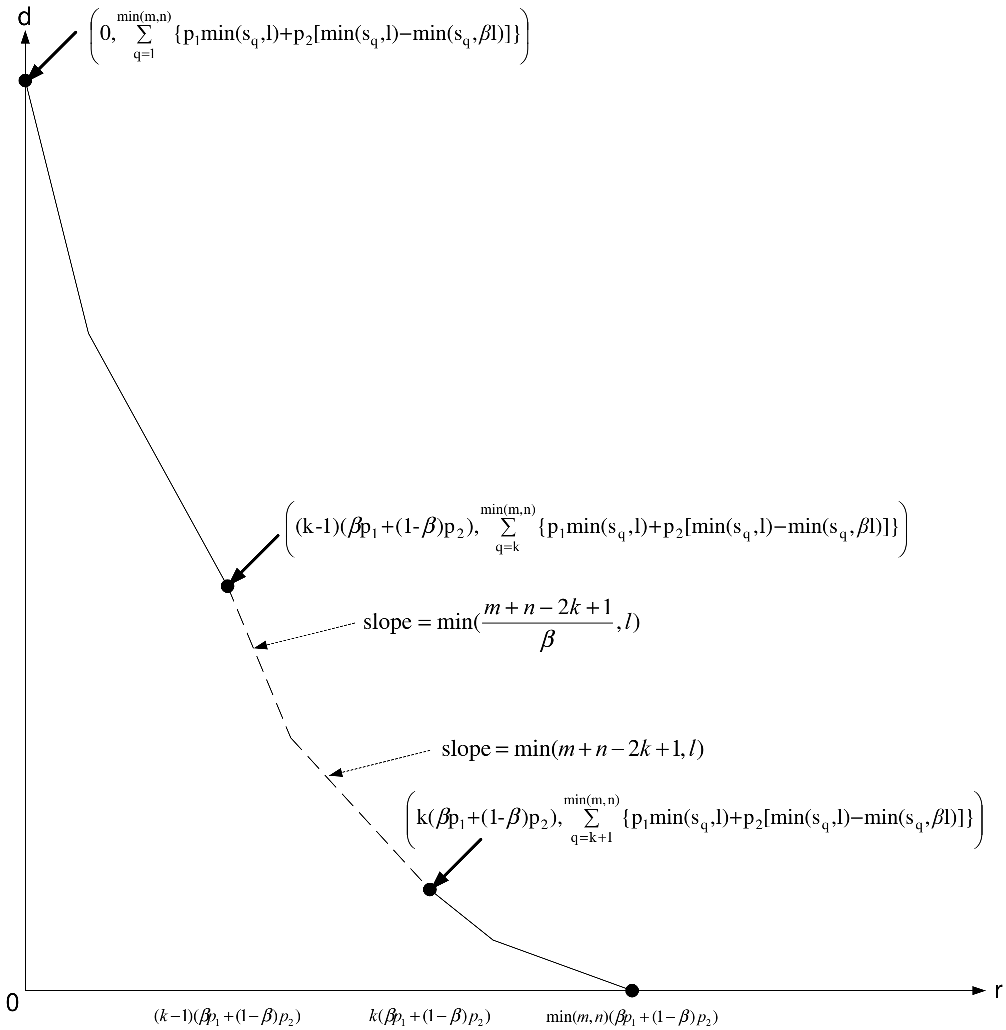


Figure 7.7: Nonuniform-power random coding diversity gain ($0 < \beta < 1$, $p_2 \leq p_1$).

an inner bound for DGR is given by

$$DGR(r_1,r_2) = \Bigg\{(d_1,d_2) : \beta = \frac{a}{l}, a \in \{0,1,\ldots,l\}, 0 \le p_1 \le 1, 0 \le p_2 \le 1,$$
$$d_1 \le \max\Big\{\min\big\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d_{m,n_1,l}(r_1+r_2)\big\}, d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta})\Big\}$$
$$d_2 \le \max\Big\{\min\big\{d^{np}_{m,n_2,l,p_2,1,\beta}(r_2), d_{m,n_2,l}(r_1+r_2)\big\}, d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta})\Big\}\Bigg\}, \tag{7.36}$$

where $d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta})$ should be interpreted as 0 for $\beta = 0$ and $d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta})$ should be interpreted as 0 for $\beta = 1$.

Proof: Let P_{e11} denote type 11 error probability, the probability that user 1 decodes (i,j) as $(\hat{i},j)$, and let P_{e13} denote type 13 error probability, the probability that user 1 decodes (i,j) as $(\hat{i},\hat{j})$, where $i \neq \hat{i}$ and $j \neq \hat{j}$. Similarly, let P_{e22} denote type 22 error probability, the probability that user 2 decodes (i,j) as $(i,\hat{j})$, and let P_{e23} denote type 23 error probability, the probability that user 2 decodes (i,j) as $(\hat{i},\hat{j})$. Applying the random coding argument, it can be shown that there exist codebooks for user 1 and user 2 using joint ML decoding such that

$$P_{e11} \le SNR^{-d^{np}_{m,n_1,l,1,p_1,\beta}(r_1)}$$
$$P_{e22} \le SNR^{-d^{np}_{m,n_2,l,p_2,1,\beta}(r_2)}$$
$$P_{e13} \le SNR^{-d_{m,n_1,l}(r_1+r_2)}$$
$$P_{e23} \le SNR^{-d_{m,n_2,l}(r_1+r_2)}. \tag{7.37}$$

The probabilities of error for user 1 and user 2 using joint ML decoding can be upper bounded by

$$\begin{aligned}
P_{e1} = P_{e11} + P_{e13} &\leq SNR^{-d^{np}_{m,n_1,l,1,p_1,\beta}(r_1)} + SNR^{-d_{m,n_1,l}(r_1+r_2)} \\
&\leq 2SNR^{-\min\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d_{m,n_1,l}(r_1+r_2)\}} \\
P_{e2} = P_{e22} + P_{e23} &\leq SNR^{-d^{np}_{m,n_2,l,p_2,1,\beta}(r_2)} + SNR^{-d_{m,n_2,l}(r_1+r_2)} \\
&\leq 2SNR^{-\min\{d^{np}_{m,n_2,l,p_2,1,\beta}(r_2), d_{m,n_2,l}(r_1+r_2)\}}. \qquad (7.38)
\end{aligned}$$

Thus the achievable diversity gains using joint ML decoding are

$$\begin{aligned}
d_1 &= \min\left\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d_{m,n_1,l}(r_1+r_2)\right\} \\
d_2 &= \min\left\{d^{np}_{m,n_2,l,p_2,1,\beta}(r_2), d_{m,n_2,l}(r_1+r_2)\right\}. \qquad (7.39)
\end{aligned}$$

When naive single-user decoding is utilized, the probability of error for user 1 can be upper bounded by

$$P_{e1} \leq SNR^{-d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta})}. \qquad (7.40)$$

Similarly, the probability of error for user 2 (using naive single-user decoding) can be upper bounded by

$$P_{e2} \leq SNR^{-d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta})}. \qquad (7.41)$$

Thus the achievable diversity gains using naive single-user decoding are

$$\begin{aligned}
d_1 &= d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta}) \\
d_2 &= d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta}). \qquad (7.42)
\end{aligned}$$

Since both users can choose either joint ML decoding or naive single-user decoding, the maximum of the corresponding diversity gains are achievable. This completes the proof. ■

Several comments are in order at this point.

- One might question the efficiency of superposition encoding when encoding over only one single block. This scheme is motivated by the fact that superposition encoding is a capacity achieving strategy for degraded broadcast channels [12, 13] (by encoding over sufficiently large N blocks). It should be clear that in the present setting we directly evaluate the corresponding error probabilities without using the property that the broadcast channel considered here is degraded.

- It is possible to improve the DGR inner bound for small block length l by expurgating codebooks and deriving a *nonuniform-power expurgated diversity gain*. The derivation of such improved bound is a straightforward extension of Theorem 7.1 and thus is omitted.

- Other encoding or decoding strategies can also be applied to provide upper bounds on the error probabilities of the two users. For instance, a decoding technique commonly used in the literature is successive cancellation. Diggavi *et al.* derived an achievable diversity gain pair for a single-user channel with two different messages (high- and low-reliability messages) using successive cancellation [23]. Our numerical results indicate that this decoding strategy does not improve the DGR inner bound given in Theorem 7.1. It can also be shown that the single-code encoding, used in Gaussian broadcast channels in Theorem 4.1, does not improve the DGR inner bound, either.

As illustrated in Fig. 7.8, the solid square is the boundary of the achievable DGR using uniform superposition and joint ML decoding, the dashed curve is the boundary of the achievable DGR using uniform superposition and a mixture of joint ML and naive single-user decoding (the dashed curve merging with the solid square at $(d_1, d_2) = (9, 4.5)$ and $(d_1, d_2) = (4.5, 9)$, and merging with the dash-dotted curve at $(d_1, d_2) = (12.5, 3.6)$ and $(d_1, d_2) = (3.6, 12.5)$), and the dotted curve is the boundary of the achievable DGR using on-off superposition. The dash-dotted curve in Fig. 7.8 is the boundary of the achievable DGR using superposition encoding, which provides a smooth transition between the achievable regions using uniform and on-off superposition.

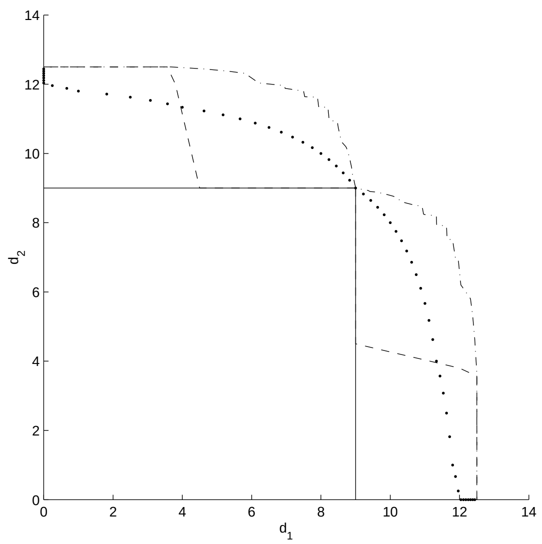


Figure 7.8: DGR inner bound using on-off superposition (dotted), uniform superposition with joint ML decoding (solid), uniform superposition with a mixture of joint ML and naive single-user decoding (dashed) and superposition (dash-dotted) for $m = 4, n_1 = 4, n_2 = 4, l = 60, r_1 = 0.5, r_2 = 0.5$.

7.3 Outer Bound for Diversity Gain Region

In this section, we derive a DGR outer bound valid for any transmission scheme over sufficiently large N blocks ($N \to \infty$), so this DGR outer bound is also valid for any scheme over finite blocks.

We summarize the result in the following theorem.

Theorem 7.2 *For a MIMO fading broadcast channel operated at a multiplexing gain pair* (r_1, r_2) *with* m *transmit antennas,* n_1*,* n_2 *receive antennas and block length* l*, an outer bound for DGR is given by*

$$d_1 \leq d_{m,n_1}^{out}(r_1) \tag{7.43a}$$

$$d_2 \leq d_{m,n_2}^{out}(r_2) \tag{7.43b}$$

$$\min\{d_1, d_2\} \leq \max\{d_{m,n_1}^{out}(r_1 + r_2), d_{m,n_2}^{out}(r_1 + r_2)\}. \tag{7.43c}$$

Proof: It is true that for any broadcast channel the probability of decoding error for user i can always be lower bounded by the probability of decoding error for user i operating over a point-to-point channel defined by the marginal distribution $P(Y_i|X)$. This implies that $d_i \leq d_{m,n_i}^{out}(r_i)$, for $i = 1, 2$.

Given an encoding and decoding scheme, it is true that

$$P_{e,sys} \leq P_{e1} + P_{e2} \leq 2\max\{P_{e1}, P_{e2}\}, \tag{7.44}$$

where the first inequality follows from the union bound. The broadcast channel considered here is a *stochastically* degraded broadcast channel [26, Chap. 14]. Since the performance of a broadcast channel depends only on the marginal distributions, we may further assume that the broadcast channel considered here is a *physically* degraded broadcast channel, i.e., $P(Y_1, Y_2|X) = P(Y_1|X)P(Y_2|Y_1)$ if $n_1 \geq n_2$. If we now allow the two receivers to cooperate, we have a single-user channel with n_1+n_2 receive antennas, whose probability of error, P_e', should be less than or equal to the probability of system error $P_{e,sys}$ in the original broadcast channel [27]. The probability of error P_e' of the new single-user channel can be lower bounded by

$$SNR^{-\max\{d^{out}_{m,n_1}(r_1+r_2),d^{out}_{m,n_2}(r_1+r_2)\}}\dot{\leq}P_e', \tag{7.45}$$

since the original broadcast channel is physically degraded. Combining (7.44) and (7.45), we have

$$SNR^{-\max\{d^{out}_{m,n_1}(r_1+r_2),d^{out}_{m,n_2}(r_1+r_2)\}}\dot{\leq}P_e' \leq P_{e,sys} \leq 2\max\{P_{e1},P_{e2}\}, \tag{7.46}$$

which implies that $\min\{d_1,d_2\} \leq \max\{d^{out}_{m,n_1}(r_1+r_2), d^{out}_{m,n_2}(r_1+r_2)\}$. ■

Note that the single-user diversity gain $d(N,r)$ is upper bounded by $d^{out}_{m,n}(r)$ for any N, so the DGR outer bound given here is valid for any N.

In Fig. 7.9, the derived inner and outer DGR bounds are shown for two channel scenarios. In this figure, the solid curve is the boundary of the DGR inner bound and the dash-dotted curve is the boundary of the DGR outer bound. Two important results are observed in Fig. 7.9: (i) the DGR inner and outer bounds are tight at the lower-right and the upper-left corners; (ii) for a symmetric MIMO fading broadcast channel, the DGR inner and outer bounds are tight at $d_1 = d_2$ (Fig. 7.9(a)). Result (i) implies that the appearance of the second user does not affect the first user (for a certain range of diversity gains for the second user) since the first user achieves the optimal single-user diversity gain $d^{out}_{m,n_1}(r_1)$ (and similarly for the second user). These results are formally expressed in the following theorem.

Theorem 7.3 *For a MIMO fading broadcast channel operated at a multiplexing gain* *pair* (r_1, r_2) *with* m *transmit antennas,* n_1*,* n_2 *receive antennas and block length* l*, the following are true:*

(a) If $r_1 < 1$ *and* $r_2 < (1 - r_1)(1 - \frac{m+n_1-1}{l}) \min(m, n_2)$*, then user 1 achieves the single-user diversity gain* $d^{out}_{m,n_1}(r_1)$*, and simultaneously user 2 achieves a diversity gain* $d_2 > 0$*. A similar result holds for user 2.*

(b) If the MIMO fading broadcast channel is symmetric $(n_1 = n_2)$ *and* $l \geq m + n_1 - 1$*, the DGR inner and outer bounds are tight at* $d_1 = d_2$*.*

Proof:

(a) Assume $r_1 < 1$. The tightness of the DGR inner and outer bounds at the lower-right corner is a direct consequence applying naive single-user decoding to user 1. From (7.23a), (7.23b), user 1 using uniform superposition encoding with $\beta = 1$ can achieve the single-user diversity gain $d^{out}_{m,n_1}(r_1)$ using naive single-user decoding as long as

$$\begin{aligned} p_2 &< 1 - \frac{m+n_1-1}{l} \\ r_1 &\leq \frac{(1-p_2)l - (m+n_1-1)}{l-(m+n_1-1)}. \end{aligned} \tag{7.47}$$

Choose $p_2^* = (1 - r_1)(1 - \frac{m+n_1-1}{l})$, then both conditions in (7.47) are satisfied. The achievable diversity gains for user 1 using naive single-user decoding and for user 2 using joint ML decoding are

$$d_1 = d^{ns}_{m,n_1,l,p_2^*}(r_1) = d^{out}_{m,n_1}(r_1) \tag{7.48a}$$

$$\begin{aligned} d_2 &= \min\{d^{np}_{m,n_2,l,p_2^*,1,\beta}(r_2), d_{m,n_2,l}(r_1 + r_2)\} \\ &= \min\{p_2^*\, d^{out}_{m,n_2}(\frac{r_2}{p_2^*}), d_{m,n_2,l}(r_1 + r_2)\}. \end{aligned} \tag{7.48b}$$

Note that the chosen p_2^* is the largest value of p_2's for which the equation $d^{ns}_{m,n_1,l,p_2}(r_1) = d^{out}_{m,n_1}(r_1)$ still holds, i.e., $d^{ns}_{m,n_1,l,p_2}(r_1) < d^{out}_{m,n_1}(r_1)$ for $p_2 > p_2^*$. Under the assumption $r_2 < p_2^* \min(m, n_2) = (1 - r_1)(1 - \frac{m+n_1-1}{l}) \min(m, n_2)$, both $\frac{r_2}{p_2^*}$ and $r_1 + r_2$ in (7.48b) are smaller than $\min(m, n_2)$, and thus $d_2 > 0$. The proof of this argument for user 2, with a multiplexing gain $r_2 < 1$, achieving the optimal single-user diversity gain $d^{out}_{m,n_2}(r_2)$ if $r_1 < (1 - r_2)(1 - \frac{m+n_2-1}{l}) \min(m, n_1)$ is similar.

(b) For a symmetric MIMO fading broadcast channel ($n_1 = n_2$), the achievable diversity gains using uniform superposition with $\beta = 1$, $p_2 = 1$ and joint ML decoding are

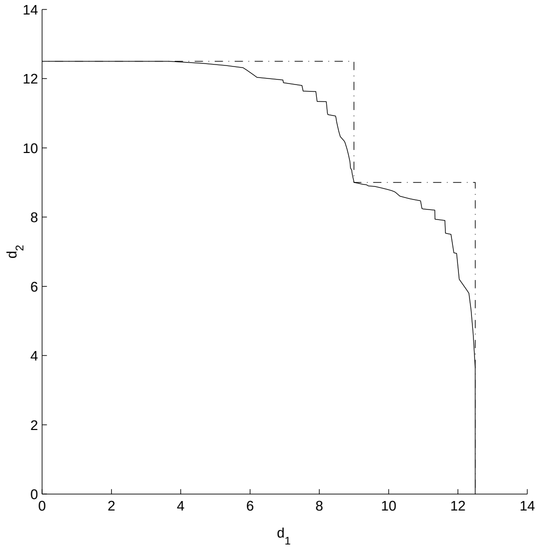
$$\begin{aligned}
d_1 &= \min\left\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d_{m,n_1,l}(r_1 + r_2)\right\} = \min\left\{d_{m,n_1,l}(r_1), d_{m,n_1,l}(r_1 + r_2)\right\} \\
&= d_{m,n_1,l}(r_1 + r_2) = d^{out}_{m,n_1}(r_1 + r_2) \\
d_2 &= \min\left\{d^{np}_{m,n_2,l,p_2,1,\beta}(r_2), d_{m,n_2,l}(r_1 + r_2)\right\} = \min\left\{d_{m,n_2,l}(r_2), d_{m,n_2,l}(r_1 + r_2)\right\} \\
&= d_{m,n_2,l}(r_1 + r_2) = d^{out}_{m,n_1}(r_1 + r_2). \qquad (7.49)
\end{aligned}$$

Comparing (7.49) with the DGR outer bound (7.43c), it is clear that the inner bound and bounds are tight at $d_1 = d_2$.

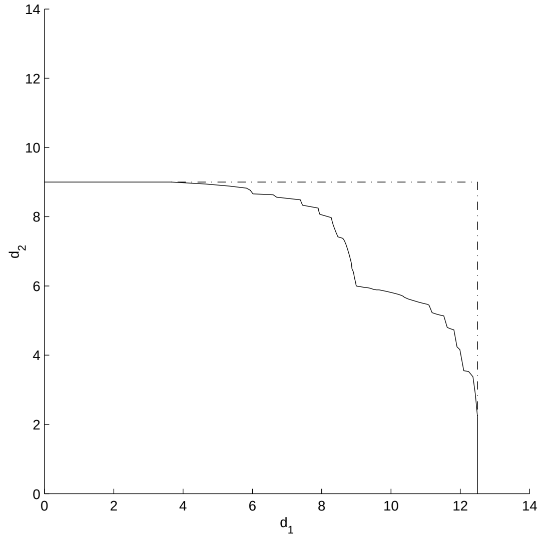
■

7.4 Multiplexing Gain Region

At last, we define the CCR counterpart in a MIMO fading broadcast channel, namely the multiplexing gain region (MGR), and derive an inner bound and an outer bound for the MGR.



(a)



(b)

Figure 7.9: DGR inner bound (solid) and outer bound (dash-dotted) for (a) $m = 4, n_1 = 4, n_2 = 4, l = 60, r_1 = 0.5, r_2 = 0.5$; (b) $m = 4, n_1 = 4, n_2 = 3, l = 55, r_1 = 0.5, r_2 = 0.5$.

Definition 7.2 *Consider a MIMO fading broadcast channel with coding over N blocks. The MGR is defined as the closure of the set of all achievable multiplexing gain pairs (r_1, r_2) with $d_1(N, r_1, r_2) > 0$ and $d_2(N, r_1, r_2) > 0$ for all encoding schemes.*

Note that in the above definition, the MGR is a function of N. We now derive an MGR inner bound and an MGR outer bound and summarize the result in the following theorem.

Theorem 7.4 *For a MIMO fading broadcast channel with m transmit antennas, n_1, n_2 receive antennas and block length l, an MGR inner bound MGR_{in} and an MGR* *outer bound MGR_{out} for any encoding scheme over N blocks are*

$$MGR_{in} = \Big\{(r_1, r_2) : \frac{r_1}{\min(m, n_1)} + \frac{r_2}{\min(m, n_2)} \leq 1\Big\} \tag{7.50}$$

$$\begin{aligned} MGR_{out} = \Big\{(r_1, r_2) :\ & 0 \leq \alpha \leq 1, \\ & r_1 \leq (\min(m, n_1) - \min(m, n_2))^+ + \alpha \min\big\{\min(m, n_1), \min(m, n_2)\big\}, \\ & r_2 \leq (\min(m, n_2) - \min(m, n_1))^+ + (1-\alpha) \min\big\{\min(m, n_1), \min(m, n_2)\big\}\Big\}. \end{aligned} \tag{7.51}$$

Proof: The inner bound is proved for $N = 1$ and the outer bound is proved for $N \to \infty$, so both bounds are valid for any N.

We first show that (7.50) is an inner bound for the MGR. Without loss of generality, we may assume $n_1 \geq n_2$. Applying uniform superposition with $\beta = 0$, we can achieve a diversity gain pair (d_1, d_2)

$$\begin{aligned} d_1 &= \min\big\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d_{m,n_1,l}(r_1 + r_2)\big\} = \min\big\{p_1 d_{m,n_1,l}(\frac{r_1}{p_1}), d_{m,n_1,l}(r_1 + r_2)\big\} \\ d_2 &= d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta}) = d^{ns}_{m,n_2,l,p_1}(r_2). \end{aligned} \tag{7.52}$$

Note that under the assumption $n_1 \geq n_2$, $d_{m,n_1,l}(r_1 + r_2) > 0$ for any interior point (r_1, r_2) of MGR_{in}, so $d_1 > 0$ as long as $d_{m,n_1,l}(\frac{r_1}{p_1}) > 0$. Since $d_{m,n_1,l}(\frac{r_1}{p_1}) > 0$ for $r_1 < \min(m, n_1)p_1$, and $d_2 = d^{ns}_{m,n_2,l,p_1}(r_2) > 0$ for $r_2 < \min(m, n_2)(1 - p_1)$, it is clear that by choosing p_1 a value between 0 and 1 we have $d_1 > 0$ and $d_2 > 0$ for any interior point of MGR_{in}.

It remains to show that (7.51) is an outer bound for the MGR. Let $\underline{\lambda}_1 = (\lambda_{1,1}\ \lambda_{1,2}\ \ldots\ \lambda_{1,\min(m,n_1)})$ and $\underline{\lambda}_2 = (\lambda_{2,1}\ \lambda_{2,2}\ \ldots\ \lambda_{2,\min(m,n_2)})$ denote the vectors of the nonzero eigenvalues of $\mathbf{H}_1'\mathbf{H}_1$ and $\mathbf{H}_2'\mathbf{H}_2$, where $\mathbf{H}_1'$ and $\mathbf{H}_2'$ are the conjugate transposes of $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. Applying singular value decomposition to $\mathbf{H}_1$

and $\mathbf{H}_2$, we can re-write the broadcast channel as

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{V}_1\mathbf{\Sigma}_1\mathbf{W}_1\mathbf{X} + \mathbf{Z}_1 \tag{7.53a}$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}}\mathbf{V}_2\mathbf{\Sigma}_2\mathbf{W}_2\mathbf{X} + \mathbf{Z}_2, \tag{7.53b}$$

where $\mathbf{\Sigma}_1 = [\sigma_{1,ij}]$ is an $n_1 \times m$ matrix with $\sigma_{1,ii} = \sqrt{\lambda_{1,i}}$ for $1 \le i \le \min(m, n_1)$ and $\sigma_{1,ij} = 0$ otherwise, and $\mathbf{\Sigma}_2 = [\sigma_{2,ij}]$ is an $n_2 \times m$ matrix with $\sigma_{2,ii} = \sqrt{\lambda_{2,i}}$ for $1 \le i \le \min(m, n_2)$ and $\sigma_{2,ij} = 0$. $\mathbf{V}_1$ and $\mathbf{V}_2$ are $n_1 \times n_1$ and $n_2 \times n_2$ unitary random matrices, and $\mathbf{W}_1$ and $\mathbf{W}_2$ are $m \times m$ unitary random matrices.

It is well-known that the capacity region of a broadcast channel depends only on the marginal distributions, so we can consider the capacity region of the following broadcast channel by replacing $\mathbf{W}_2$ with $\mathbf{W}_1$ in (7.53b)

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{V}_1\mathbf{\Sigma}_1\mathbf{W}_1\mathbf{X} + \mathbf{Z}_1 \tag{7.54a}$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}}\mathbf{V}_2\mathbf{\Sigma}_2\mathbf{W}_1\mathbf{X} + \mathbf{Z}_2. \tag{7.54b}$$

If we now assume that the matrix $\mathbf{W}_1$ is also known at the transmitter in the broadcast channel (7.54), we can consider the following equivalent broadcast channel

$$\mathbf{Y}_1 = \sqrt{\frac{SNR}{m}}\mathbf{\Sigma}_1\mathbf{X} + \mathbf{Z}_1 \tag{7.55a}$$

$$\mathbf{Y}_2 = \sqrt{\frac{SNR}{m}}\mathbf{\Sigma}_2\mathbf{X} + \mathbf{Z}_2, \tag{7.55b}$$

whose MGR is an outer bound for the MGR of the original broadcast channel.

For any encoding scheme over N blocks, define the following notations.

$\mathbf{U}_1$, $\mathbf{U}_2$: input messages for user 1 and user 2.

$\mathbf{X}^N$: $m \times lN$ channel input matrix. $\mathbf{X}^N = f(\mathbf{U}_1, \mathbf{U}_2)$ for some deterministic function $f(\cdot)$ depending on the encoding scheme.

$\mathbf{Y}_1^N$: $n_1 \times lN$ channel output matrix of user 1.

$\mathbf{H}_1^N$: $n_1 \times mN$ channel fading matrix of user 1.

$\mathbf{H}_{1,k}$: $n_1 \times m$ channel fading matrix of user 1 for the k^{th} block, i.e., $\mathbf{H}_1^N = [\mathbf{H}_{1,1}\ \mathbf{H}_{1,2}\ \ldots\ \mathbf{H}_{1,N}]$.

$\mathbf{\Sigma}_{1,k}$: $n_1 \times m$ channel singular value matrix of user 1 for the k^{th} block $\mathbf{H}_{1,k}$.

$\mathbf{\Sigma}_1^N$: $\mathbf{\Sigma}_1^N = [\mathbf{\Sigma}_{1,1}\ \mathbf{\Sigma}_{1,2}\ \ldots\ \mathbf{\Sigma}_{1,N}]$.

$\underline{\lambda}_{1,k}$: $1 \times \min(m, n_1)$ vector of the nonzero eigenvalues of $\mathbf{H}'_{1,k}\mathbf{H}_{1,k}$, where $\mathbf{H}'_{1,k}$ is the conjugate transpose of $\mathbf{H}_{1,k}$.

$\underline{\lambda}_1^N$: $\underline{\lambda}_1^N = [\underline{\lambda}_{1,1}\ \underline{\lambda}_{1,2}\ \ldots\ \underline{\lambda}_{1,N}]$.

$\mathcal{A} \triangleq [SNR^{-\epsilon}\ \ SNR^{\epsilon}]^{\min(m,n_1)}$: $\min(m, n_1)^{th}$-fold Cartesian product of the interval $[SNR^{-\epsilon}\ \ SNR^{\epsilon}]$.

$\mathcal{A}^N$: N^{th}-fold Cartesian product of $\mathcal{A}$, i.e., $\mathcal{A}^N = [SNR^{-\epsilon}\ \ SNR^{\epsilon}]^{\min(m,n_1)N}$.

Since $\mathbf{U}_1$ and $\mathbf{\Sigma}_1^N$ are independent, the achievable rate R_1 for user 1 can be written as

$$
\begin{aligned}
R_1 &= \frac{1}{N} I(\mathbf{U}_1; \mathbf{Y}_1^N, \mathbf{\Sigma}_1^N) && \text{(7.56a)}\\
&= \frac{1}{N} I(\mathbf{U}_1; \mathbf{Y}_1^N | \mathbf{\Sigma}_1^N) && \text{(7.56b)}\\
&= \frac{1}{N} \int_{\underline{\lambda}_1^N \in \mathcal{A}^N} I(\mathbf{U}_1; \mathbf{Y}_1^N | \mathbf{\Sigma}_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N + \\
&\qquad \frac{1}{N} \int_{\underline{\lambda}_1^N \in (\mathcal{A}^N)^c} I(\mathbf{U}_1; \mathbf{Y}_1^N | \mathbf{\Sigma}_1^N = \Sigma_1^N) P(\Sigma_1^N) d\Sigma_1^N. && \text{(7.56c)}
\end{aligned}
$$

The second term in (7.56c) can be upper bounded by

$$
\begin{aligned}
\frac{1}{N}\int_{\underline{\lambda}_1^N\in(\mathcal{A}^N)^c} & I(\mathbf{U}_1;\mathbf{Y}_1^N|\mathbf{\Sigma}_1^N=\Sigma_1^N)P(\Sigma_1^N)d\Sigma_1^N \\
&\le \frac{1}{N}\int_{\underline{\lambda}_1^N\in(\mathcal{A}^N)^c} I(\mathbf{X}^N;\mathbf{Y}_1^N|\mathbf{\Sigma}_1^N=\Sigma_1^N)P(\Sigma_1^N)d\Sigma_1^N \\
&\le \int_{\underline{\lambda}_1\in\mathcal{A}^c} \log|I+\frac{SNR}{m}\Sigma_1'\Sigma_1|P(\Sigma_1)d\Sigma_1 \\
&= \int_{\underline{\lambda}_1\in\mathcal{A}^c} \sum_{i=1}^{\min\{m,n_1\}} \log(1+\frac{SNR}{m}\lambda_{1,i})P(\underline{\lambda}_1)d\underline{\lambda}_1, \qquad (7.57)
\end{aligned}
$$

where the first inequality in (7.57) is due to $\mathbf{U}_1-\mathbf{X}^N-\mathbf{Y}_1^N$ forming a Markov chain and the second inequality in (7.57) is based on the fact that $I(\mathbf{X}^N;\mathbf{Y}_1^N|\mathbf{\Sigma}_1^N=\Sigma_1^N)$ is maximized when $\mathbf{X}^N$ has i.i.d. complex Gaussian entries. Note that $P(\underline{\lambda}_1)$ goes to zero exponentially for large $\underline{\lambda}_1$ but the other component of the integrand in (7.57) is in the order of $\log(SNR)$, so the last term in (7.57) goes to zero as $SNR\to\infty$. Thus R_1 is equal to the first term in (7.56c) at high SNR. In other words, regarding the data rate for user 1, we may assume that $\underline{\lambda}_1^N\in\mathcal{A}^N$. Let $I_1=[I_{1,ij}]$ be an $n_1\times m$ matrix with $I_{1,ii}=1$ for $1\le i\le\min(m,n_1)$ and $I_{1,ij}=0$ otherwise, and let $I_2=[I_{2,ij}]$ be an $n_2\times m$ matrix with $I_{2,ii}=1$ for $1\le i\le\min(m,n_2)$ and $I_{2,ij}=0$ otherwise. Since $I(\mathbf{U}_1;\mathbf{Y}_1^N|\mathbf{\Sigma}_1^N=SNR^{-\epsilon}I_1)\le I(\mathbf{U}_1;\mathbf{Y}_1^N|\mathbf{\Sigma}_1^N=\Sigma_1^N)\le I(\mathbf{U}_1;\mathbf{Y}_1^N|\mathbf{\Sigma}_1^N=SNR^{\epsilon}I_1)$ for $\underline{\lambda}_1^N\in\mathcal{A}^N$ and ϵ is any positive constant, we can take ϵ arbitrarily small and on the scale of our interest (multiplexing gain) we may assume that $\underline{\lambda}_1$ is a vector with every entry equal to one. A similar argument also applies to user 2. Therefore, regarding multiplexing gains, we can consider the following broadcast channel

$$
\begin{aligned}
\mathbf{Y}_1 &= \sqrt{\frac{SNR}{m}}I_1\mathbf{X}+\mathbf{Z}_1 \\
\mathbf{Y}_2 &= \sqrt{\frac{SNR}{m}}I_2\mathbf{X}+\mathbf{Z}_2. \qquad (7.58)
\end{aligned}
$$

The MGR of the broadcast channel defined in (7.58) is an outer bound of the MGR of the original fading broadcast channel. However, it is easy to see that the MGR of the broadcast channel defined in (7.58) is exactly (7.51). This completes the proof. ■

Note that the inner bound (7.50) is derived using uniform superposition. It can also be shown that the achievable MGR derived using (nonuniform) superposition is the same region as given in (7.50). Again, without loss of generality, we may assume $n_1 \geq n_2$. Since $d_{m,n_2,l}(r_1+r_2) = 0$ for $r_1+r_2 > \min(m, n_2)$, it's sufficient to consider only naive single-user decoding for user 2 because not all the interior points of MGR_{in} are achievable using joint ML decoding for user 2 when $\min(m, n_1) > \min(m, n_2)$ ($d_2 \leq d_{m,n_2,l}(r_1 + r_2) = 0$ for some interior points in this case). The following diversity gains

$$\begin{aligned} d_1 &= \min\left\{d^{np}_{m,n_1,l,1,p_1,\beta}(r_1), d_{m,n_1,l}(r_1+r_2)\right\} \\ d_2 &= d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta}) \end{aligned} \tag{7.59}$$

are achievable, where we assume using joint ML decoding for user 1. Note that under the assumption $n_1 \geq n_2$, $d_{m,n_1,l}(r_1+r_2) > 0$ for any interior point (r_1, r_2) of MGR_{in}, so $d_1 > 0$ as long as $d^{np}_{m,n_1,l,1,p_1,\beta}(r_1) > 0$. Since

$$d^{np}_{m,n_1,l,1,p_1,\beta}(r_1) > 0 \qquad \text{for} \quad r_1 < \min(m, n_1)(\beta + (1-\beta)p_1) \tag{7.60a}$$

$$d^{ns}_{m,n_2,(1-\beta)l,p_1}(\frac{r_2}{1-\beta}) > 0 \qquad \text{for} \quad r_2 < \min(m, n_2)(1-\beta)(1-p_1), \tag{7.60b}$$

it is clear that by choosing appropriate p_1 and β we have $d_1 > 0$ and $d_2 > 0$ for any interior point of MGR_{in}, and $d_1 = 0$ or $d_2 = 0$ for any point outside MGR_{in}. Finally, note that $d^{ns}_{m,n_1,\beta l,p_2}(\frac{r_1}{\beta}) > 0$ only for $r_1 < \min(m, n_1)\beta(1-p_2)$, which is less than $\min(m, n_1)(\beta + (1-\beta)p_1)$ in (7.60a), so it is sufficient to consider only joint

ML decoding for user 1 in the above derivation.

Examples of typical MGR inner and outer bounds are shown in Fig. 7.10. Although the MGR might be a function of N, the inner bound (derived for $N = 1$) and the outer bound (derived for $N \to \infty$) given in Theorem 7.4 are valid for any N.

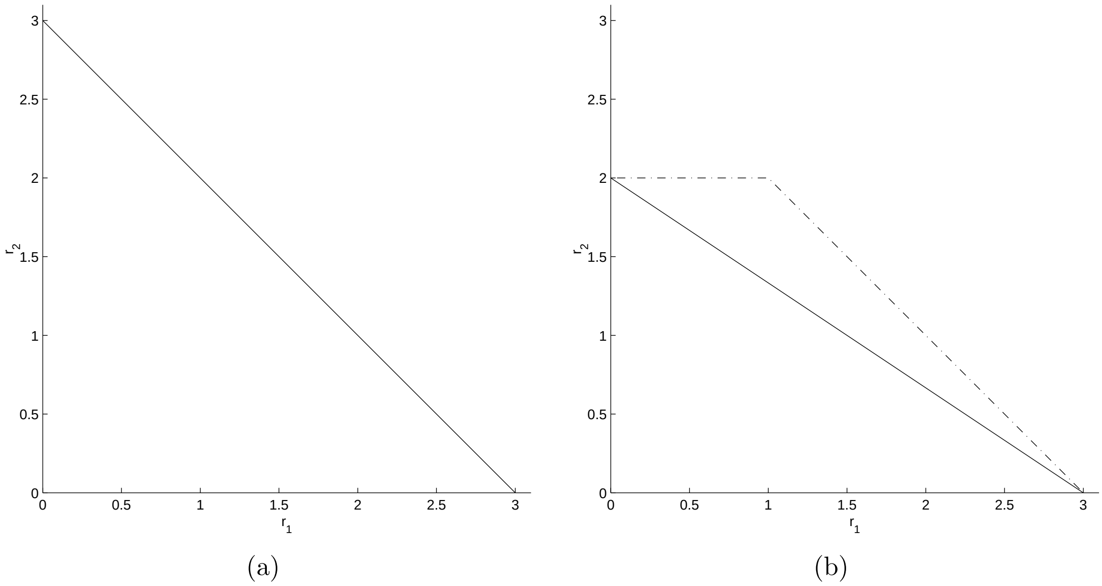


Figure 7.10: MGR inner bound (solid) and outer bound (dash-dotted) for (a) $m = 3$, $n_1 = 4$, $n_2 = 4$ (solid = dash-dotted); (b) $m = 3$, $n_1 = 3$, $n_2 = 2$.

CHAPTER 8

Diversity Gain Regions for MIMO Fading Multiple Access Channels

Consider a MIMO fading multiple access channel with m_1, m_2 transmit antennas for user 1 and user 2, respectively and n receive antennas. The channel model is given by

$$\mathbf{Y} = \sqrt{\frac{SNR}{m_1}}\mathbf{H}_1\mathbf{X}_1 + \sqrt{\frac{SNR}{m_2}}\mathbf{H}_2\mathbf{X}_2 + \mathbf{Z}. \tag{8.1}$$

The channel fading matrices between transmitter 1, transmitter 2 and the receiver are represented by an $n \times m_1$ matrix $\mathbf{H}_1$ and an $n \times m_2$ matrix $\mathbf{H}_2$, respectively. We assume that $\mathbf{H}_1$ and $\mathbf{H}_2$ remain constant over a block with length l, and change to a new independent realization in the next block. $\mathbf{H}_1$ and $\mathbf{H}_2$ have i.i.d. entries and each entry is distributed as $\mathcal{CN}(0,1)$. We assume that the fading matrices are known by the receiver but not known by the transmitters. The channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ are $m_1 \times l$ and $m_2 \times l$ matrices, respectively, and are normalized such that the average power at each transmit antenna is 1, which means that the average power of each user's signal at each receive antenna is SNR. The noise $\mathbf{Z}$ is an $n \times l$ matrix with i.i.d. entries distributed as $\mathcal{CN}(0,1)$. The channel output $\mathbf{Y}$ is an $n \times l$ matrix.

Similar to a MIMO fading broadcast channel, an encoding scheme for a MIMO fading multiple access channel over N blocks is said to achieve a multiplexing gain pair (r_1, r_2) and a diversity gain pair (d_1, d_2) if

$$\begin{aligned}\lim_{SNR\to\infty} \frac{R_1(SNR)}{\log SNR} &= r_1 \\ \lim_{SNR\to\infty} \frac{R_2(SNR)}{\log SNR} &= r_2,\end{aligned} \tag{8.2}$$

where $R_1(SNR)$ and $R_2(SNR)$ are the rates for user 1 and user 2, respectively, and

$$\begin{aligned}d_1(N, r_1, r_2) &\triangleq \lim_{SNR\to\infty} -\frac{\log P_{e1}(N, R_1(SNR), R_2(SNR))}{N \log SNR} \\ d_2(N, r_1, r_2) &\triangleq \lim_{SNR\to\infty} -\frac{\log P_{e2}(N, R_1(SNR), R_2(SNR))}{N \log SNR},\end{aligned} \tag{8.3}$$

where $P_{e1}(N, R_1(SNR), R_2(SNR))$ and $P_{e2}(N, R_1(SNR), R_2(SNR))$ are the probabilities of error for user 1 and user 2, respectively. We use $d_1(r_1, r_2)$ and $d_2(r_1, r_2)$ as shorthand notations for $d_1(1, r_1, r_2)$ and $d_2(1, r_1, r_2)$, respectively. Given a multiplexing gain pair (r_1, r_2), the DGR is defined as the set of all achievable diversity gain pairs $(d_1(N, r_1, r_2), d_2(N, r_1, r_2))$ for all encoding schemes. Similarly, the MGR is defined as the closure of the set of all multiplexing gain pairs (r_1, r_2) with $d_1(N, r_1, r_2) > 0$ and $d_2(N, r_1, r_2) > 0$ for all encoding schemes. The MGR for a MIMO fading multiple access channel with m_1, m_2 transmit antennas and n receive antennas is

$$\begin{aligned}r_1 &\leq \min(m_1, n) \\ r_2 &\leq \min(m_2, n) \\ r_1 + r_2 &\leq \min(m_1 + m_2, n).\end{aligned} \tag{8.4}$$

The achievability of (8.4) for $N = 1$ is given in [19]. The proof that (8.4) is also an outer bound for sufficiently large N ($N \to \infty$) can be derived easily from the channel capacity for a general multiple access channel [26, Chap. 14], and the channel capacity for a MIMO fading single-user channel [20].

8.1 Inner Bound for Diversity Gain Region

As in a MIMO fading broadcast channel, an inner bound is developed for $N = 1$, and thus is valid for any value of N.

Motivated by (nonuniform) superposition encoding in a MIMO fading broadcast channel, it is reasonable to consider a similar technique for a MIMO fading multiple access channel. The technique that enables an efficient bound using superposition strategy in a MIMO fading broadcast channel is based on a combination of joint ML and naive single-user decoding. However, this bounding technique does not provide a tight bound in the case of multiple access channels. To see this, recall that in a MIMO fading broadcast channel, when the intended signal of one of the two users suffers from the channel fading, the side interference (from the other user) also suffers from *the same* channel fading. In a MIMO fading multiple access channel, on the other hand, the intended signal and the side interference go through *different* and *independent* fading paths. This implies that sometimes the intended user's signal might suffer deep fading but the side interference is "unfaded" at the channel output. Thus naive single-user decoding might work poorly in a MIMO fading multiple access channel.

Using our bounding techniques, it is observed that (nonuniform) superposition using joint ML decoding does not provide an advantage over uniform and on-off superposition. Therefore, due to lack of an effective decoding strategy to close the gap between individual and joint ML decoding, we only derive an inner bound using two special cases of superposition encoding - uniform superposition and on-off superposition. We summarize the result in the following theorem.

Theorem 8.1 *For a MIMO fading multiple access channel operated at a multiplexing gain pair* (r_1, r_2) *with* m_1, m_2 *transmit antennas,* n *receive antennas and block length* l*, an inner bound for DGR is* $DGR_{us}(r_1, r_2) \cup DGR_{os}(r_1, r_2)$*, where* $DGR_{us}(r_1, r_2)$ *and* $DGR_{os}(r_1, r_2)$ *are given by*

$$\begin{aligned} DGR_{us}(r_1, r_2) = \Big\{ &(d_1, d_2) : \\ &d_1 \leq \min\big\{ \max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\},\ d_{m_1+m_2,n,l}(r_1+r_2)\big\} \\ &d_2 \leq \min\big\{ \max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\},\ d_{m_1+m_2,n,l}(r_1+r_2)\big\}\Big\} \end{aligned} \tag{8.5}$$

$$\begin{aligned} DGR_{os}(r_1, r_2) = \Big\{ &(d_1, d_2) : \beta = \frac{a}{l}, a \in \{1, 2, \ldots, l-1\}, \\ &d_1 \leq \max\big\{d_{m_1,n,\beta l}(\frac{r_1}{\beta}), d^{ex}_{m_1,n,\beta l}(\frac{r_1}{\beta})\big\} \\ &d_2 \leq \max\big\{d_{m_2,n,(1-\beta)l}(\frac{r_2}{1-\beta}), d^{ex}_{m_2,n,(1-\beta)l}(\frac{r_2}{1-\beta})\big\}\Big\}. \end{aligned} \tag{8.6}$$

Proof: In uniform superposition encoding, the channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ have i.i.d. entries $\mathcal{CN}(0, 1)$. The bounds on the users' error probabilities are obtained using joint ML decoding. Following [17], we define three types of error events, type 1, type 2, and type 3. It can be shown by using random coding arguments that there exist codebooks for user 1 and user 2 such that

$$\begin{aligned} P_{e,t1} &\leq SNR^{-\max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\}} \\ P_{e,t2} &\leq SNR^{-\max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\}} \\ P_{e,t3} &\leq SNR^{-d_{m_1+m_2,n,l}(r_1+r_2)}. \end{aligned} \tag{8.7}$$

The probabilities of error for user 1 and user 2 can be upper bounded by

$$
\begin{aligned}
P_{e1} = P_{e,t1} + P_{e,t3} &\leq SNR^{-\max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\}} + SNR^{-d_{m_1+m_2,n,l}(r_1+r_2)} \\
&\leq 2SNR^{-\min\{\max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\},\ d_{m_1+m_2,n,l}(r_1+r_2)\}} \\
P_{e2} = P_{e,t2} + P_{e,t3} &\leq SNR^{-\max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\}} + SNR^{-d_{m_1+m_2,n,l}(r_1+r_2)} \\
&\leq 2SNR^{-\min\{\max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\},\ d_{m_1+m_2,n,l}(r_1+r_2)\}}. \qquad (8.8)
\end{aligned}
$$

Thus the achievable diversity gains derived from joint ML decoding are

$$d_1^{us} = \min\big\{ \max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\},\ d_{m_1+m_2,n,l}(r_1+r_2)\big\} \qquad (8.9a)$$

$$d_2^{us} = \min\big\{ \max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\},\ d_{m_1+m_2,n,l}(r_1+r_2)\big\}, \qquad (8.9b)$$

where the superscript "us" denotes uniform superposition.

In on-off superposition, the channel input $\mathbf{X}_1$ has i.i.d. entries $\mathcal{CN}(0,1)$ for the first βl transmissions (on) and is zero for the remaining $(1-\beta)l$ transmissions (off), where $\beta = \frac{a}{l}$ for some $a \in \{1, 2, \ldots, l-1\}$. Similarly, the channel input $\mathbf{X}_2$ is zero for the first βl transmissions (off) and has i.i.d. entries $\mathcal{CN}(0,1)$ for the remaining $(1-\beta)l$ transmissions (on). We can derive achievable diversity gains for this strategy as

$$d_1^{os} = \max\big\{d_{m_1,n,\beta l}(\frac{r_1}{\beta}), d^{ex}_{m_1,n,\beta l}(\frac{r_1}{\beta})\big\} \qquad (8.10a)$$

$$d_2^{os} = \max\big\{d_{m_2,n,(1-\beta)l}(\frac{r_2}{1-\beta}), d^{ex}_{m_2,n,(1-\beta)l}(\frac{r_2}{1-\beta})\big\}, \qquad (8.10b)$$

where the superscript "os" denotes on-off superposition. ■

In Fig. 8.1(a) we depict the achievable DGR using on-off superposition. In this figure, each point is a diversity gain pair corresponding to one particular $\beta = \frac{a}{l}$ in (8.10), and the achievable DGR using on-off superposition is shown in Fig. 8.1(b).

Note that the boundary in Fig. 8.1(b) is not simply a smooth curve connecting the corresponding achievable diversity gain pairs in Fig. 8.1(a). This is because the value of βl assigned to user 1 must be an integer and each consecutive point in Fig. 8.1(a) corresponds to one consecutive integer.

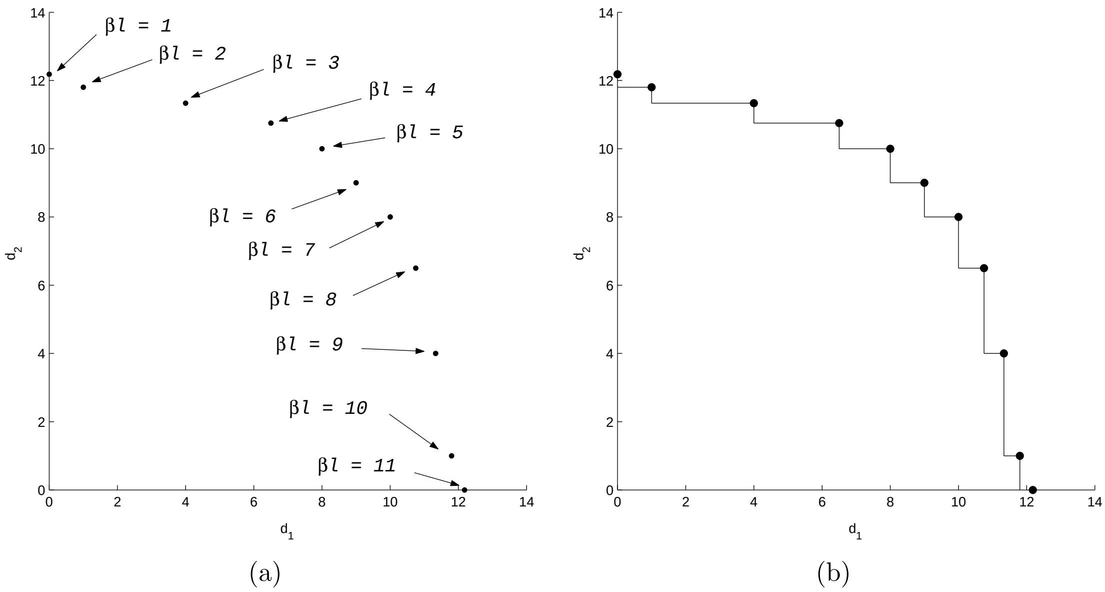


Figure 8.1: On-off superposition for $m_1 = 4, m_2 = 4, n = 4, l = 12, r_1 = 0.5, r_2 = 0.5$ (a) achievable diversity gain pairs; (b) DGR inner bound.

Based on Theorem 8.1, the MGR of a MIMO fading multiple access channel can be divided into four regions r_{12}, r_{13}, r_{23}, and r_3

$$\begin{aligned}
r_{12} &\triangleq \big\{(r_1, r_2):\ \max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\} \le d_{m_1+m_2,n,l}(r_1+r_2), \\
&\qquad \max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\} \le d_{m_1+m_2,n,l}(r_1+r_2)\big\} \\
r_{13} &\triangleq \big\{(r_1, r_2):\ \max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\} \le d_{m_1+m_2,n,l}(r_1+r_2), \\
&\qquad d_{m_1+m_2,n,l}(r_1+r_2) \le \max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\}\big\} \\
r_{23} &\triangleq \big\{(r_1, r_2):\ \max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\} \le d_{m_1+m_2,n,l}(r_1+r_2), \\
&\qquad d_{m_1+m_2,n,l}(r_1+r_2) \le \max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\}\big\} \\
r_3 &\triangleq \big\{(r_1, r_2): d_{m_1+m_2,n,l}(r_1+r_2) \le \max\{d_{m_1,n,l}(r_1), d^{ex}_{m_1,n,l}(r_1)\}, \\
&\qquad d_{m_1+m_2,n,l}(r_1+r_2) \le \max\{d_{m_2,n,l}(r_2), d^{ex}_{m_2,n,l}(r_2)\}\big\},
\end{aligned} \tag{8.11}$$

depending on whether the bound for type 1 error, type 2 error, or type 3 error dominates when using uniform superposition (see an illustration in Fig. 8.2). In region r_{12}, each user attains the optimal single-user diversity gain, and thus there is no diversity gain tradeoff between the users. In region r_{13}, the first user achieves the optimal single-user diversity gain, while the second user's error probability is dominated by type 3 error. Thus it is possible to provide a tradeoff of diversity gains between the users using on-off superposition. In Fig. 8.3 we show the DGRs for two channel scenarios. Observe that in Fig. 8.3(a), user 2's diversity gain d_2 can go beyond $d_2^{us} = 5$ by reducing user 1's diversity gain, where the solid curve is the boundary of the achievable DGR using uniform superposition, and the dotted curve is the boundary of the achievable DGR using on-off superposition. A similar result also holds for region r_{23}. In region r_3, type 3 error is dominant over both type 1 and type 2 errors, so it is possible to provide a tradeoff of diversity gains between the two users. This is illustrated in Fig. 8.3(b).

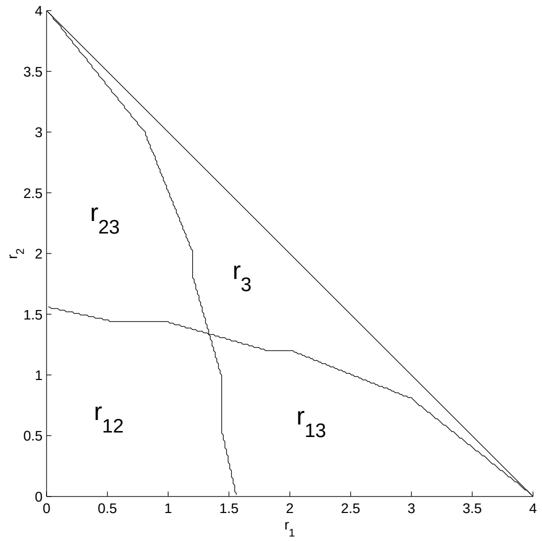


Figure 8.2: MGR for $m_1 = 4, m_2 = 4, n = 4$.

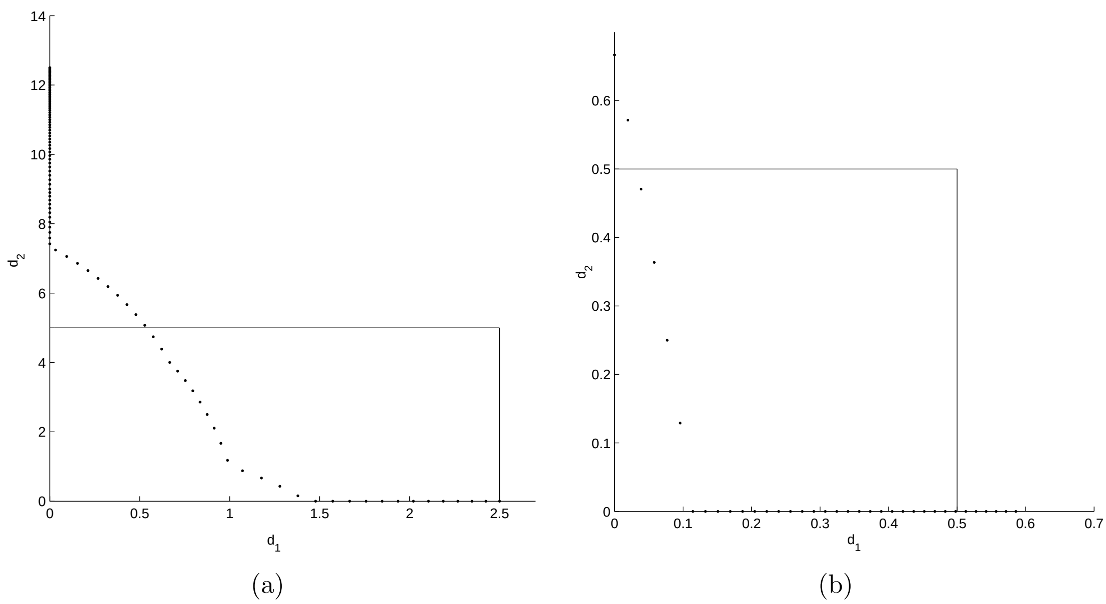


Figure 8.3: DGR inner bound using on-off superposition (dotted) and uniform superposition (solid) for (a) $m_1 = 4, m_2 = 4, n = 4, l = 60, r_1 = 2.5, r_2 = 0.5$ ($(r_1, r_2) \in$ region r_{13}); (b) $m_1 = 4, m_2 = 4, n = 4, l = 240, r_1 = 3.4, r_2 = 0.5$ ($(r_1, r_2) \in$ region r_3).

8.2 Outer Bound for Diversity Gain Region

As in a MIMO fading broadcast channel, an outer bound is developed for sufficiently large N ($N \to \infty$), and thus is valid for any finite value of N.

For a MIMO fading multiple access channel, the probabilities of decoding error for user 1 and user 2 can always be lower bounded by the probabilities of decoding error for user 1 and user 2 operating over the point-to-point channels defined by $\mathbf{Y} = \sqrt{\frac{SNR}{m_i}}\mathbf{H}_i\mathbf{X}_i + \mathbf{Z}$, for $i = 1, 2$. Furthermore, if we allow the two transmitters in the MIMO fading multiple access channel to cooperate, we have a MIMO fading single-user channel with $m_1 + m_2$ transmit antennas and n receive antennas, whose probability of error (using an optimal receiver) should be less than or equal to the probability of system error in the original multiple access channel. Note that the diversity gain corresponding to the probability of system error is equal to $\min\{d_1, d_2\}$. Collecting all these ideas, we have the following outer bound for the DGR

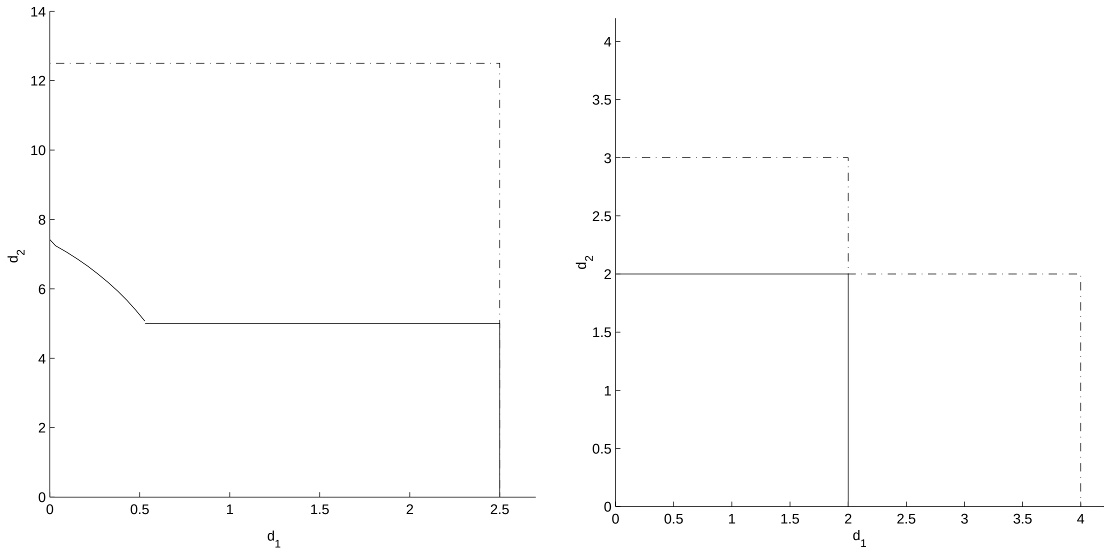
$$d_1 \leq d^{out}_{m_1,n}(r_1) \tag{8.12a}$$

$$d_2 \leq d^{out}_{m_2,n}(r_2) \tag{8.12b}$$

$$\min\{d_1, d_2\} \leq d^{out}_{m_1+m_2,n}(r_1 + r_2). \tag{8.12c}$$

If the block length $l \geq m_1 + m_2 + n - 1$, we can conclude from (8.9a), (8.9b), (8.12a), (8.12b) that all the operating points inside the region r_{12} in Fig. 8.2 have tight DGR inner and outer bounds.

In Fig. 8.4, two examples are considered for illustrating the derived inner and outer bounds. The solid curve is the boundary of the achievable DGR, and the dashed-dotted curve is the DGR outer bound. It is noted that in these particular examples, it is not clear whether a tradeoff between the optimal diversity gains of the two users exists. Furthermore, it is noted in Fig. 8.4(b) that the inner bound achieved by uniform superposition completely contains that achieved by on-off superposition.



(a) (b)

Figure 8.4: DGR inner bound (solid) and outer bound (dash-dotted) for (a) $m_1 = 4, m_2 = 4, n = 4, l = 60, r_1 = 2.5, r_2 = 0.5$; (b) $m_1 = 5, m_2 = 2, n = 4, l = 55, r_1 = 2.5, r_2 = 1$.

CHAPTER 9

Future Directions

In this work, we only consider two-user broadcast and multiple access channels. One natural generalization is to extend these results to cases with more than two users. For example, suppose there are three users in a broadcast channel. User 1 can decode his/her own message using joint ML decoding, or decode his/her own message regarding the other two messages as noise, or decode his/her own message regarding only user 3's message as noise, etc. The achievable region is a union of all the above decoding strategies.

Another possible direction is to extend our results to other multi-terminal channels, for example, a relay channel [31]. A relay channel combines a broadcast channel and a multiple access channel. The capacity is known for the special case of the physically degraded relay channel. One open problem is to find the error exponent for a relay channel. It is interesting to derive new error exponent inner and outer bounds for a relay channel based on our existing results in broadcast and multiple access channels.

Finally, we consider the problem of error exponent regions for discrete memoryless multi-user channels. In contrast to Gaussian or MIMO fading channels, where at least one capacity achieving input distribution is known, e.g. Gaussian distribution, there is no fixed input distribution to achieve capacity in discrete memoryless channels. For a channel with small input and output alphabets, a brute-force search for the optimum input distribution might be possible. As the sizes of the input and output alphabets increase, the optimum input distribution might be difficult to find.

The problem becomes even more complicated in discrete memoryless multi-user channels when EERs are considered. In the following, we derive an EER outer bound for discrete memoryless multiple access channels (DMMAC). Consider a DMMAC with channel inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ for user 1 and user 2, respectively, and channel output $\mathbf{Y}$. The channel transition matrix Q is given by $Q_{k|ij} = P(\mathbf{Y} = k|\mathbf{X}_1 = i, \mathbf{X}_2 = j)$, where $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $k \in \mathcal{K}$, and $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ are the input alphabets (with finite elements) for user 1 and user 2 and the output alphabet (with finite elements).

An EER outer bound for the DMMAC is summarized in the following theorem.

Theorem 9.1 *For a discrete memoryless multiple access channel with transition matrix Q and data rates R_1 and R_2 for user 1 and user 2, respectively, an outer bound for EER is*

$$
\begin{aligned}
EER(R_1, R_2) = \Big\{(E_1, E_2) : & \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} = 1 \\
E_1 \le \max_{\tilde{s}\ge 0} \min_{\tilde{q}_{k|j}} & \Big[-\tilde{s}R_1 - (1+\tilde{s}) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log\big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+\tilde{s}}} \tilde{q}_{k|j}^{\frac{\tilde{s}}{1+\tilde{s}}}\big)\Big] \\
E_2 \le \max_{\hat{s}\ge 0} \min_{\hat{q}_{k|i}} & \Big[-\hat{s}R_2 - (1+\hat{s}) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log\big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+\hat{s}}} \hat{q}_{k|i}^{\frac{\hat{s}}{1+\hat{s}}}\big)\Big] \\
\min\{E_1, E_2\} \le \max_{s\ge 0} \min_{q_k} & \Big[-s(R_1+R_2) - (1+s) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log\big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k}^{\frac{s}{1+s}}\big)\Big]\Big\},
\end{aligned}
\tag{9.1}
$$

where the minimizations are over all the (conditional) probability distributions $\tilde{q}$, $\hat{q}$ and q, i.e., $\sum_{k\in\mathcal{K}} \tilde{q}_{k|j} = 1$ for $\forall j \in \mathcal{J}$, $\sum_{k\in\mathcal{K}} \hat{q}_{k|i} = 1$ for $\forall i \in \mathcal{I}$, and $\sum_{k\in\mathcal{K}} q_k = 1$.

Proof: The proof is given in Appendix B. ■

For illustration, consider a DMMAC with input alphabets $\mathcal{I} = \{0, 1\}$, $\mathcal{J} = \{0, 1\}$, output alphabet $\mathcal{K} = \{0, 1\}$, and channel transition matrix Q given by

$$Q_{0|00} = 0.80,\ Q_{0|01} = 0.05,\ Q_{0|10} = 0.10,\ Q_{0|11} = 0.85,$$
$$Q_{1|00} = 0.20,\ Q_{1|01} = 0.95,\ Q_{1|10} = 0.90,\ Q_{0|11} = 0.15.$$

In Fig. 9.1, the dash-dotted curve is the boundary of the EER outer bound. On the other hand, the achievable error exponents using joint ML decoding can be derived from [17, Theorem 2] as

$$\begin{aligned} E_1 &= \min\{E_{t1}(R_1, p_1, p_2), E_{t3}(R_1 + R_2, p_1, p_2)\} \\ E_2 &= \min\{E_{t2}(R_2, p_1, p_2), E_{t3}(R_1 + R_2, p_1, p_2)\}, \end{aligned} \tag{9.2}$$

where

$$\begin{aligned} E_{t1}(R_1, p_1, p_2) &\triangleq \max_{0 \le \tilde{s} \le 1} \Big\{ -\tilde{s}R_1 - \log\big(\sum_{j \in \mathcal{J}, k \in \mathcal{K}} p_{2j} \big[\sum_{i \in \mathcal{I}} p_{1i} Q_{k|ij}^{\frac{1}{1+\tilde{s}}} \big]^{1+\tilde{s}} \big) \Big\} \\ E_{t2}(R_2, p_1, p_2) &\triangleq \max_{0 \le \hat{s} \le 1} \Big\{ -\hat{s}R_2 - \log\big(\sum_{i \in \mathcal{I}, k \in \mathcal{K}} p_{1i} \big[\sum_{j \in \mathcal{J}} p_{2j} Q_{k|ij}^{\frac{1}{1+\hat{s}}} \big]^{1+\hat{s}} \big) \Big\} \\ E_{t3}(R_1 + R_2, p_1, p_2) &\triangleq \max_{0 \le s \le 1} \Big\{ -s(R_1 + R_2) - \log\big(\sum_{k \in \mathcal{K}} \big[\sum_{i \in \mathcal{I}, j \in \mathcal{J}} p_{1i} p_{2j} Q_{k|ij}^{\frac{1}{1+s}} \big]^{1+s} \big) \Big\}, \end{aligned} \tag{9.3}$$

and p_1 and p_2 are the probability distributions used to construct the random codebooks for user 1 and user 2, respectively. In Fig. 9.1, the solid curve is the EER inner bound.

Arutyunyan developed a type of sphere packing bound for the probability of

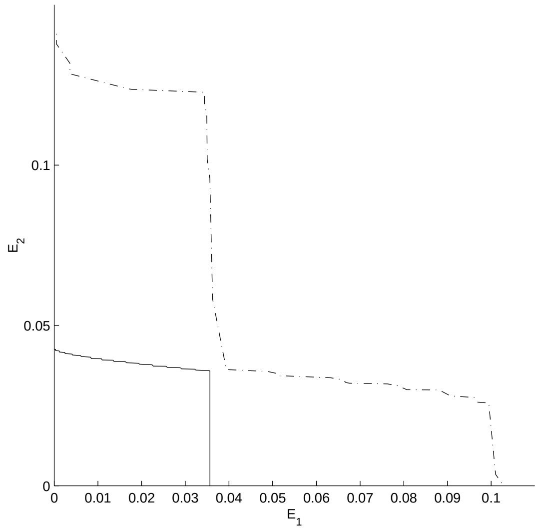


Figure 9.1: EER inner bound (solid) and outer bound (dashed-dotted) for $R_1 = 0.1$, $R_2 = 0.1$.

system error in [32]. This type of sphere packing bound can be written as

$$E_{sys}(R_1, R_2) \leq \max_{p} \ \min\{E_{u1}(R_1, p), E_{u2}(R_2, p), E_{u3}(R_1 + R_2, p)\}, \tag{9.4}$$

where

$$\begin{aligned} E_{u1}(R_1, p) &\triangleq \max_{\tilde{s}\geq 0}\Big\{-\tilde{s}R_1 - \log\big(\sum_{j\in\mathcal{J},k\in\mathcal{K}} p_j\big[\sum_{i\in\mathcal{I}} p_{i|j} Q_{k|ij}^{\frac{1}{1+\tilde{s}}}\big]^{1+\tilde{s}}\big)\Big\} \\ E_{u2}(R_2, p) &\triangleq \max_{\hat{s}\geq 0}\Big\{-\hat{s}R_2 - \log\big(\sum_{i\in\mathcal{I},k\in\mathcal{K}} p_i\big[\sum_{j\in\mathcal{J}} p_{j|i} Q_{k|ij}^{\frac{1}{1+\hat{s}}}\big]^{1+\hat{s}}\big)\Big\} \\ E_{u3}(R_1 + R_2, p) &\triangleq \max_{s\geq 0}\Big\{-s(R_1 + R_2) - \log\big(\sum_{k\in\mathcal{K}}\big[\sum_{i\in\mathcal{I},j\in\mathcal{J}} p_{ij} Q_{k|ij}^{\frac{1}{1+s}}\big]^{1+s}\big)\Big\}, \end{aligned} \tag{9.5}$$

and $p_i = \sum_{j\in\mathcal{J}} p_{ij}$ and $p_j = \sum_{i\in\mathcal{I}} p_{ij}$ are the marginal probability distributions, and $p_{i|j} = \frac{p_{ij}}{p_j}$ and $p_{j|i} = \frac{p_{ij}}{p_i}$ are the conditional probability distributions. We claim that, in the sense of probability of system error, the bound in Theorem 9.1 and Arutyunyan's bound are the same. In other words, the EER outer bound given in Theorem 9.1 includes Arutyunyan's bound as a special case. We summarize the result in the following theorem.

Theorem 9.2 *The system error exponent upper bound derived from Theorem 9.1 is the same as Arutyunyan's bound, i.e., if we define the right hand side of the inequalities in Theorem 9.1 as*

$$
\begin{aligned}
E_{v1}(R_1,p) &\triangleq \max_{\tilde{s}\geq 0}\min_{\tilde{q}_{k|j}} \Big[-\tilde{s}R_1 - (1+\tilde{s}) \sum_{i\in\mathcal{I},j\in\mathcal{J}} p_{ij} \log\big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+\tilde{s}}} \tilde{q}_{k|j}^{\frac{\tilde{s}}{1+\tilde{s}}} \big)\Big] \\
E_{v2}(R_2,p) &\triangleq \max_{\hat{s}\geq 0}\min_{\hat{q}_{k|i}} \Big[-\hat{s}R_2 - (1+\hat{s}) \sum_{i\in\mathcal{I},j\in\mathcal{J}} p_{ij} \log\big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+\hat{s}}} \hat{q}_{k|i}^{\frac{\hat{s}}{1+\hat{s}}} \big)\Big] \\
E_{v3}(R_1+R_2,p) &\triangleq \max_{s\geq 0}\min_{q_k} \Big[-s(R_1+R_2) - (1+s) \sum_{i\in\mathcal{I},j\in\mathcal{J}} p_{ij} \log\big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_k^{\frac{s}{1+s}} \big)\Big],
\end{aligned}
\tag{9.6}
$$

then

$$
\begin{aligned}
&\max_p \ \min\{E_{v1}(R_1,p), E_{v2}(R_2,p), E_{v3}(R_1+R_2,p)\} = \\
&\qquad \max_p \ \min\{E_{u1}(R_1,p), E_{u2}(R_2,p), E_{u3}(R_1+R_2,p)\},
\end{aligned}
\tag{9.7}
$$

where the left hand side of the equation is the system error exponent upper bound derived directly from Theorem 9.1, and the right hand side of the equation is Arutyunyan's bound.

Proof: From [33, Theorem 10.1.5: Step 3], we have

$$
\begin{aligned}
E_{u1}(R_1,p) &\leq E_{v1}(R_1,p) \\
E_{u2}(R_2,p) &\leq E_{v2}(R_2,p) \\
E_{u3}(R_1+R_2,p) &\leq E_{v3}(R_1+R_2,p),
\end{aligned}
\tag{9.8}
$$

so

$$\max_p \; \min\{E_{u1}(R_1,p), E_{u2}(R_2,p), E_{u3}(R_1+R_2,p)\} \leq$$
$$\max_p \; \min\{E_{v1}(R_1,p), E_{v2}(R_2,p), E_{v3}(R_1+R_2,p)\}. \tag{9.9}$$

In addition, it is shown in [33, Theorem 10.1.5] that

$$\begin{aligned}
\max_p E_{u1}(R_1,p) &= \max_p E_{v1}(R_1,p) \\
\max_p E_{u2}(R_2,p) &= \max_p E_{v2}(R_2,p) \\
\max_p E_{u3}(R_1+R_2,p) &= \max_p E_{v3}(R_1+R_2,p),
\end{aligned} \tag{9.10}$$

so

$$\begin{aligned}
&\min\{\max_p E_{u1}(R_1,p), \max_p E_{u2}(R_2,p), \max_p E_{u3}(R_1+R_2,p)\} \\
&\qquad = \min\{\max_p E_{v1}(R_1,p), \max_p E_{v2}(R_2,p), \max_p E_{v3}(R_1+R_2,p)\}.
\end{aligned} \tag{9.11}$$

Combining (9.9) and (9.11), we have

$$\begin{aligned}
&\min\{\max_p E_{u1}(R_1,p), \max_p E_{u2}(R_2,p), \max_p E_{u3}(R_1+R_2,p)\} \\
&\qquad = \min\{\max_p E_{v1}(R_1,p), \max_p E_{v2}(R_2,p), \max_p E_{v3}(R_1+R_2,p)\} \\
&\qquad \geq \max_p \; \min\{E_{v1}(R_1,p), E_{v2}(R_2,p), E_{v3}(R_1+R_2,p)\} \\
&\qquad \geq \max_p \; \min\{E_{u1}(R_1,p), E_{u2}(R_2,p), E_{u3}(R_1+R_2,p)\}.
\end{aligned} \tag{9.12}$$

Finally, it is shown in the proof of [32, Lemma 1] that

$$\min\{\max_p E_{u1}(R_1, p), \max_p E_{u2}(R_2, p), \max_p E_{u3}(R_1 + R_2, p)\}$$
$$= \max_p \ \min\{E_{u1}(R_1, p), E_{u2}(R_2, p), E_{u3}(R_1 + R_2, p)\}, \tag{9.13}$$

so

$$\max_p \ \min\{E_{v1}(R_1, p), E_{v2}(R_2, p), E_{v3}(R_1 + R_2, p)\} =$$
$$\max_p \ \min\{E_{u1}(R_1, p), E_{u2}(R_2, p), E_{u3}(R_1 + R_2, p)\} \tag{9.14}$$

by (9.12), (9.13). ∎

APPENDICES

APPENDIX A

Proof of Theorem 6.1

We now derive the EER outer bound given in Theorem 6.1. Without loss of generality, we assume $P = 1$ and noise variance $\sigma^2 = \frac{1}{SNR}$, so we consider only spherical codes. Moreover, in most cases we consider a sequence of spherical codes as the dimension $N \to \infty$, i.e., one spherical code for each N, so "a spherical code" mentioned here should be understood as "a sequence of spherical codes" whenever necessary.

The proof for Theorem 6.1 is outlined in the following.

- In Section A.1, we consider a spherical code with two message sets and with user 1's error exponent given by E_1, i.e., $P_{e1} = e^{-NE_1}$. In Lemma A.1, we find an index i^* such that the area of the union of certain cones associated with this index i^* is bounded, i.e., $A(\bigcup_j Cone^s(\theta_{E_1}, C_{i^*,j})) \leq 4\frac{\Omega(N,\pi)}{M_1}$, where $\theta_{E_1} = \sin^{-1}\left(\sqrt{\frac{2E_1}{SNR}}\right)$. When θ_{E_1} is large, it is necessary for the codewords $\bigcup_j \{C_{i^*,j}\}$ to concentrate at the center of the region $\bigcup_j Cone^s(\theta_{E_1}, C_{i^*,j})$ due to the area constraint $4\frac{\Omega(N,\pi)}{M_1}$.

- Our goal is to find an upper bound for the minimum distance $d_{min}(\bigcup_j \{C_{i^*,j}\})$. In Section A.2, we first consider the densest spherical code $\mathcal{S}^* \subset \partial Cone^s(\theta'_1, W)$

satisfying the sum area constraint $\frac{\Omega(N,\pi)}{M_1}$, where θ_1' is some positive angle and W is the north pole of S^{N-1}. Based on Conjecture 6.1, we can upper bound $d_{min}(\bigcup_j \{C_{i^*,j}\})$ by $d_{min}(\mathcal{S}^*)$. Since $d_{min}(\mathcal{S}^*) = \sin\theta_1'\ d_{min}(R_2)$, we want to find an upper bound for $\sin\theta_1'$. To do this, it turns out that we only need to find an upper bound for the ratio of $\Omega(N,\theta_1')$ to $A(\bigcup_k Cone^s(\theta_{E_1}, S_k^*))$. We derive a lower bound for $A(\bigcup_k Cone^s(\theta_{E_1}, S_k^*))$ in Lemma A.2.

- In Section A.3, based on Lemma A.2, we derive an upper bound for the ratio of $\Omega(N,\theta_1')$ to $A(\bigcup_k Cone^s(\theta_{E_1}, S_k^*))$. Then we show that $d_{min}(\bigcup_j \{C_{i^*,j}\})$ is upper bounded by $\sin\eta_1\ d_{min}(R_2)$, where $\eta_1 = \eta(R_1, R_2, E_1, SNR)$. Hence there exist a pair of codewords C_{i^*,j_1} and C_{i^*,j_2} such that P_{e2,i^*j_1} and P_{e2,i^*j_2} are lower bounded by $e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)}$. Finally, applying the standard technique that an error exponent upper bound for the *maximum* probability of error is also an error exponent upper bound for the *average* probability of error, the average probability of error for message 2 is also upper bounded by $e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)}$.

A.1 Union of Cones under Sum Area Constraint

For a two-message spherical code $\mathcal{C} = \{C_{i,j} \mid 1 \le i \le M_1, 1 \le j \le M_2\} \subset S^{N-1}$, the probabilities of error for message 1 and for message 2 given that $C_{i,j}$ is transmitted are denoted as $P_{e1,ij}$ and $P_{e2,ij}$, respectively. Define

$$P_{e1,i} \triangleq \frac{1}{M_2}\sum_{j=1}^{M_2} P_{e1,ij}$$
$$P_{e2,j} \triangleq \frac{1}{M_1}\sum_{i=1}^{M_1} P_{e2,ij} \qquad \text{(A.1)}$$

as the probabilities of error for message 1 and for message 2 averaged out over the codewords with the same first and the same second indices, respectively. Thus the (average) probabilities of error for message 1 and message 2 can be written as

$$P_{e1} = \frac{1}{M_1}\sum_{i=1}^{M_1} P_{e1,i} = \frac{1}{M_1 M_2}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} P_{e1,ij}$$
$$P_{e2} = \frac{1}{M_2}\sum_{j=1}^{M_2} P_{e2,j} = \frac{1}{M_1 M_2}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} P_{e2,ij}. \tag{A.2}$$

Assume $P_{e1} = e^{-NE_1}$ for the spherical code $\mathcal{C}$. Our goal is to show that there exist two different codewords C_{i^*,j_1} and C_{i^*,j_2} such that the distance between C_{i^*,j_1} and C_{i^*,j_2} is at most $\sin\eta(R_1, R_2, E_1, SNR)\; d_{min}(R_2)$. This, with an additional argument given later, implies that the error exponent for message 2 is upper bounded by $\sin^2\eta(R_1, R_2, E_1, SNR)\; E_{md}(R_2, SNR)$. Thus there is a tradeoff between the error exponents of message 1 and message 2 through the quantity $\eta(R_1, R_2, E_1, SNR)$. On the other hand, we can also start with the assumption $P_{e2} = e^{-NE_2}$ for any spherical code and derive an upper bound for message 1's error exponent. Since these two methods are identical, we only work on the first case.

Under the assumption $P_{e1} = e^{-NE_1}$, we show that the surface caps $Cone^s(\theta_{E_1}, C_{i,j})$ are subject to the sum area constraint $4\frac{\Omega(N,\pi)}{M_1}$. We summarize the result in the following lemma.

Lemma A.1 *For any spherical code with two message sets satisfying* $P_{e1} = e^{-NE_1}$*, there exist* $\tilde{\mathcal{I}} \subset \{1, 2, \ldots, M_1\}$ *and* $\mathcal{J}_i \subset \{1, 2, \ldots, M_2\}$ *such that* $|\tilde{\mathcal{I}}| = \frac{M_1}{4}$, $|\mathcal{J}_i| = \frac{M_2}{2}$ *and*

$$A\Big(\bigcup_{j\in\mathcal{J}_i} Cone^s(\theta_{E_1}, C_{i,j})\Big) \le 4\frac{\Omega(N,\pi)}{M_1} \tag{A.3}$$

for all $i \in \tilde{\mathcal{I}}$.

Proof: Define

$$C_{i,j}^{\min} \triangleq \operatorname*{arg\,min}_{C_{i',j'}:\; i' \neq i} d(C_{i,j}, C_{i',j'}), \tag{A.4}$$

i.e., $d(C_{i,j}, C_{i',j'}) \geq d(C_{i,j}, C_{i,j}^{\min})$ for all $i' \neq i$ and j', where $1 \leq i' \leq M_1$ and $1 \leq j' \leq M_2$. If there are more than two codewords satisfying (A.4) for the codeword $C_{i,j}$, we choose one and denote the chosen one as $C_{i,j}^{\min}$.

Now consider the genie-aided receiver for message 1 in Fig. A.1. Assume that the codeword $C_{i,j}$ is transmitted. In addition to the channel output $\mathbf{Y}^N$, a genie tells the receiver that one of the two codewords $\{C_{i,j}, C_{i,j}^{min}\}$ is transmitted. Denote $P'_{e1,ij}$ as the probability of error in the genie-aided channel when the codeword $C_{i,j}$ is transmitted, and define $P'_{e1,i}$ and P'_{e1} correspondingly. Clearly, $P'_{e1} \leq P_{e1}$, since the extra information from the genie can only improve the performance for message 1.

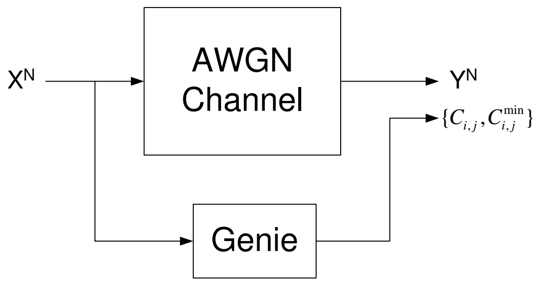


Figure A.1: Genie-aided receiver.

Define $\mathcal{I}$ as the subset (with $\frac{M_1}{2}$ elements) of the values of the first index i with smaller probability of error $P'_{e1,i}$, i.e., $P'_{e1,i_1} \leq P'_{e1,i_2}$ for any $i_1 \in \mathcal{I}$ and any $i_2 \notin \mathcal{I}$. Given an index i with $1 \leq i \leq M_1$, define $\mathcal{J}_i$ as the subset (with $\frac{M_2}{2}$ elements) of the values of the second index j with smaller probability of error $P'_{e1,ij}$, i.e., $P'_{e1,ij_1} \leq P'_{e1,ij_2}$ for any $j_1 \in \mathcal{J}_i$ and any $j_2 \notin \mathcal{J}_i$. Under the assumption $P_{e1} = e^{-NE_1}$, we have $P'_{e1,i} \leq 2e^{-NE_1}$ for all $i \in \mathcal{I}$, since $P'_{e1} \leq P_{e1} = e^{-NE_1}$. Similarly, we have $P'_{e1,ij} \leq 4e^{-NE_1}$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}_i$, since $P'_{e1,i} \leq 2e^{-NE_1}$ for all $i \in \mathcal{I}$.

Given any codeword $C_{i,j}$, the error probability $P'_{e1,ij}$ is

$$P'_{e1,ij} = \int_{\frac{d(C_{i,j},C_{i,j}^{\min})}{2}}^{\infty} \sqrt{N\frac{SNR}{2\pi}}\ e^{-N\frac{SNR}{2}x^2}dx \cong e^{-N\frac{SNR}{8}d^2(C_{i,j},C_{i,j}^{\min})}. \tag{A.5}$$

Hence we have

$$\frac{SNR}{8}d^2(C_{i,j},C_{i,j}^{min}) = \lim_{N\to\infty} -\frac{\log P'_{e1,ij}}{N} \geq \lim_{N\to\infty} -\frac{\log(4e^{-NE_1})}{N} = E_1, \tag{A.6}$$

i.e.,

$$d(C_{i,j},C_{i,j}^{min}) \geq \sqrt{\frac{8E_1}{SNR}} \tag{A.7}$$

for all $i \in \mathcal{I}$ and $j \in \mathcal{J}_i$. Define

$$\theta_{E_1} \triangleq \sin^{-1}\Big(\sqrt{\frac{2E_1}{SNR}}\Big), \tag{A.8}$$

then the interiors of any two surface caps $Cone^s(\theta_{E_1}, C_{i,j})$ and $Cone^s(\theta_{E_1}, C_{i',j'})$ are disjoint, where $i, i' \in \mathcal{I}$, $j \in \mathcal{J}_i$, $j' \in \mathcal{J}_{i'}$, and $i \neq i'$. This implies that the interiors of $\bigcup_{j\in\mathcal{J}_i} Cone^s(\theta_{E_1}, C_{i,j})$ and $\bigcup_{j\in\mathcal{J}_{i'}} Cone^s(\theta_{E_1}, C_{i',j})$ are disjoint, where $i, i' \in \mathcal{I}$ and $i \neq i'$. Therefore, we have

$$\sum_{i\in\mathcal{I}} A\big(\bigcup_{j\in\mathcal{J}_i} Cone^s(\theta_{E_1}, C_{i,j})\big) = A\big(\bigcup_{i\in\mathcal{I}}\bigcup_{j\in\mathcal{J}_i} Cone^s(\theta_{E_1}, C_{i,j})\big) \leq A(S^{N-1}) = \Omega(N,\pi). \tag{A.9}$$

Define $\tilde{\mathcal{I}}$ as the subset (with $\frac{M_1}{4}$ elements) of $\mathcal{I}$ with smaller surface area, i.e.,

$$A\Big(\bigcup_{j\in\mathcal{J}_i} Cone^s(\theta_{E_1}, C_{i,j})\Big) \leq A\Big(\bigcup_{j\in\mathcal{J}_{i'}} Cone^s(\theta_{E_1}, C_{i',j})\Big) \tag{A.10}$$

for all $i \in \tilde{\mathcal{I}}$ and $i' \in \mathcal{I} \setminus \tilde{\mathcal{I}}$. From (A.9), (A.10) we have

$$A\Big(\bigcup_{j\in\mathcal{J}_i} Cone^s(\theta_{E_1}, C_{i,j})\Big) \leq 4\frac{\Omega(N,\pi)}{M_1} \tag{A.11}$$

for all $i \in \tilde{\mathcal{I}}$. This completes the proof. ■

A.2 Minimum Distance Under Sum Area Constraint

We now derive the EER outer bound. The basic idea behind this EER outer bound is illustrated in Fig. A.2. Pick an element from the set $\tilde{\mathcal{I}}$ in Lemma A.1 and denote it as i^*. In Fig. A.2, the solid curve is the boundary of $\bigcup_{j\in\mathcal{J}_{i^*}} Cone^s(\theta_{E_1}, C_{i^*,j})$ and the black dots are the codewords $C_{i^*,j}$. When $2\sin\frac{\theta_{E_1}}{2}$ is large compared to $d_{min}(\bigcup_{j\in\mathcal{J}_{i^*}}\{C_{i^*,j}\})$, it is necessary to include enough "empty" space around the boundary. The extra empty area inside $\bigcup_{j\in\mathcal{J}_{i^*}} Cone^s(\theta_{E_1}, C_{i^*,j})$ reduces the minimum distance $d_{min}(\bigcup_{j\in\mathcal{J}_{i^*}}\{C_{i^*,j}\})$ due to the total area constraint $4\frac{\Omega(N,\pi)}{M_1}$ imposed on $A(\bigcup_{j\in\mathcal{J}_{i^*}} Cone^s(\theta_{E_1}, C_{i^*,j}))$, and this increases the probability of error for message 2 since all the codewords $C_{i^*,j}$'s considered above have different values for the second index.

In order to estimate (find an upper bound for) $d_{min}(\bigcup_{j\in\mathcal{J}_{i^*}}\{C_{i^*,j}\})$, we consider the following problem:

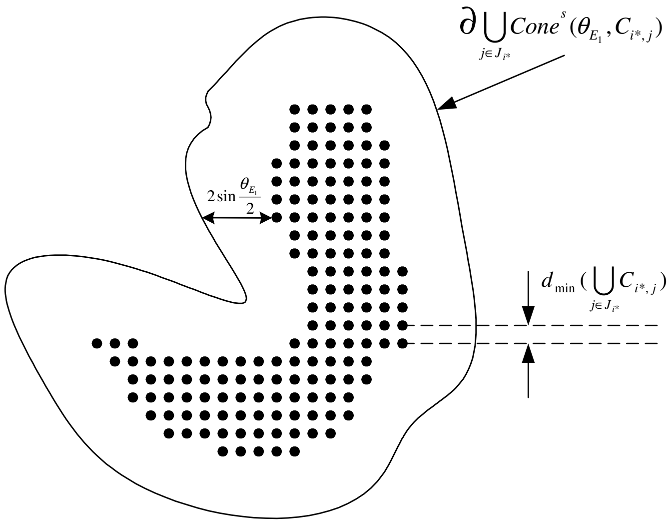


Figure A.2: $\bigcup_{j\in\mathcal{J}_{i^*}} Cone^s(\theta_{E_1}, C_{i^*,j})$ and codewords $C_{i^*,j}$'s (black dots).

"What is the maximum of $d_{min}(\mathcal{S})$ under the area constraint $A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k)) \leq \frac{\Omega(N,\pi)}{M_1}$, where the maximization is over all spherical codes $\mathcal{S} = \{S_1, \ldots, S_{M_2}\} \subset S^{N-1}$?"

Note that $d_{min}(\bigcup_{j\in\mathcal{J}_{i^*}}\{C_{i^*,j}\})$ is upper bounded by $\max_{\mathcal{S}\subset S^{N-1}} d_{min}(\mathcal{S})$, since $|\mathcal{J}_{i^*}| = \frac{M_2}{2} \cong M_2$ and $A\big(\bigcup_{j\in\mathcal{J}_{i^*}} Cone^s(\theta_{E_1}, C_{i^*,j})\big) \leq 4\frac{\Omega(N,\pi)}{M_1} \cong \frac{\Omega(N,\pi)}{M_1}$ by Lemma A.1. Equivalently, we may consider the following problem:

"What is the minimum of $A\big(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k)\big)$ under the distance constraint $d_{min}(\mathcal{S}) \geq d$, where the minimization is over all spherical codes $\mathcal{S} = \{S_1, \ldots, S_{M_2}\} \subset S^{N-1}$?"

Clearly, these two problems are equivalent. Denote $\mathcal{S}^* \triangleq \text{argmax}_{\mathcal{S}\subset S^{N-1}} d_{min}(\mathcal{S})$ as the solution of the optimum spherical code satisfying the area constraint in the problem and define $d_3' \triangleq d_{min}(\mathcal{S}^*)$. Based on Conjecture 6.1, we may assume that the solution $\mathcal{S}^*$ is the densest spherical code in a surface cap $Cone^s(\theta_1', W)$ for some

$\theta_1' > 0$, where W is the north pole, or is the densest spherical code in the boundary $\partial Cone^s(\theta_1', W)$, since the area around the boundary of $Cone^s(\theta_1', W)$ is dominant in high dimensions. Here the densest spherical code $\mathcal{S}^*$ means that for any point $V \in \partial Cone^s(\theta_1', W)$, the surface cap $Cone^s(2\sin^{-1}\frac{d_{min}(S^*)}{2}, V)$ must include at least one codeword of $\mathcal{S}^*$. We regard the densest spherical code in the surface cap ($\mathcal{S}^* \subset Cone^s(\theta_1', W)$) and the densest spherical code in the surface cap boundary ($\mathcal{S}^* \subset \partial Cone^s(\theta_1', W)$) the same and use these two terms interchangeably.

We now formulate the tradeoff between E_1 and E_2. Our goal is to find an upper bound for the distance d_3'. Note that $d_3' = \sin\theta_1' \ d_{min}(N-1, \frac{N}{N-1}R_2)$ from Conjecture 6.1, so we only need to find an upper bound for $\sin\theta_1'$. Since

$$\begin{aligned}
\sin^N \theta_1' &\cong \frac{\Omega(N, \theta_1')}{\Omega(N, \pi)} \\
&= \frac{\Omega(N, \theta_1')}{A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*))} \times \frac{A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*))}{\Omega(N, \pi)} \\
&\leq \frac{\Omega(N, \theta_1')}{A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*))} \times \frac{1}{M_1},
\end{aligned} \tag{A.12}$$

we need to find an upper bound for the ratio of $\Omega(N, \theta_1')$ to $A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*))$.

We now derive a lower bound for $A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*))$, and summarize the result in the following lemma.

Lemma A.2 *For the densest spherical code $\mathcal{S}^*$ in $\partial Cone^s(\theta_1', W)$ with the assumption $2\sin\frac{\theta_{E_1}}{2} > d_3' = \sin\theta_1' \ d_{min}(N-1, \frac{N}{N-1}R_2)$, we have the following properties.*
(a) The surface cap $Cone^s(\psi, W) \subset \bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*)$, where $\psi(R_2, \theta_1', \theta_{E_1})$ is defined as

$$\psi(R_2, \theta_1', \theta_{E_1}) = \sin^{-1}\Big(\frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\Big), \tag{A.13}$$

where

$$\alpha = \sin^2\theta_1'\cos^2\phi + \cos^2\theta_1' \tag{A.14a}$$

$$\beta = (2 - 4\sin^2\frac{\theta_{E_1}}{2})\sin\theta_1'\cos\phi \tag{A.14b}$$

$$\gamma = 1 - 4\sin^2\frac{\theta_{E_1}}{2} + 4\sin^4\frac{\theta_{E_1}}{2} - \cos^2\theta_1', \tag{A.14c}$$

and

$$\phi = \cos^{-1}\big(1 - \frac{1}{2}d^2_{min}(N-1, \frac{N}{N-1}R_2)\big). \tag{A.15}$$

(b) The function $d_{min}(\cdot)$ *in* (A.15) *can be substituted by any upper bound for* $d_{min}(\cdot)$*, i.e., if define*

$$\tilde{\psi}(R_2, \theta_1', \theta_{E_1}) = \sin^{-1}\big(\frac{\tilde{\beta} + \sqrt{\tilde{\beta}^2 - 4\tilde{\alpha}\tilde{\gamma}}}{2\tilde{\alpha}}\big) \tag{A.16a}$$

$$\tilde{\alpha} = \sin^2\theta_1'\cos^2\tilde{\phi} + \cos^2\theta_1' \tag{A.16b}$$

$$\tilde{\beta} = (2 - 4\sin^2\frac{\theta_{E_1}}{2})\sin\theta_1'\cos\tilde{\phi} \tag{A.16c}$$

$$\tilde{\gamma} = 1 - 4\sin^2\frac{\theta_{E_1}}{2} + 4\sin^4\frac{\theta_{E_1}}{2} - \cos^2\theta_1' \tag{A.16d}$$

$$\tilde{\phi} = \cos^{-1}\big(1 - \frac{1}{2}d^2_{min,u}(N-1, \frac{N}{N-1}R_2)\big), \tag{A.16e}$$

where $d_{min,u}(\cdot)$ *is any upper bound for* $d_{min}(\cdot)$*, then* $\tilde{\psi} \leq \psi$*, which implies that* $Cone^s(\tilde{\psi}, W) \subset \bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*)$*.*

(c) The function $\delta(R_2, \theta_1', \theta_{E_1})$*, defined as*

$$\delta(R_2, \theta_1', \theta_{E_1}) \triangleq \psi(R_2, \theta_1', \theta_{E_1}) - \theta_1', \tag{A.17}$$

decreases as the argument θ_1' *increases, while* R_2 *and* θ_{E_1} *are fixed.*

Proof: Consider the following four points on S^{N-1} with coordinates given in the following (see Fig. A.3).

$W = (0, 0, 0, \ldots, 0, 1)$. North pole.

$A = (\sin\psi, 0, 0, \ldots, 0, \cos\psi)$. Angle $\angle AOC = \theta_{E_1}$.

$B = (\sin\theta_1', 0, 0, \ldots, 0, \cos\theta_1')$.

$C = (\sin\theta_1' \cos\phi, \sin\theta_1' \sin\phi, 0, \ldots, 0, \cos\theta_1')$. Length $\overline{BC} = d_3'$.

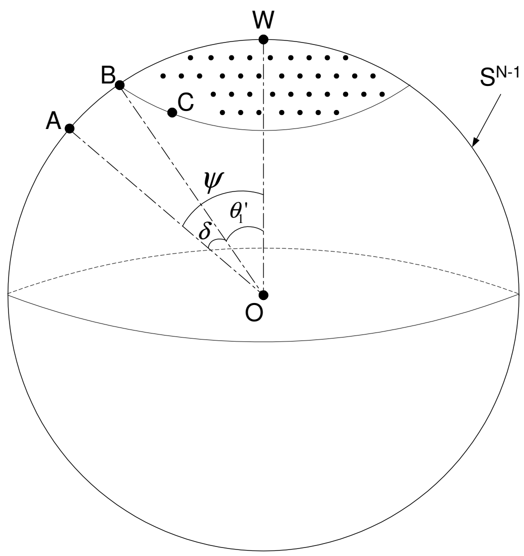


Figure A.3: Spherical code in surface cap $Cone^s(\theta_1', W)$. $\overline{AC} = 2\sin\frac{\theta_{E_1}}{2}$, $\overline{BC} = d_3'$.

Here we assume that $2\sin\frac{\theta_{E_1}}{2} = \overline{AC} > \overline{BC} = d_3'$, so $\psi > \theta_1'$. Since $\mathcal{S}^*$ is the densest code in the boundary $\partial Cone^s(\theta_1', W)$, the surface cap $Cone^s(2\sin^{-1}\frac{d_3'}{2}, B)$ must include at least one codeword of $\mathcal{S}^*$, and denote this codeword as S_m^* ($S_m^* \in \partial Cone^s(\theta_1', W)$). Note that $\angle AOC = \theta_{E_1}$ and $\overline{BC} = d_3'$, so $\angle AOS_m^* \leq \theta_{E_1}$. This implies that $A \in Cone^s(\theta_{E_1}, S_m^*)$ and consequently $A \in \bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*)$. The above argument can be applied to any point on the boundary $\partial Cone^s(N, \psi)$, so we conclude that $Cone^s(\psi, W) \subset \bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*)$.

The angle ψ can be expressed as a function of R_2, θ'_1 and θ_{E_1}. From

$$d'_3 = \overline{BC} = \sqrt{(1-\cos\phi)^2 + \sin^2\phi}\ \sin\theta'_1, \tag{A.18}$$

we have

$$\phi = \cos^{-1}\big(1 - \frac{1}{2}(\frac{d'_3}{\sin\theta'_1})^2\big) = \cos^{-1}\big(1 - \frac{1}{2}d^2_{min}(N-1, \frac{N}{N-1}R_2)\big). \tag{A.19}$$

From

$$2\sin\frac{\theta_{E_1}}{2} = \overline{AC} = \sqrt{(\sin\psi - \sin\theta'_1\cos\phi)^2 + (\sin\theta'_1\sin\phi)^2 + (cos\psi - \cos\theta'_1)^2}, \tag{A.20}$$

we have

$$\begin{aligned}\big(\sin^2\theta'_1\cos^2\phi + \cos^2\theta'_1\big)\sin^2\psi - \big((2 - 4\sin^2\frac{\theta_{E_1}}{2})\sin\theta'_1\cos\phi\big)\sin\psi \\ + \big(1 - 4\sin^2\frac{\theta_{E_1}}{2} + 4\sin^4\frac{\theta_{E_1}}{2} - \cos^2\theta'_1\big) = 0.\end{aligned} \tag{A.21}$$

Define

$$\alpha = \sin^2\theta'_1\cos^2\phi + \cos^2\theta'_1 \tag{A.22a}$$

$$\beta = (2 - 4\sin^2\frac{\theta_{E_1}}{2})\sin\theta'_1\cos\phi \tag{A.22b}$$

$$\gamma = 1 - 4\sin^2\frac{\theta_{E_1}}{2} + 4\sin^4\frac{\theta_{E_1}}{2} - \cos^2\theta'_1, \tag{A.22c}$$

then

$$\psi = \sin^{-1}\big(\frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\big) \tag{A.23}$$

is the desired root of the equation (A.21).

It is easy to verify that ψ decreases as we substitute the $d_{min}(\cdot)$ in (A.15) for any upper bound for $d_{min}(\cdot)$. Finally, since the length $\overline{AC}$

$$
\begin{aligned}
2\sin\frac{\theta_{E_1}}{2} &= \overline{AC} \\
&= \sqrt{(\sin\psi - \sin\theta_1'\cos\phi)^2 + (\sin\theta_1'\sin\phi)^2 + (\cos\psi - \cos\theta_1')^2} \\
&= \sqrt{2(1 - \sin\psi\sin\theta_1'\cos\phi - \cos\psi\cos\theta_1')} \\
&= \sqrt{2(1 - \sin\psi\sin\theta_1' - \cos\psi\cos\theta_1' + \sin\psi\sin\theta_1'(1-\cos\phi))} \\
&= \sqrt{2(1 - \cos\delta + \sin(\theta_1'+\delta)\sin\theta_1'(1-\cos\phi))} \qquad \text{(A.24)}
\end{aligned}
$$

is a constant (depending only on θ_{E_1}) and the angle ϕ is a constant (depending only on R_2), the angle δ must decrease as θ_1' increases. ■

A.3 EER Outer Bound

We are now ready to derive the EER outer bound. Applying Lemma A.2, we have

$$\frac{\Omega(N,\theta_1')}{A(\bigcup_{k=1}^{M_2} Cone^s(\theta_{E_1}, S_k^*))} \leq \frac{\Omega(N,\theta_1')}{\Omega(N,\theta_1'+\delta(R_2,\theta_1',\theta_{E_1}))} \cong \frac{\sin^N\theta_1'}{\sin^N(\theta_1'+\delta(R_2,\theta_1',\theta_{E_1}))}. \qquad \text{(A.25)}$$

Define θ_1 as the root θ of the equation $\Omega(N,\theta) = \frac{\Omega(N,\pi)}{M_1}$, so $\theta_1' \leq \theta_1$ since $\Omega(N,\theta_1') \leq \frac{\Omega(N,\pi)}{M_1}$. Thus

$$\frac{\sin\theta_1'}{\sin(\theta_1'+\delta(R_2,\theta_1',\theta_{E_1}))} \leq \frac{\sin\theta_1}{\sin(\theta_1+\delta(R_2,\theta_1',\theta_{E_1}))} \leq \frac{\sin\theta_1}{\sin(\theta_1+\delta(R_2,\theta_1,\theta_{E_1}))}, \qquad \text{(A.26)}$$

where the last inequality comes from the fact that $\delta(R_2, \theta_1', \theta_{E_1})$ decreases as θ_1' increases.

From (A.12), (A.25), (A.26), we have

$$\log(\sin\theta_1') = \lim_{N\to\infty} \frac{1}{N} \log \frac{\Omega(N,\theta_1')}{\Omega(N,\pi)} \leq \log \frac{\sin\theta_1}{\sin(\theta_1 + \delta(R_2,\theta_1,\theta_{E_1}))} - R_1, \tag{A.27}$$

i.e.,

$$\sin\theta_1' \leq \frac{\sin\theta_1\ e^{-R_1}}{\sin(\theta_1 + \delta(R_2,\theta_1,\theta_{E_1}))}. \tag{A.28}$$

Define

$$\eta_1 \triangleq \sin^{-1}\Big(\frac{\sin\theta_1\ e^{-R_1}}{\sin(\theta_1 + \delta(R_2,\theta_1,\theta_{E_1}))}\Big), \tag{A.29}$$

thus

$$d_{min}(\bigcup_{j\in\mathcal{J}_{i^*}} \{C_{i^*,j}\}) \leq d_3' = \sin\theta_1'\ d_{min}(N-1, \frac{N}{N-1}R_2) \leq \sin\eta_1\ d_{min}(N-1, \frac{N}{N-1}R_2). \tag{A.30}$$

Therefore, we have shown that there exist a pair of codewords C_{i^*,j_1} and C_{i^*,j_2} such that P_{e2,i^*j_1} and P_{e2,i^*j_2} are lower bounded by $e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)}$ for sufficiently large N.

A standard technique used in single-user channels is that an error exponent upper bound for the *maximum* probability of error of the codewords is also an error exponent upper bound for the *average* probability of error of the codewords. [1] A proof for this can be found in [33, Sec. 10.6]. Basically, this technique is shown by applying the upper bound, say E_u, to half of the codewords with smaller prob-

[1]This is not true in multi-user channels. An counter-example can be found in [34]

ability of error in the codebook, then the probability of error of every codeword in the other half of the codebook (with larger probability of error) is lower bound by e^{-NE_u}. Therefore, the average probability of error of the codebook is lower bounded by $\frac{1}{2}e^{-NEu}$ by adding only the probabilities of error of the codewords from the other half of the codebook (with larger probability of error), and a factor of $\frac{1}{2}$ has no effect on the error exponent.

Apply this argument to our case, we can write

$$\frac{1}{|\mathcal{J}_i|}\sum_{j\in\mathcal{J}_i} P_{e2,ij} \geq \frac{1}{2}e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)} \tag{A.31}$$

for all $i \in \tilde{\mathcal{I}}$. This can also be written as

$$\sum_{j\in\mathcal{J}_i} P_{e2,ij} \geq \frac{M_2}{4}e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)}, \tag{A.32}$$

since $|\mathcal{J}_i| = \frac{M_2}{2}$. Thus the average probability of error for message 2 is lower bounded by

$$\begin{aligned}
P_{e2} &= \frac{1}{M_3}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} P_{e2,ij} \\
&\geq \frac{1}{M_3}\sum_{i\in\tilde{\mathcal{I}}}\sum_{j\in\mathcal{J}_i} P_{e2,ij} \\
&\geq \frac{1}{M_3}\sum_{i\in\tilde{\mathcal{I}}} \frac{M_2}{4}e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)} \\
&= \frac{1}{M_3}\frac{M_1}{4}\frac{M_2}{4}e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)} \\
&= \frac{1}{16}e^{-N\sin^2\eta_1 E_{md}(R_2,SNR)},
\end{aligned} \tag{A.33}$$

so $E_2 \leq \sin^2\eta_1 E_{md}(R_2, SNR)$.

Finally, note that

$$\begin{aligned}\theta_{E_1} &= \sin^{-1}\left(\sqrt{\frac{2E_1}{SNR}}\right) \\ e^{-NR_1} &= \frac{\Omega(N,\theta_1)}{\Omega(N,\pi)} \cong \sin^N\theta_1, \text{ i.e., } \sin\theta_1 = e^{-R_1} \text{ as } N\to\infty,\end{aligned} \tag{A.34}$$

so we have

$$\eta_1 = \eta(R_1, R_2, E_1, SNR). \tag{A.35}$$

Thus

$$E_2 \le \sin^2\eta(R_1, R_2, E_1, \frac{P}{\sigma^2})\ E_{md}(R_2, \frac{P}{\sigma^2}) \tag{A.36}$$

as claimed in Theorem 6.1.

The proof for

$$E_1 \le \sin^2\eta(R_2, R_1, E_2, \frac{P}{\sigma^2})\ E_{md}(R_1, \frac{P}{\sigma^2}) \tag{A.37}$$

is identical as the proof given above except exchanging the roles of message 1 and message 2. This completes the proof.

APPENDIX B

Proof of Theorem 9.1

The proof for Theorem 9.1 basically follows the derivation of the sphere packing bound given in [33, Sec. 10.1 & Sec. 10.2] with a few modifications. We first review the Neyman-Pearson theorem in Section B.1. Next, we derive an error exponent upper bound for user 1 in Section B.2 (Lemma B.3) assuming that the input for user 2 is fixed for some codeword C_{2,n^*} and is known at the receiver. Based on Lemma B.3, we derive an error exponent upper bound for user 1 in Section B.3 by transforming the original DMMAC into a genie-aided system. An error exponent upper bound for user 2 can be derived similarly.

B.1 Neyman-Pearson Theorem

Consider a binary hypothesis testing problem with hypotheses H_0 and H_1 and the corresponding output probabilities Q_0 and Q_1, i.e., $P(\mathbf{Y} = k|H_0) = Q_{0k}$ and $P(\mathbf{Y} = k|H_1) = Q_{1k}$. Accepting hypothesis H_1 when H_0 is actually true is called a type I error, and the probability of this event is denoted by α. Accepting hypothesis H_0 when H_1 is actually true is called a type II error, and the probability of this event is denoted by β. Denote $\mathcal{D} \subset \mathcal{K}$ as the decision region for hypothesis H_0, then

$$\alpha = \sum_{k \notin \mathcal{D}} Q_{0k}, \ \ \beta = \sum_{k \in \mathcal{D}} Q_{1k}. \tag{B.1}$$

We state the Neyman-Pearson theorem in the following lemma.

Lemma B.1 *Define the Neyman-Pearson decision region as*

$$\mathcal{D}^{NP} = \{k : \ \log \frac{Q_{0k}}{Q_{1k}} \geq T\}, \tag{B.2}$$

where T is the decision threshold, and let α^{NP} and β^{NP} be the probabilities of type I and type II errors corresponding to this region. Suppose α and β are the probabilities of type I and type II errors corresponding to some other decision region. Then $\alpha < \alpha^{NP}$ implies $\beta > \beta^{NP}$, and $\alpha = \alpha^{NP}$ implies $\beta \geq \beta^{NP}$.

Proof: See [33, Theorem 4.1.1]. ∎

Now consider a binary hypothesis testing with N independent but not identical channel outputs. The l^{th} channel output $\mathbf{Y}_l$ is said to be in mode $u \in \mathcal{U}$ if $P(\mathbf{Y}_l = k|H_0) = Q_{0k|u}$ and $P(\mathbf{Y}_l = k|H_1) = Q_{1k|u}$, where $\mathcal{U}$ is the mode alphabet (with finite elements), and $Q_{0k|u}$ and $Q_{1k|u}$ are the corresponding probability distributions for H_0 and H_1, respectively. The probability distributions of the channel output $\mathbf{Y}^N$ are given by

$$H_0 : \ P(\mathbf{Y}^N = Y^N|H_0) = \prod_{l=1}^{N} Q_{0Y_l|u_l} \tag{B.3a}$$

$$H_1 : \ P(\mathbf{Y}^N = Y^N|H_1) = \prod_{l=1}^{N} Q_{1Y_l|u_l}, \tag{B.3b}$$

where the l^{th} channel output is assumed in mode u_l.

Define p as the composition of the modes, i.e., $p_u \triangleq \frac{|\{l:\ u_l=u\}|}{N}$. Given the mode vector $u^N = \{u_1, \ldots, u_N\}$, the Neyman-Pearson decision region for the binary hypothesis testing with N independent channel outputs is

$$\begin{aligned}
\mathcal{D}^{NP} &= \{Y^N:\ \log \frac{P(\mathbf{Y}^N = Y^N|H_0)}{P(\mathbf{Y}^N = Y^N|H_1)} \geq NT\} \\
&= \{Y^N:\ \frac{1}{N}\sum_{l=1}^{N} \log \frac{Q_{0Y_l|u_l}}{Q_{1Y_l|u_l}} \geq T\} \\
&= \{Y^N:\ \sum_{u\in\mathcal{U}} p_u \sum_{k\in\mathcal{K}} \frac{|\{l:\ Y_l = k, u_l = u\}|}{|\{l:\ u_l = u\}|} \log \frac{Q_{0k|u}}{Q_{1k|u}} \geq T\} \\
&= \{Y^N:\ \sum_{u\in\mathcal{U}} p_u \sum_{k\in\mathcal{K}} Q_{k|u}(Y^N)\ \log \frac{Q_{0k|u}}{Q_{1k|u}} \geq T\}, \qquad \text{(B.4)}
\end{aligned}$$

where $Q_{k|u}(Y^N) \triangleq \frac{|\{l:\ Y_l=k, u_l=u\}|}{|\{l:\ u_l=u\}|}$. Denote $D(\tilde{Q}||\hat{Q}|p)$ as the conditional discrimination function (Kullback-Leibler distance) between the probability distributions $\tilde{Q}$ and $\hat{Q}$, i.e.,

$$D(\tilde{Q}||\hat{Q}|p) = \sum_{u\in\mathcal{U}} p_u \sum_{k\in\mathcal{K}} \tilde{Q}_{k|u} \log \frac{\tilde{Q}_{k|u}}{\hat{Q}_{k|u}}. \qquad \text{(B.5)}$$

We summarize the result regarding the probabilities of type I and type II errors using Neyman-Pearson decision region in the following lemma.

Lemma B.2 *The probabilities of type I and type II errors based on Neyman-Pearson decision region satisfy*

$$e^{-ND(Q_\lambda||Q_0|p)-o(N)} \leq \alpha^{NP} \leq e^{-ND(Q_\lambda||Q_0|p)} \qquad \text{(B.6a)}$$

$$e^{-ND(Q_\lambda||Q_1|p)-o(N)} \leq \beta^{NP} \leq e^{-ND(Q_\lambda||Q_1|p)}, \qquad \text{(B.6b)}$$

where $\lim_{N\to\infty} \frac{o(N)}{N} = 0$*,* p *is the composition of the modes,* Q_λ *is the "tilted" distri-*

bution given by

$$Q_{\lambda k|u} \triangleq \frac{Q_{0k|u}^{1-\lambda} Q_{1k|u}^{\lambda}}{\sum_{k \in \mathcal{K}} Q_{0k|u}^{1-\lambda} Q_{1k|u}^{\lambda}} \tag{B.7}$$

with $0 \leq \lambda \leq 1$*, and the decision threshold* T *in* (B.4) *is chosen equal to* $D(Q_\lambda || Q_1|p) - D(Q_\lambda || Q_0|p)$.

Proof: See [33, Theorem 4.5.2 & Theorem 4.5.3]. ■

Define

$$E(R, p, Q_0, Q_1) \triangleq \min_{D(\hat{Q}||Q_1|p) \leq R} D(\hat{Q}||Q_0|p), \tag{B.8}$$

where the minimization is over all distributions $\hat{Q}$ satisfying $D(\hat{Q}||Q_1|p) \leq R$. Applying the Lagrange multiplier method, it is shown in [33, Theorem 4.6.3] that the optimal distribution $\hat{Q}$ minimizing $D(\hat{Q}||Q_0|p)$ under the constraint $D(\hat{Q}||Q_1|p) \leq R$ is the tilted distribution Q_λ with λ satisfying $D(Q_\lambda||Q_1|p) = R$. Therefore, Lemma B.2 can also be written as

$$e^{-NE(R,p,Q_0,Q_1)-o(N)} \leq \alpha^{NP} \leq e^{-NE(R,p,Q_0,Q_1)} \tag{B.9a}$$

$$e^{-NR-o(N)} \leq \beta^{NP} \leq e^{-NR}, \tag{B.9b}$$

and the decision threshold T in (B.4) is chosen equal to $T = R - E(R, p, Q_0, Q_1)$. In addition, applying the Lagrange multiplier method for the minimization in (B.8), $E(R, p, Q_0, Q_1)$ can be simplified to

$$E(R, p, Q_0, Q_1) = \max_{s \geq 0} \Big[- sR - (1+s) \sum_{u \in \mathcal{U}} p_u \log \big(\sum_{k \in \mathcal{K}} Q_{0k|u}^{\frac{1}{1+s}} Q_{1k|u}^{\frac{s}{1+s}} \big) \Big]. \tag{B.10}$$

A proof for this can be found in [33, Theorem 4.6.4].

B.2 Error Exponent Upper Bound with User 2's Input Fixed

Consider two codebooks $CB_1 = \{C_{1,1}, \ldots, C_{1,M_1}\}$ and $CB_2 = \{C_{2,1}, \ldots, C_{2,M_2}\}$ with codeword length N for user 1 and user 2 in the DMMAC. Denote $C_{1,m}(l)$ and $C_{2,n}(l)$ as the l^{th} elements in the codewords $C_{1,m}$ and $C_{2,n}$, respectively. Define the compositions $p(C_{1,m})$ and $p(C_{2,n})$ of the codewords $C_{1,m}$ and $C_{2,n}$ as

$$p_i(C_{1,m}) \triangleq \frac{|\{l : \ C_{1,m}(l) = i\}|}{N} \tag{B.11a}$$

$$p_j(C_{2,n}) \triangleq \frac{|\{l : \ C_{2,n}(l) = j\}|}{N}, \tag{B.11b}$$

and the (joint) composition $p(C_{1,m}, C_{2,n})$ of $C_{1,m}$ and $C_{2,n}$ as

$$p_{ij}(C_{1,m}, C_{2,n}) \triangleq \frac{|\{l : \ C_{1,m}(l) = i, C_{2,n}(l) = j\}|}{N}. \tag{B.12}$$

Clearly, $p_i(C_{1,m}) = \sum_{j \in \mathcal{J}} p_{ij}(C_{1,m}, C_{2,n})$ and $p_j(C_{2,n}) = \sum_{i \in \mathcal{I}} p_{ij}(C_{1,m}, C_{2,n})$.

Now assume that the input for user 2 is fixed with some codeword C_{2,n^*} and this information is known at the receiver (see Fig. B.1). We derive an error exponent upper bound for user 1 in this channel under an additional assumption that the joint compositions of the codewords $C_{1,m}$'s and the codeword C_{2,n^*} are all the same, i.e.,

$$p(C_{1,m}, C_{2,n^*}) = p \tag{B.13}$$

for some composition p, where $1 \leq m \leq M_1$. We summarize the result in the following Lemma.

Lemma B.3 *For the user 2's input fixed DMMAC with a constant joint composition codebook CB_1, i.e., the joint composition of every codeword $C_{1,m}$ and user 2's input*

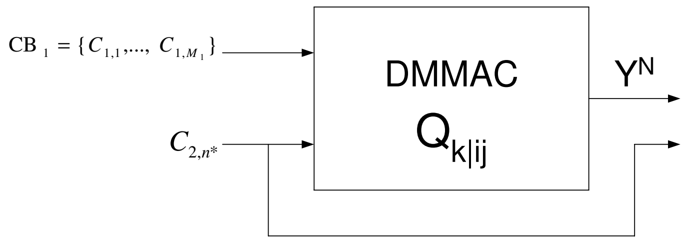


Figure B.1: DMMAC with user 2's input fixed.

C_{2,n^*} is the same composition p, the probability of error for user 1 is lower bounded by

$$P_{e1,n^*} = \frac{1}{M_1} \sum_{m=1}^{M_1} P_{e1,mn^*} \geq e^{-NE(R_1,p,Q)-o(N)}, \tag{B.14}$$

where $E(R_1, p, Q)$ is defined as

$$E(R_1, p, Q) \triangleq \min_{q_{k|j}} \max_{s\geq 0} \Big[- sR_1 - (1+s) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log \big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}} \big) \Big]. \tag{B.15}$$

Moreover, we have the following equality

$$\min_{q_{k|j}} \max_{s\geq 0} \Big[- sR_1 - (1+s) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log \big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}} \big) \Big] =$$
$$\max_{s\geq 0} \min_{q_{k|j}} \Big[- sR_1 - (1+s) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log \big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}} \big) \Big], \tag{B.16}$$

which means that we can exchange the order of min *and* max *in* (B.15).

Proof: The proof basically follows the derivation of the sphere packing bound given in [33, Sec. 10.1 & Sec. 10.2].

Given any conditional probability distribution $q = \{q_{k|j} | \sum_{k\in\mathcal{K}} q_{k|j} = 1, \forall j \in \mathcal{J}\}$, construct a probability distribution q^N on the channel output Y^N as

$$q^N(Y^N) = \prod_{l=1}^{N} q_{Y_l|C_{2,n^*}(l)}. \tag{B.17}$$

Suppose $\{D_m :\ m = 1, \ldots, M_1\}$ is the partition of the output space $\mathcal{K}^N$ into M_1 decision regions. Select a decision region $\mathcal{D}_{m^*}$ such that $q^N(\mathcal{D}_{m^*}) \leq \frac{1}{M_1}$. Consider the following binary hypothesis testing

$$\begin{aligned} H_0 :\ P(\mathbf{Y}^N = Y^N|H_0) &= \prod_{l=1}^{N} Q_{Y_l|C_{1,m^*}(l)C_{2,n^*}(l)} \\ H_1 :\ P(\mathbf{Y}^N = Y^N|H_1) &= q^N(Y^N) = \prod_{l=1}^{N} q_{Y_l|C_{2,n^*}(l)}. \end{aligned} \tag{B.18}$$

The probabilities of type I and type II errors based on decision region $\mathcal{D}_{m^*}$ are

$$\begin{aligned} \alpha &= \sum_{Y^N \notin \mathcal{D}_{m^*}} P(\mathbf{Y}^N = Y^N|H_0) \\ \beta &= \sum_{Y^N \in \mathcal{D}_{m^*}} P(\mathbf{Y}^N = Y^N|H_1) = q^N(\mathcal{D}^*_m) \leq \frac{1}{M_1}. \end{aligned} \tag{B.19}$$

Note that $\alpha = P_{e1,m^*n^*}$. From Lemma B.2, the probabilities of type I and type II errors based on Neymain-Pearson decision region $\mathcal{D}^{NP}$ are

$$\begin{aligned} e^{-NE(R_1,p,Q,q)-o(N)} &\leq \alpha^{NP} \leq e^{-NE(R_1,p,Q,q)} \\ e^{-NR_1-o(N)} &\leq \beta^{NP} \leq e^{-NR_1}, \end{aligned} \tag{B.20}$$

where the decision threshold T in (B.4) is chosen equal to $T = R_1 - E(R_1, p, Q, q)$. Combining (B.19) and (B.20), we have

$$\beta^{NP} \cong \frac{1}{M_1} \geq \beta, \tag{B.21}$$

which implies

$$P_{e1,m^*n^*} = \alpha \gtrsim \alpha^{NP} \geq e^{-NE(R_1,p,Q,q)-o(N)} \tag{B.22}$$

by Lemma B.1. Since the conditional distribution q is chosen arbitrarily, we can choose q to maximize the right hand side of (B.22), so

$$P_{e1,m^*n^*} \geq e^{-N\min_{q_{k|j}} E(R_1,p,Q,q)-o(N)}. \tag{B.23}$$

From (B.10) and (B.15), we have

$$E(R_1, p, Q) = \min_{q_{k|j}} E(R_1, p, Q, q), \tag{B.24}$$

so (B.23) can also be written as

$$P_{e1,m^*n^*} \geq e^{-NE(R_1,p,Q)-o(N)}. \tag{B.25}$$

A standard technique used in single-user channels is that an error exponent upper bound for the *maximum* probability of error of the codebook is also an error exponent upper bound for the *average* probability of error of the codebook. A proof for this can be found in [33, Sec. 10.6]. Therefore, (B.25) implies that

$$P_{e1,n^*} \geq e^{-NE(R_1,p,Q)-o(N)}. \tag{B.26}$$

At last, we need to prove the equality in (B.16). For simplicity, define

$$f_R(q, s) \triangleq -sR - (1+s) \sum_{i\in\mathcal{I},j\in\mathcal{J}} p_{ij} \log\Big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}}\Big). \tag{B.27}$$

Note that from (B.10), (B.15) and (B.27), we have

$$\begin{aligned} E(R,p,Q) &= \min_{q_{k|j}} E(R,p,Q,q) \\ E(R,p,Q,q) &= \max_{s\geq 0} f_R(q,s). \end{aligned} \tag{B.28}$$

Assume

$$(\breve{q},\breve{s}) = \arg\ \min_{q_{k|j}} \max_{s\geq 0} f_{R_1}(q,s), \tag{B.29}$$

i.e.,

$$E(R_1,p,Q) = E(R_1,p,Q,\breve{q}) = f_{R_1}(\breve{q},\breve{s}). \tag{B.30}$$

We want to show that $(\breve{q},\breve{s})$ is a saddle point, i.e., $f_{R_1}(\breve{q},\breve{s})$ satisfies

$$f_{R_1}(\breve{q},s) \leq f_{R_1}(\breve{q},\breve{s}) \leq f_{R_1}(q,\breve{s}) \tag{B.31}$$

for all $s \geq 0$ and conditional probabilities q. This implies that

$$\min_{q_{k|j}} \max_{s\geq 0} f_{R_1}(q,s) = \max_{s\geq 0} \min_{q_{k|j}} f_{R_1}(q,s) \tag{B.32}$$

as claimed in (B.16).

From (B.28) and (B.30), we have

$$f_{R_1}(\breve{q},s) \leq \max_{s\geq 0} f_{R_1}(\breve{q},s) = E(R_1,p,Q,\breve{q}) = f_{R_1}(\breve{q},\breve{s}), \tag{B.33}$$

so it remains to prove that $f_{R_1}(\breve{q},\breve{s}) \leq f_{R_1}(q,\breve{s})$.

It is shown in [33, Theorem 4.6.2] and in [33, Section 10.1] that $E(R,p,Q,q)$ and

$E(R,p,Q)$ are convex, nonincreasing and continuous as a function of R. $E(R,p,Q)$ is convex, so there is a line tangent to $E(R,p,Q)$ at the value of R_1. Since $E(R,p,Q) = \min_{q_{k|j}} E(R,p,Q,q) \leq E(R,p,Q,\breve{q})$ and $E(R_1,p,Q) = E(R_1,p,Q,\breve{q})$, this line is also tangent to $E(R,p,Q,\breve{q})$ at the value of R_1 (see Fig. B.2). From [33, Theorem 4.6.4], the slope of the line tangent to $E(R,p,Q,\breve{q})$ at the value of R_1 is $-\breve{s}$. Hence, we have the following inequality

$$E(R_1,p,Q) - \breve{s}(R - R_1) \leq E(R,p,Q) \tag{B.34}$$

for all $R \geq 0$.

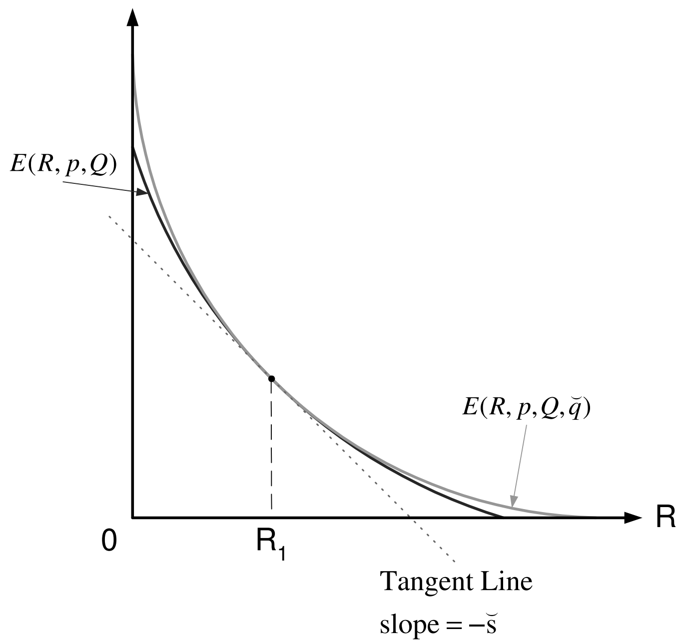


Figure B.2: $E(R,p,Q,\breve{q})$ and $E(R,p,Q)$.

Define the tilted distribution $\Lambda(Q,q,s)$ as

$$\Lambda_{k|ij}(Q,q,s) \triangleq \frac{Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}}}{\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}}}. \tag{B.35}$$

Define the rate function $\mathtt{R}(p, Q, q, s)$ as

$$\mathtt{R}(p, Q, q, s) \triangleq D(\Lambda(Q, q, s)||q|p). \tag{B.36}$$

From (B.8), (B.10) and (B.28), we have

$$\begin{aligned}
E(\mathtt{R}(p, Q, q, s), p, Q) &= \min_{\hat{q}_{k|j}} E(\mathtt{R}(p, Q, q, s), p, Q, \hat{q}) \\
&= \min_{\hat{q}_{k|j}} \min_{D(\hat{Q}||\hat{q}|p) \le \mathtt{R}(p,Q,q,s)} D(\hat{Q}||Q|p) \\
&\le \min_{D(\hat{Q}||q|p) \le \mathtt{R}(p,Q,q,s)} D(\hat{Q}||Q|p) \\
&\le D(\Lambda(Q, q, s)||Q|p)
\end{aligned} \tag{B.37}$$

for all $s \ge 0$ and conditional probabilities q, where the last inequality is because the probability distribution $\Lambda(Q, q, s)$ satisfies the constraint in the minimization ($D(\Lambda(Q, q, s)||q|p) = \mathtt{R}(p, Q, q, s)$ by definition). Note that $f_{R_1}(q, s)$ can also be written as

$$\begin{aligned}
f_{R_1}(q, s) &= -sR_1 + D(\Lambda(Q, q, s)||Q|p) + sD(\Lambda(Q, q, s)||q|p) \\
&= D(\Lambda(Q, q, s)||Q|p) - s(R_1 - \mathtt{R}(p, Q, q, s)),
\end{aligned} \tag{B.38}$$

where the first equality is obtained by substituting the condition discrimination function defined in (B.5) into the right hand side of the equation. Combining (B.37) and (B.38), we have

$$\begin{aligned}
&E(\mathtt{R}(p, Q, q, \breve{s}), p, Q) - \breve{s}(R_1 - \mathtt{R}(p, Q, q, \breve{s})) \\
&\qquad \le D(\Lambda(Q, q, \breve{s})||Q|p) - \breve{s}(R_1 - \mathtt{R}(p, Q, q, \breve{s})) \\
&\qquad = f_{R_1}(q, \breve{s}).
\end{aligned} \tag{B.39}$$

From (B.30), (B.34) and (B.39), we have

$$\begin{aligned} f_{R_1}(\breve{q}, \breve{s}) &= E(R_1, p, Q) \\ &\le E(\mathtt{R}(p, Q, q, \breve{s}), p, Q) - \breve{s}(R_1 - \mathtt{R}(p, Q, q, \breve{s})) \\ &\le f_{R_1}(q, \breve{s}). \end{aligned} \tag{B.40}$$

This completes the proof. ■

B.3 Proof of Theorem 9.1

For any two codebooks $CB_1 = \{C_{1,1}, \dots, C_{1,M_1}\}$ and $CB_2 = \{C_{2,1}, \dots, C_{2,M_2}\}$ with codeword length N, there are less than $(N+1)^{|\mathcal{I}||\mathcal{J}|}$ different joint compositions of the codeword pairs. Let p be the most frequently occurring joint composition, i.e.,

$$\begin{aligned} &|\{(m,n) : p(C_{1,m}, C_{1,n}) = \tilde{p}, 1 \le m \le M_1, 1 \le n \le M_2\}| \le \\ &\qquad |\{(m,n) : \ p(C_{1,m}, C_{1,n}) = p, 1 \le m \le M_1, 1 \le n \le M_2\}| \end{aligned} \tag{B.41}$$

for all joint compositions $\tilde{p}$. Define

$$\Phi \triangleq \{(m,n) : \ p(C_{1,m}, C_{1,n}) = p, 1 \le m \le M_1, 1 \le n \le M_2\}, \tag{B.42}$$

i.e., Φ is the (index) set of codeword pairs with joint composition p.

Since

$$\begin{aligned}M_3 &= \sum_{\tilde{p}} |\{(m,n):\ p(C_{1,m}, C_{1,n}) = \tilde{p}, 1 \le m \le M_1, 1 \le n \le M_2\}| \\ &\le \sum_{\tilde{p}} |\Phi| \\ &\le (N+1)^{|\mathcal{I}||\mathcal{J}|}|\Phi|,\end{aligned} \tag{B.43}$$

we have $|\Phi| \cong M_3$.

Define

$$\begin{aligned}\Psi_n &\triangleq \{m:\ (m,n) \in \Phi, 1 \le m \le M_1\} \\ \Upsilon &\triangleq \{n:\ |\Psi_n| \cong M_1, 1 \le n \le M_2\},\end{aligned} \tag{B.44}$$

i.e., Ψ_n is the (index) set of codewords $C_{1,m}$'s in CB_1 with joint composition p with the codeword $C_{2,n}$, and Υ is the (index) set such that each codeword $C_{2,n}$ in Υ has roughly M_1 codewords $C_{1,m}$'s in CB_1 with joint composition p. We claim that $|\Upsilon| \cong M_2$. Otherwise, there exist some positive numbers ϵ and δ such that

$$\begin{aligned}|\Phi| &= \sum_{n=1}^{M_2} |\Psi_n| \\ &= \sum_{n\in\Upsilon} |\Psi_n| + \sum_{n\notin\Upsilon} |\Psi_n| \\ &\le \sum_{n\in\Upsilon} e^{NR_1} + \sum_{n\notin\Upsilon} e^{N(R_1-\delta)} \\ &\le e^{N(R_2-\epsilon)}e^{NR_1} + e^{NR_2}e^{N(R_1-\delta)} \\ &\le 2e^{N(R_1+R_2-\min\{\epsilon,\delta\})},\end{aligned} \tag{B.45}$$

but this contradicts $|\Phi| \cong M_3$.

Now focus on the average error probability P_{e1} for user 1. Further, we consider a genie-aided system such that the index value of the transmitted codeword for user 2 is notified at the receiver by the genie. Denote the probability of error for user 1 in this new genie-aided system as $P'_{e1,mn}$ when the codewords $C_{1,m}$ and $C_{2,n}$ are transmitted. Clearly, the average error probability for user 1 in the new genie-aided system is no greater than that in the original DMMAC, since the extra information from the genie can only improve the performance for user 1. Thus the average error probability for user 1 can be lower bounded by

$$
\begin{aligned}
P_{e1} &= \frac{1}{M_1 M_2} \sum_{n=1}^{M_2} \sum_{m=1}^{M_1} P_{e1,mn} && \text{(B.46a)} \\
&\geq \frac{1}{M_1 M_2} \sum_{n=1}^{M_2} \sum_{m=1}^{M_1} P'_{e1,mn} && \text{(B.46b)} \\
&\geq \frac{1}{M_1 M_2} \sum_{n\in\Upsilon} \sum_{m\in\Psi_n} P'_{e1,mn} && \text{(B.46c)} \\
&\gtrsim \frac{1}{M_1 M_2} \sum_{n\in\Upsilon} M_1 e^{-NE(R_1,p,Q)} && \text{(B.46d)} \\
&\cong \frac{1}{M_1 M_2} M_2 M_1 e^{-NE(R_1,p,Q)} && \text{(B.46e)} \\
&= e^{-NE(R_1,p,Q)}, && \text{(B.46f)}
\end{aligned}
$$

where the inequality "$\gtrsim$" in (B.46d) is due to Lemma B.3 by noting that $|\Psi_n| \cong M_1$ for all $n \in \Upsilon$, and the equality "$\cong$" in (B.46e) is due to $|\Upsilon| \cong M_2$.

Hence we have proved that

$$
E_1 \leq \max_{s\geq 0} \min_{q_{k|j}} \Big[-sR_1 - (1+s) \sum_{i\in\mathcal{I}, j\in\mathcal{J}} p_{ij} \log \big(\sum_{k\in\mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|j}^{\frac{s}{1+s}} \big) \Big]. \tag{B.47}
$$

The proof for

$$E_2 \leq \max_{s \geq 0} \min_{q_{k|i}} \Big[-sR_2 - (1+s) \sum_{i \in \mathcal{I}, j \in \mathcal{J}} p_{ij} \log \big(\sum_{k \in \mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_{k|i}^{\frac{s}{1+s}} \big) \Big] \tag{B.48}$$

is similar.

Finally, if we allow the two transmitters in the DMMAC to cooperate, we have a single-user channel whose probability of error P_e'' should be less than or equal to the probability of system error $P_{e,sys}$ in the original DMMAC. Thus

$$\begin{aligned} \min\{E_1, E_2\} &= E_{sys} \\ &\leq E'' \\ &\leq \max_{s \geq 0} \min_{q_k} \Big[-s(R_1+R_2) - (1+s) \sum_{i \in \mathcal{I}, j \in \mathcal{J}} p_{ij} \log \big(\sum_{k \in \mathcal{K}} Q_{k|ij}^{\frac{1}{1+s}} q_k^{\frac{s}{1+s}} \big) \Big] \end{aligned} \tag{B.49}$$

where E'' is the error exponent of the transmitter-cooperate channel, and the last inequality can be derived by applying [33, Theorem 10.1.5]. This completes the proof of Theorem 9.1 by combining (B.47), (B.48), (B.49), and noting that the $EER(R_1, R_2)$ given in Theorem 9.1 is a union of the above three upper bounds over all joint compositions p.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] V. K. Garg, *IS-95 CDMA and cdma2000.* Prentice-Hall, 2000.

[2] M. R. Karim and M. Sarraf, *W-CDMA and cdma2000 for 3G Mobile Networks.* McGraw-Hill, 2002.

[3] L. F. Wei, "Coded modulation with unequal error protection," *IEEE Trans. Communications*, vol. 41, no. 10, pp. 1439–1449, Oct. 1993.

[4] S. Gadkari and K. Rose, "Time-division versus superposition coded modulation schemes for unequal error protection," *IEEE Trans. Communications*, vol. 47, no. 3, pp. 370–379, Mar. 1999.

[5] A. Feinstein, "Error bounds in noisy channels without memory," *IEEE Trans. Information Theory*, vol. 1, no. 2, pp. 13–14, Sept. 1955.

[6] P. Elias, "Coding for noisy channels," *IRE Convention Record*, no. 4, pp. 37–46, 1955.

[7] C. E. Shannon, "Probability of error for optimal codes in a Gaussian channel," *Bell System Tech. J.*, vol. 38, pp. 611–656, 1959.

[8] R. M. Fano, *Transmission of Information: A Statistical Theory of Communication.* MIT Press, 1961.

[9] R. L. Dobrusin, "Asymptotic bounds of the probability of error for the transmission of messages over a discrete memoryless channel with a symmetric transition probability matrix," *Teor. Veroyatnost. i Primenen*, vol. 7, pp. 283–311, 1962.

[10] R. G. Gallager, "A simple derivation of the coding theorem and some applications," *IEEE Trans. Information Theory*, vol. 11, no. 1, pp. 3–18, Jan. 1965.

[11] C. E. Shannon, R. G. Gallager, and E. R. Berlekamp, "Lower bounds to error probability for coding on discrete memoryless channels," *Inform. and Control*, vol. 10, pp. 65–103 (Part I), 522–552 (Part II), 1967.

[12] T. M. Cover, "Broadcast channels," *IEEE Trans. Information Theory*, vol. 18, no. 1, pp. 2–14, Jan. 1972.

[13] P. P. Bergmans, “A simple converse for braodcast channels with additive white Gaussian noise,” *IEEE Trans. Information Theory*, vol. 20, no. 2, pp. 279–280, Mar. 1974.

[14] R. Ahlswede, “Multi-way communication channels,” in *Proc. International Symposium on Information Theory*, Tsahkadsor, Armenian S.S.R., 1971.

[15] H. Liao, “Multiple access channels,” Ph.D. dissertation, University of Hawaii, Honolulu, 1972.

[16] L. Zheng and D. N. C. Tse, “Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels,” *IEEE Trans. Information Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[17] R. G. Gallager, “A perspective on multiaccess channels,” *IEEE Trans. Information Theory*, vol. 31, no. 2, pp. 124–142, Mar. 1985.

[18] T. Guess and M. K. Varanasi, “Error exponents for maximum-likelihood and successive decoder for the Gaussian CDMA channel,” *IEEE Trans. Information Theory*, vol. 46, no. 4, pp. 1683–1691, July 2000.

[19] D. N. C. Tse, P. Viswanath, and L. Zheng, “Diversity-multiplexing tradeoff for multiple-access channels,” *IEEE Trans. Information Theory*, vol. 50, no. 9, pp. 1859–1874, Sept. 2004.

[20] E. Telatar, “Capacity of multi-antenna Gaussian channels,” *European Trans. Telecommun.*, vol. 10, no. 6, pp. 585–595, November-December 1999.

[21] J. G. Proakis, *Digital Communications*, 4th ed. New York: McGraw-Hill, 2001.

[22] J. Korner and A. Sgarro, “Universal attainable error exponents for broadcast channels with degraded message sets,” *IEEE Trans. Information Theory*, vol. 26, no. 6, pp. 670–679, Nov. 1980.

[23] S. N. Diggavi, N. Al-Dhahir, and A. R. Calderbank, “Diversity-embedded space-time codes,” in *Proc. Globecom Conf.*, San Francisco, CA, Dec. 2003.

[24] ——, “Diversity embedding in multiple antenna communications,” in *DIMACS Workshop Network Information Theory*, Sept. 2004, pp. 285–302.

[25] R. G. Gallager, *Information Theory and Reliable Communication.* John Wiley & Sons, 1968.

[26] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* New York: John Wiley & Sons, 1991.

[27] H. Sato, “An outer bound on the capacity region of broadcast channels,” *IEEE Trans. Information Theory*, vol. 24, no. 3, pp. 374–377, May 1978.

[28] G. A. Kabatyanskii and V. I. Levenshtein, "Bounds for packings on a sphere and in the space," *Problemy Peredachi Informatsii*, vol. 14, no. 1, pp. 3–25, 1978.

[29] A. E. Ashikhmin, A. Barg, and S. N. Litsyn, "A new upper bound on the reliability funcion of the Gaussian channel," *IEEE Trans. Information Theory*, vol. 46, no. 6, pp. 1945–1961, Sept. 2000.

[30] R. A. Rankin, "The closest packing of spherical caps in n dimensions," *Proc. Glasgow Math. Assoc.*, vol. 2, pp. 139–144, 1955.

[31] T. M. Cover and A. A. E. Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Information Theory*, vol. 25, no. 5, pp. 572–584, Sept. 1979.

[32] E. A. Arutyunyan, "Lower bound for the error probability of multiple-access channels," *Problemy Peredachi Informatsii*, vol. 11, no. 2, pp. 23–36, Apr. 1975.

[33] R. E. Blahut, *Principles and Practice of Information Theory.* New York: Addison-Wesley, 1987.

[34] G. Dueck, "Maximal error capacity regions are smaller than average error capacity regions for multi-user channels," *Problems of Control and Information Theory*, vol. 7, no. 1, pp. 11–19, 1978.