Variable-length codes for channels with memory and feedback: error-exponent lower bounds

Achilleas Anastasopoulos and Jui Wu

Abstract

The reliability function of memoryless channels with noiseless feedback and variable-length coding has been found to be a linear function of the average rate in the classic work of Burnashev. In this work we consider unifilar channels with noiseless feedback and study specific transmission schemes, the performance of which provides lower bounds for the channel reliability function. In unifilar channels the channel state evolves in a deterministic fashion based on the previous state, input, and output, and is known to the transmitter but is unknown to the receiver. We consider two transmission schemes with increasing degree of complexity. The first one is a single-stage sequential transmission scheme, where both transmitter and receiver summarize their common information in an M-dimensional vector with elements in the state space of the unifilar channel and an M-dimensional probability mass function, with M being the number of messages. The second one is a two-stage transmission scheme where the first stage is as described above, and the second stage, which is entered when one of the messages is sufficiently reliable, is resolving a binary hypothesis testing problem. This stage is different from the one employed by Burnashev for discrete memoryless channels. The analysis assumes the presence of some common randomness shared by the transmitter and receiver, and is based on the study of the log-likelihood ratio of the transmitted message posterior belief, and in particular on the study of its multi-step drift. Simulation results for the trapdoor, chemical, and other binary input/state/output unifilar channels confirm that the bounds are tight compared to the upper bounds derived in a companion paper.

I. Introduction

There exists a substantial body of literature on transmission schemes for memoryless channels with noiseless feedback. Horstein [1] proposed a simple sequential transmission scheme which is capacity-achieving and provides larger error exponents than traditional fixed-length block-coding for discrete memoryless channels (DMCs). Similarly, Schalkwijk and Kailath [2] showed that capacity and a double exponentially decreasing error probability can be achieved by a simple sequential transmission scheme for the additive white Gaussian noise (AWGN) channel with average power constraint. A remarkable result was derived by Burnashev in [3], where error exponent matching upper and lower bounds were derived for DMCs with feedback and variable-length codes. The error exponent has a simple form $E(\overline{R}) = C_1(1 - \overline{R}/C)$, where $\overline{R}$ is the average rate, C is the channel capacity and C_1 is the maximum divergence that can be obtained in the channel for a binary hypothesis testing problem. Recently, Shayevitz and Feder [4] identified an underlying principle shared by the aforementioned schemes and introduced a simple encoding scheme, namely the posterior matching (PM) scheme for general memoryless channels and showed that it achieves capacity. The above transmission schemes can be contrasted to those inspired by the work in [5] where a variable-length transmission scheme was proposed and its error exponent was found to achieve the Burnashev upper bound. This scheme (and others inspired by it) is not explicit in the sense that it assumes that some unspecified capacity-achieving codes are used in the "communication" stage of the transmission.

For channels with memory and noiseless feedback, there exists a rich literature on the capacity characterization [6], [7], [8], [9], [10]. Recently, the capacity of the trapdoor channel was found in closed form in [9], and extended to a subset of chemical channels in [11], while the capacity of the binary unit memory channel on the output was found in closed form in [12]. A number of "explicit" transmission schemes have been recently studied in the literature [13], [14] but no results on error exponents are reported. In the case of channels with memory and feedback, an error exponent analysis is performed in [10] for fixed length coding. The only work that studies error exponents for variable-length codes for channels with memory and feedback is [15] where the authors consider a finite state channels with channel state known causally to both the transmitter and the receiver. The transmission scheme presented therein is inspired by that of [5] and as a result it is based on an otherwise unspecified capacity-achieving code for this channel.

In this work, we consider channels with memory and feedback, and propose and analyze variable-length transmission schemes. We specifically look at unifilar channels since for this family, the capacity has been characterized in an elegant way through the use of Markov decision processes (MDPs) [9]. We consider two transmission schemes with increasing degree of complexity. The first one is a single-stage sequential transmission scheme, similar to the one proposed in [3]. The encoding is a time-invariant function that depends on a summary of the available common information between the transmitter and the receiver in the form of two M-dimensional vectors: one is the vector of current states conditioned on each message and the other is the posterior probability mass function of the message given the observation (with M being the number of messages). The second one is a two-stage transmission scheme where the first stage is as described above, and the second stage, which is entered when one of the messages is sufficiently reliable, is resolving a binary hypothesis testing problem much like the original scheme of Burnashev. Following the hints from our upper bound analysis in the companion paper [16], the second stage employs a more sophisticated transmission scheme (compared to that for DMCs) in order to achieve the error-exponent upper bound. The analysis assumes the presence of some common randomness shared by the transmitter and

The authors are with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI, 48105 USA e-mail: {anastas, juiwu}@umich.edu

receiver[1], and is based on the study of the log-likelihood ratio of the transmitted message posterior belief, and in particular on the study of its multi-step drift. Our results are derived under an additional assumption/conjecture on the concentration of an appropriately defined Markov process. Although this part is currently unresolved, we provide strong evidence supporting this assumption through numerical evaluations. Simulation results for the trapdoor, chemical, and other binary input/state/output unifilar channels confirm that the bounds are tight compared to the upper bounds derived in [16].

The main difference between our work and that in [15] is that for unfilar channels, the channel state is not observed at the receiver. In addition, in this work, an "explicit" transmission scheme is proposed and analyzed.

The remaining part of this paper is organized as follows. In section II, we describe the channel model for the unifilar channel and its capacity characterization. In section III, we propose and analyze a single-stage transmission scheme with common randomness. In section IV, we propose and analyze a two-stage transmission scheme with common randomness. Section V presents numerical evidence for the performance of the proposed schemes for several unifilar channels. Final conclusions are given in section VI. **All proofs are omitted due to space limitations. They can be found on the extended version of the paper [17].**

II. Channel Model and preliminaries

Consider a family of finite-state point-to-point channels with inputs $X_t \in \mathcal{X}$, output $Y_t \in \mathcal{Y}$ and state $S_t \in \mathcal{S}$ at time t, with all alphabets being finite and S_1 known to both the transmitter and the receiver. The channel conditional probability is

$$P(Y_t, S_{t+1}|X^t, Y^{t-1}, S^t) = Q(Y_t|X_t, S_t)\delta_{g(S_t,X_t,Y_t)}(S_{t+1}), \tag{1}$$

for a given stochastic kernel $Q \in \mathcal{X} \times \mathcal{S} \to \mathcal{P}(\mathcal{Y})$ and deterministic function $g \in \mathcal{S} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{S}$, where $\mathcal{P}(\mathcal{Y})$ denotes the space of all probability measure on $\mathcal{Y}$, and $\delta_{\cdot}(\cdot)$ is the Kronecker delta function.

This family of channels is referred to as unifilar channels [9]. The authors in [9] have derived the capacity C in the form of

$$C = \lim_{N\to\infty} \sup_{\{p(x_t|s_t,y^{t-1},s_1)\}_{t\geqslant 1}} \frac{1}{N}\sum_{i=1}^{N} I(X_t, S_t; Y_t|Y^{t-1}, S_1). \tag{2}$$

The capacity can be written as an optimal reward per unit time of an appropriately defined MDP [9], [11]. For channels with ergodic behavior, the capacity has a single-letter expression

$$C = \sup_{P_{X|SB}} I(X_t, S_t; Y_t|B_{t-1}), \tag{3}$$

where $B_{t-1} \in \mathcal{P}(\mathcal{S})$ is the posterior belief on the current state given (Y^{t-1}, S_1) at time t and the mutual information is evaluated using the distribution

$$\begin{aligned} &P(X_t, S_t, Y_t, B_{t-1}) \\ &\quad = Q(Y_t|X_t, S_t)P_{X|SB}(X_t|S_t, B_{t-1})B_{t-1}(S_t)\pi_B(B_{t-1}), \end{aligned} \tag{4}$$

where π_B is the stationary distribution of the Markov chain $\{B_{t-1}\}$ with transition kernel

$$\begin{aligned} &P(B_t|B_{t-1}) \\ &\quad = \sum_y \delta_{\phi(B_{t-1},y)}(B_t)\sum_{x,s} Q(y|x,s)P(x|s,B_{t-1})B_{t-1}(s), \end{aligned} \tag{5}$$

where the update function $B_t = \phi(B_{t-1}, Y_t)$ is defined through the recursion

$$\begin{aligned} &B_t(s) \\ &\quad = \frac{\sum_{x,\hat{s}} \delta_{g(\hat{s},x,Y_t)}(s)Q(Y_t|x,\hat{s})P_{X|SB}(x|\hat{s}, B_{t-1})B_{t-1}(\hat{s})}{\sum_{x,\hat{s}} Q(Y_t|x,\hat{s})P_{X|SB}(x|\hat{s}, B_{t-1})B_{t-1}(\hat{s})}. \end{aligned} \tag{6}$$

In this paper, we restrict our attention to such channels with strictly positive $Q(y|x,s)$ for any $(y,x,s) \in \mathcal{Y} \times \mathcal{X} \times \mathcal{S}$ and ergodic behavior so that the above capacity characterization is indeed valid.

III. A single-stage transmission scheme

In this section, we adapt Burnashev's scheme [3] to unifilar channels. To introduce randomness into our encoding strategy, in addition to the structure described in section II, the transmitter and the receiver have access to a common set of random variables $\{V_t\}_{t\geqslant 1}$ in a causal way.

[1] It is interesting to note that the analysis of Burnashev in [3] also assumes that common randomness is present although this is not explicitly stated.

A. Transmission scheme

Let $W \in \{1, 2, 3, \cdots, M = 2^K\}$ and P_e be the message to be transmitted and the target error probability. In this system, the transmitter receives perfect feedback of the output with unit delay and determines the input $X_t = e_t(W, Y^{t-1}, V^t, S_1)$ with (deterministic) encoding strategies $\{e_t\}_{t \geqslant 1}$ at time t. Observe that the encoding strategies are deterministic and this is a crucial part of the analysis, since it allows the receiver to have an estimate of the current state and input for any hypothesized message. The common random variables are utilized to induce the appropriate maximizing input distribution. Define the filtration $\{\mathcal{F}_t = (Y^t, V^t, S_1)\}_{t \geqslant 1}$ and define the posterior probability of the massage, Π_{t-1} as

$$\Pi_{t-1}(i) = P(W = i|\mathcal{F}_{t-1}) \qquad \forall i \in \{1, 2, \cdots, M\}, \tag{7}$$

We now specify in more detail the encoding strategy. Suppose we are given a collection of input distributions $\{P_X(\cdot|s, b) \in \mathcal{P}(\mathcal{X})\}_{s,b \in \mathcal{S} \times \mathcal{P}(\mathcal{S})}$ that maximize (3). We define the vector $\underline{S}_t = (S_t^i)_{i=1}^M$, where S_t^i is the state at time t conditioning on $W = i$ and thus can be written as $S_t^i = g_t(i, S_1, Y^{t-1}, V^{t-1})$ since deterministic strategies are used. We further define for any $s \in \mathcal{S}$ and $i \in \{1, \cdots, M\}$ the quantities

$$\hat{B}_{t-1}(s) = \sum_{i=1}^{M} \Pi_{t-1}(i) 1_{\{S_t^i = s\}} \tag{8}$$

$$\Pi_{t-1}^s(i) = \frac{\Pi_{t-1}(i) 1_{\{S_t^i = s\}}}{\hat{B}_{t-1}(s)}, \tag{9}$$

which are almost surely equal to $P(S_t = s|\mathcal{F}_{t-1})$ and $P(W = i|S_t = s, \mathcal{F}_{t-1})$, respectively. The common random variables $\{V_t = (V_t^1, \cdots, V_t^{|\mathcal{S}|}) \in [0, 1]^{|\mathcal{S}|}\}$ are generated as

$$P(V_t|V^{t-1}, X^{t-1}, S^t, Y^{t-1}, W) = \prod_{i=1}^{|\mathcal{S}|} u(V_t^i), \tag{10}$$

where $u(\cdot)$ denotes the uniform distribution. The transmission scheme is a generalization of Burnashev's scheme [3] and also very similar to the PM scheme [4]. First, based on the quantity $\Pi_{t-1}, \underline{S}_t, \hat{B}_{t-1}$, and $S_t^W = S_t$ the conditional distribution on the state is evaluated as in (9). Then the input signal X_t is generated exactly as in DMC from $\Pi_{t-1}^{S_t^W}$ to "match" the input distribution $P_X(\cdot|S_t, \hat{B}_{t-1})$. Figure 1 illustrates the encoding strategy for binary unifilar channels with $M = 4$. At that situation, X_t is deterministic when $W = 1, 3$, or 4. When $W = 2$, the output X_t depends on $V_t^{S_t^W}$. If $V_t^{S_t} < a$, then $X_t = 0$. Otherwise the transmitter sends $X_t = 1$.

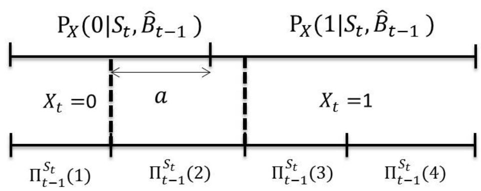

Fig. 1: Illustration of the encoding strategy for binary unifilar channels with $M = 4$.

If there exists an $x \in \mathcal{X}$ such that

$$\sum_{k=1}^{x-1} P_X(x|S_t^W, \hat{B}_{t-1}) < \sum_{i=1}^{W-1} \Pi_{t-1}^{S_t^W}(i) \tag{11a}$$

$$\sum_{i=1}^{W} \Pi_{t-1}^{S_t^W}(i) \leqslant \sum_{k=1}^{x} P_X(x|S_t^W, \hat{B}_t), \tag{11b}$$

then $X_t = x$, otherwise

$$X_t = \begin{cases} x - 1, & V_t^{S_t^W} < \frac{\sum_{k=1}^{x-1} P_X(x|S_t^W, \hat{B}_{t-1}) - \sum_{i=1}^{W-1} \Pi_{t-1}^{S_{t-1}^W}(i)}{\Pi_t^{S_{t-1}^W}(W)} \\ x, & \text{otherwise.} \end{cases} \tag{12}$$

We can write $X_t = e(W, \underline{S}_t, \Pi_{t-1}, V_t^{S_t^W})$. For any $i \in \{1, 2, \cdots, M\}$, the quantities $\underline{S}_t$ and Π_t can be updated as

$$\Pi_t(i) = \frac{Q(Y_t|e(i, \underline{S}_t, \Pi_{t-1}, V_t^{S_t^i}), S_t^i)\Pi_{t-1}(i)}{\sum_j Q(Y_t|e(j, \underline{S}_t, \Pi_{t-1}, V_t^{S_t^j}), S_t^j)\Pi_{t-1}(j)} \tag{13a}$$

$$S_{t+1}^i = g(S_t^i, e(i, \underline{S}_t, \Pi_{t-1}, V_t^{S_t^i}), Y_t), \tag{13b}$$

which can be concisely written as

$$\underline{S}_{t+1} = \phi_s(\underline{S}_t, \Pi_{t-1}, Y_t, V_t) \tag{14a}$$
$$\Pi_t = \phi_\pi(\underline{S}_t, \Pi_{t-1}, Y_t, V_t), \tag{14b}$$

and $\hat{B}_t$ can be generated according to (8). Regarding decoding, the receiver estimates the message by $\hat{W}_t = \arg\max_i \Pi_t(i)$ at time t. There is a stopping time T_{max} which is the first time that $\max_i \Pi_t(i)$ is greater than $1-P_e$, where Pe is a pre-specified system parameter (the target probability of error). At this time transmission stops and the decoded message is declared to be $\hat{W}_{T_{max}}$.

B. Error analysis

The average transmission rate of this system is defined as $\overline{R} = \frac{K}{E[T]}$. The error probability of this scheme is given by

$$\begin{aligned}
&P(err) \\
&= P(T_{max} = \infty) + P(\hat{W}_{T_{max}} \neq W) \\
&= P(T_{max} = \infty) + 1 - P(\hat{W}_{T_{max}} = W) \\
&= P(T_{max} = \infty) + 1 \\
&\quad - \sum_{t=1}^{\infty} \sum_{y^t} P(T_{max} = t, y^t) P(\hat{w}_t = W | T_{max} = t, Y^t = y^t) \\
&\overset{(a)}{\leqslant} P(T_{max} = \infty) + 1 \\
&\quad - \sum_{t=1}^{\infty} \sum_{y^t, s_1} P(T_{max} = t, y^t, s_1)(1 - Pe) \\
&= P(T_{max} = \infty) + Pe.
\end{aligned} \tag{15}$$

where (a) is due to that $\hat{W}_t = \arg\max_i \Pi_t(i)$ and $\max_i \Pi_{T_{max}}(i) > 1 - Pe$. The drift analysis provided below shows that T_{max} is almost surely finite and therefore $P(err) < Pe$. Inspired by Burnashev's method, we analyze the log-likelihood ratio L_t at time t, defined by

$$L_t = \log \frac{\Pi_t(W)}{1 - \Pi_t(W)}, \tag{16}$$

and stopping time T given by

$$T = \min\{t | L_t > \log \frac{1 - Pe}{Pe}\}. \tag{17}$$

By definition we have $T_{max} \leqslant T$ almost surely. To derive a lower bound on error-exponent, we would like to have an upper bound on T_{max}. In the following we derive an upper bound on T instead of T_{max} since T is almost surely greater than T_{max} and directly connected with the message W. We first analyze the drift of L_t w.r.t $\{\mathcal{F}_t\}_{t\geqslant 0}$ and then apply the optional sampling theorem to a proposed submartingale, in order to derive the desired result. A major difference between unifilar channels and DMCs is the presence of memory. In order to capture channel memory, we eventually need to analyze multi-step instead of one-step drift of L_t.

Lemma 1. *The one-step drift of $\{L_t\}_{t\geqslant 0}$ w.r.t. $\{\mathcal{F}_t\}_{t\geqslant 0}$ is lower-bounded by*

$$E[L_t - L_{t-1} | \mathcal{F}_{t-1}] \geqslant I(\hat{B}_{t-1}), \tag{18}$$

where $I(\hat{B}_{t-1})$ is given by

$$\begin{aligned}
I(\hat{B}_{t-1}) = &\sum_{s,x,y} Q(y|x,s) P_{X|SB}(x|s, \hat{B}_{t-1}) \hat{B}_{t-1}(s) \\
&\log \frac{Q(y|x,s)}{\sum_{\tilde{s},\tilde{x}} Q(y|\tilde{x},\tilde{s}) P_{X|SB}(\tilde{x}|\tilde{s}, \hat{B}_{t-1}) \hat{B}_{t-1}(\tilde{s})}.
\end{aligned} \tag{19}$$

Proof: See Appendix A. ■

The following remark is crucial in the subsequent development.

Remark 1. *First, unlike the case in DMC where the one-step drift is shown to be greater than the channel capacity, here the one step drift is a random variable. This is exactly the reason we consider multi-step drift analysis: under an ergodicity assumption, the arithmetic mean of these random variables will converge almost surely to their mean. This raises the question of what the mean of the process $\{I(\hat{B}_{t-1})\}_t$ is. If the process $\{\hat{B}_{t-1}\}_t$ had the same statistics as those of the Markov chain $\{B_{t-1}\}_t$ defined in* (5)*, then convergence to C would be guaranteed with rate independent of the parameter K. However, the two processes have different statistics. This is because of the introduction of common randomness! Indeed, $B_{t-1}(s) = P(s_t = s|Y^{t-1}, S_1)$, while $\hat{B}_{t-1}(s) = P(S_t = s|Y^{t-1}, V^{t-1}, S_1)$ and they are related according to $B_{t-1} = E[\hat{B}_{t-1}|Y^{t-1}, S_1]$. In fact, $\{\hat{B}_{t-1}\}_t$ is* *not a Markov chain, but is measurable w.r.t. the state of the Markov chain* $\{(\underline{S_t}, \Pi_{t-1})\}_t$ *as shown in* (8). *We conjecture that* $\{\hat{B}_{t-1}\}_t$ *converges to its steady state independently of* K. *We argue that this is so since the first part of the state* $(\underline{S_t}, \Pi_{t-1})$ *has a finite state space and its transition matrix is very sparse and only depends loosely on* Π_{t-1}. *This implies (from random matrix theory) that* $\underline{S}_{t-1}$ *converges exponentially fast to its steady state, independently of* Π *and thus independently of* K. *Furthermore, this induces a steady-state distribution on* $\hat{B}$ *that is exactly that of* B. *Our results presented in Section V provide strong support for this conjecture.*

Based on the above discussion we have

$$\tilde{C} = \lim_{N\to\infty} \frac{1}{N} \sum I(\hat{B}_{t-1}) \quad \text{almost surely.} \tag{20}$$

Furthermore, due to the assumption that the transition kernel $Q(y|x,s)$ positive for any $(s,x,y) \in \mathcal{S}\times\mathcal{X}\times\mathcal{Y}$, the quantity $|L_t - L_{t-1}|$ can be shown to be upper bounded by a certain constant C_2 as in [3, Lemma 4]. Now we have the result of drift analysis of our system. The relation between the drift and the stopping time, T is given through the following result

Fact 1. *[18, Lemma, p. 230] Let* $\{Z_t\}_{t\geqslant 0}$ *be a submartingale w.r.t.* $\{Z_t\}_{t\geqslant 0}$ *which has the following properties*

$$E[Z_{t+1} - Z_t|F_t] \geqslant K_1 \qquad (K_1 > 0) \tag{21a}$$

$$|Z_{t+1} - Z_t|F_t| \leqslant K_3 \tag{21b}$$

and define a stopping time $T = \min\{t|Z_t \geqslant B\}$. *Then*

$$E[T] \leqslant \frac{B - Z_0 + K_3}{K_1}. \tag{22}$$

Combining the above results, we are ready to state a lower bound on the error exponent.

Proposition 1. *With* $M = 2^K$ *messages and target error probability* Pe, *given any* $\epsilon > 0$ *we have*

$$-\frac{\log Pe}{E[T]} \geqslant \tilde{C}(1 - \frac{\overline{R}}{\tilde{C}}) + U(\epsilon, K, \overline{R}, \tilde{C}, C_2), \tag{23}$$

where $\lim_{K\to\infty} U(\epsilon, K, \overline{R}, \tilde{C}, C_2) = o_\epsilon(1)$.

Proof: See Appendix B. ■

IV. A TWO-STAGE TRANSMISSION SCHEME

A. Transmission scheme

Following the ideas from [3], we now consider a two-stage scheme that has the potential to achieve the upper bounds derived in [16]. Suppose we are given the optimizing strategies relating to the MDP discussed in [16]. In particular we are given a policy $X^0 : \mathcal{S}\times\mathcal{P}(\mathcal{S}) \to \mathcal{X}$ and a policy $X^1 : \mathcal{S}\times\mathcal{P}(\mathcal{S}) \to (\mathcal{S} \to \mathcal{P}(\mathcal{X}))$. For now we fix a threshold p_0 that will be specified in the analysis. The transmission starts at stage one which is exactly the one described in the previous Section. If $\max_i \Pi_{t-1}(i)$ exceeds the threshold p_0, this implies the receiver has very high confidence that a certain message is the transmitted one. Therefore the transmitter enters the second stage that determines whether the estimated message of the receiver is the true message, which is a binary hypothesis testing problem. Define $\hat{W} = \arg\max_i \Pi_{t-1}(i)$ whenever $\max_i \Pi_{t-1}(i) \geqslant p_0$ and $\max_i \Pi_{t-2}(i) < p_0$. Now we specify our encoding strategy. Let H_0 be the hypothesis that the estimation at the receiver is correct (i.e., $W = \hat{W}$), and H_1 be the opposite. We define the quantities $\hat{B}^1_{t-1} \in \mathcal{P}(\mathcal{S})$ and $\Pi^1_{t-1} \in \mathcal{P}(\{1,2,\cdots,M\})^{|\mathcal{S}|}$ similarly to stage-one related quantities, with the only difference being that they represent posterior beliefs conditioned on H_1

$$\begin{aligned}\hat{B}^1_{t-1}(s) &= P(S_t = s|\mathcal{F}_{t-1}, H_1)\\ &= \frac{\sum_{i\neq\hat{W}} \Pi_{t-1}(i)1_{\{S^i_t = s\}}}{1 - \Pi_{t-1}(\hat{W})}\end{aligned} \tag{24}$$

$$\begin{aligned}\Pi^{1,s}_{t-1}(i) &= P(W = i|S_t = s, \mathcal{F}_{t-1}, H_1)\\ &= \frac{\Pi_{t-1}(i)1_{\{i\neq\hat{W}\}}1_{\{S^i_t = s\}}}{\hat{B}^1_{t-1}(s)(1 - \Pi_{t-1}(\hat{W}))}.\end{aligned} \tag{25}$$

Under H_0, the input signal is $X_t = X^0[S^{\hat{W}}_t, \hat{B}^1_{t-1}]$. Under H_1, the input signal X_t is generated in a similar fashion as in stage one (as illustrated in Fig. 1), expect that now, the message distribution $\Pi^{1,S^W_t}_{t-1}(\cdot)$ is used (instead of $\Pi^{S^W_t}_{t-1}(\cdot)$) and the input distribution $X^1[S^{\hat{W}}_t, \hat{B}^1_{t-1}](\cdot|S^W_t)$ is to be "matched" (instead of $P_{X|SB}(\cdot|S_t, \hat{B}_{t-1})$). We use $e^0(S^{\hat{W}}_t, \hat{B}^1_{t-1})$ and $e^1(S^{\hat{W}}_t, \hat{B}^1_{t-1}, W, S^W_t, V_t)$ to denote the encoding functions for H_0 and H_1 respectively, where we make explicit the dependence on the common random variable V_t used in H_1. With this encoding strategy, we can update $\underline{S}_t$ and Π_t by

$$\begin{aligned}S^{\hat{W}}_{t+1} &= g(S^{\hat{W}}_t, e^0(S^{\hat{W}}_t, \hat{B}^1_{t-1}), Y_t)\\ S^i_{t+1} &= g(S^i_t, e^1(S^{\hat{W}}_t, \hat{B}^1_{t-1}, i, S^i_t, V_t), Y_t) \qquad \forall i \neq \hat{W}\end{aligned} \tag{26}$$

$$\Pi_t(\hat{W}) = \frac{Q(Y_t|e^0(S_t^{\hat{W}}, \hat{B}_{t-1}^1), S_t^{\hat{W}})\Pi_{t-1}(\hat{W})}{P_Y(\underline{S}_t, \Pi_{t-1})}$$

$$\Pi_t(j) = \frac{Q(Y_t|e^1(S_t^{\hat{W}}, \hat{B}_{t-1}^1, j, S_t^j, V_t), S_t^j)\Pi_{t-1}(j)}{P_Y(\underline{S}_t, \Pi_{t-1})}, \tag{27}$$

where P_Y is given by

$$P_Y(\underline{S}_t, \Pi_{t-1}) = \sum_{i \neq \hat{W}} Q(Y_t|e^1(S_t^{\hat{W}}, \hat{B}_{t-1}^1, i, S_t^i, V_t), S_t^i)\Pi_{t-1}(i) + Q(Y_t|e^0(S_t^{\hat{W}}, \hat{B}_{t-1}^1), S_t^{\hat{W}})\Pi_{t-1}(\hat{W}), \tag{28}$$

and $\hat{B}_t^1$, Π_t^1 can be updated according to (24), and (25), respectively. We conclude the description of the encoding scheme by noting that if during stage two the posterior belief of message $\hat{W}$ drops below the threshold then the system reverts to stage one.

B. Error analysis

The analysis follows the same structure as in the first stage: we first analyze the one-step drift of L_t under both hypotheses and then, arguing in the same way we did above, we consider the asymptotic behaviour of the N-step drift.

Lemma 2. *For any $\epsilon > 0$, there exist an $N = N(\epsilon)$ such that*

$$E[L_{t+N} - L_t|\mathcal{F}_t] \geqslant N(\tilde{C}_1 - \epsilon) \quad \text{if } L_t \geqslant \log\frac{p_0}{1-p_0} \tag{29a}$$

$$E[L_{t+N} - L_t|\mathcal{F}_t] \geqslant N(\tilde{C} - \epsilon) \quad \text{if } L_t < \log\frac{p_0}{1-p_0}. \tag{29b}$$

Proof: See Appendix C. ∎

Lemma 2 shows that in the second stage, the likelihood ratio grows faster than in the first stage if the estimation at the receiver is correct. Even if the estimation is wrong, the likelihood ratio maintains the increasing rate as in the first stage. For this to be true we have assumed that $\tilde{C}_1^* > \tilde{C}$. If this is not the case then an alternative scheme can be proposed as in [3, p. 261]. We omit this description due to space limitations. As in the previous section, the following result connects the drift and the stopping time.

Fact 2. *[19, Lemma, p. 50] Suppose Z_t is a process with the following properties*

$$E[Z_{t+1} - Z_t|F_t] \geqslant K_1 \qquad \text{if } Z_t < 0, K_1 > 0 \tag{30a}$$

$$E[Z_{t+1} - Z_t|F_t] \geqslant K_2 \qquad \text{if } Z_t \geqslant 0, K_2 > K_1 \tag{30b}$$

$$|Z_{t+1} - Z_t| \leqslant K_3 \tag{30c}$$

$$Z_0 < 0. \tag{30d}$$

Define a stopping time $T = \min\{t|Z_t \geqslant B\}$. Then we have

$$E[T] \leqslant \frac{B}{K_2} + \frac{|Z_0|}{K_1} + D(K_1, K_2, K_3), \tag{31}$$

where $D(K_1, K_2, K_3)$ is a bounded constant.

The final result is given in the following proposition.

Proposition 2. *With $M = 2^K$ messages and target error probability Pe, given any $\epsilon > 0$ we have*

$$-\frac{\log Pe}{E[T]} \geqslant \tilde{C}_1(1 - \frac{\overline{R}}{\tilde{C}}) + U(\epsilon, K, Pe, \tilde{C}, \tilde{C}_1, C_2), \tag{32}$$

where $\lim_{K\to\infty} U(\epsilon, K, Pe, \tilde{C}, \tilde{C}_1, C_2) = o_\epsilon(1)$.

Proof: See Appendix D. ∎

V. Numerical results

In this section, we provide simulation results for the error exponents achieved by the two proposed transmission schemes for some binary input/output/state unifilar channels. We consider the trapdoor channel (denoted as channel A), chemical channel (denoted as channel $B(p_0 = 0.9)$), two symmetric unifilar channels (denoted as $C(p_0 = 0.5, q_0 = 0.1)$, and $C(p_0 = 0.9, q_0 = 0.1)$, respectively), and two asymmetric unifilar channels (denoted as $D(p_0 = 0.5, q_0 = 0.1, p_1 = 0.1, q_1 = 0.1)$, and $D(p_0 = 0.9, q_0 = 0.1, p_1 = 0.1, q_1 = 0.1)$, respectively). All of these channels have $g(s, x, y) = s \oplus x \oplus y$ and kernel Q as shown in Table I.

TABLE I: Kernel definition for binary unifilar channels

Channel	$Q(0\|0,0)$	$Q(0\|1,0)$	$Q(0\|0,1)$	$Q(0\|1,1)$
Trapdoor	1	0.5	0.5	0
Chemical(p_0)	1	p_0	$1-p_0$	0
Symmetric(p_0, q_0)	$1-q_0$	p_0	$1-p_0$	q_0
Asymmetric(p_0, q_0, p_1, q_1)	$1-q_0$	p_0	$1-p_1$	q_1

We simulated a system with message length $K = 10, 20, 30, 40$ (bits) and target error rates $P_e = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}$. In each simulation sufficient experiments were run to have a convergent average rate, since the error probability is guaranteed to be bellow the target. Infinite precision arithmetic was used in all evaluations through the "GNU Multiple Precision Arithmetic Library" (GMP). The results are shown in Fig. 2, Fig. 3, and Fig. 4, respectively. Each curve in these figures corresponds to a value of K. All two-stage schemes were run with the optimal policy in the MDP discussed in [16]. Also shown on the same figures are the error exponent upper bounds obtained in [16], as well as the parameters (C, C_1, C_1^*) for each channel.

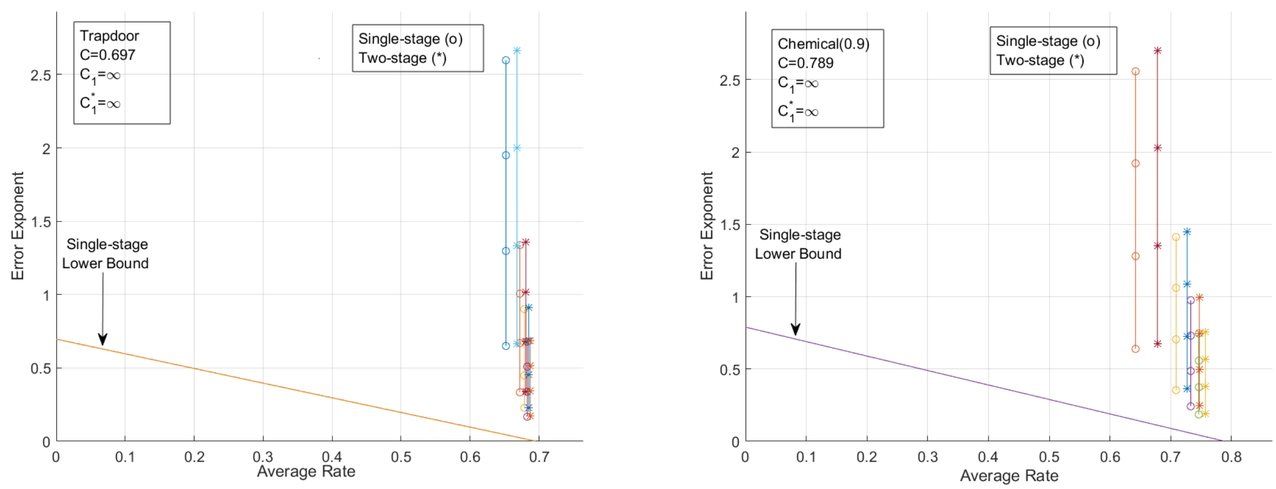

Fig. 2: Error exponent for the trapdoor channel A and B(0.9).

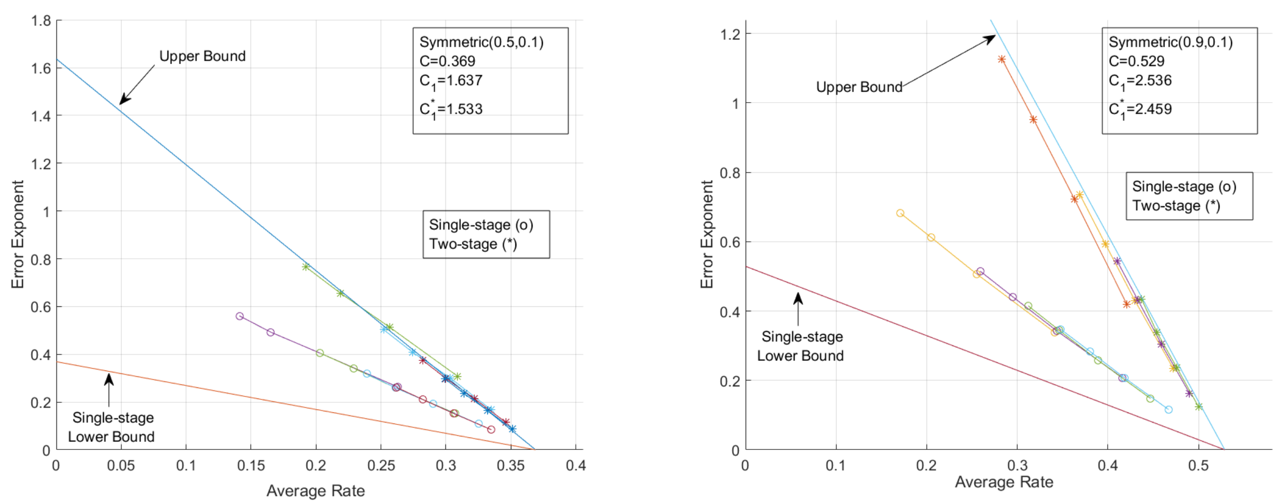

Fig. 3: Error exponent for the symmetric unifilar channels C(0.5,0.1) and C(0.9,0.1).

We make two main observations regarding these results. The first observation is that for the trapdoor channels there is strong evidence that the error exponents are infinite. This is consistent with the findings in [9] where a zero-error capacity-achieving scheme is proposed, and the discussion in [16] regarding channels with zeros in their transition kernels. Interestingly enough, the infinite exponent is achieved even with a single-stage receiver. Similar comments are valid for the chemical channel. The second observation is the remarkable agreement between simulation results of the proposed (two-stage) scheme and the upper bound derived in [16] for channels C and D. The analysis of the one-stage scheme seems to be quite conservative and the true drift is higher than the lower bound C. These results represent very strong evidence for the validity of the conjecture stated earlier.

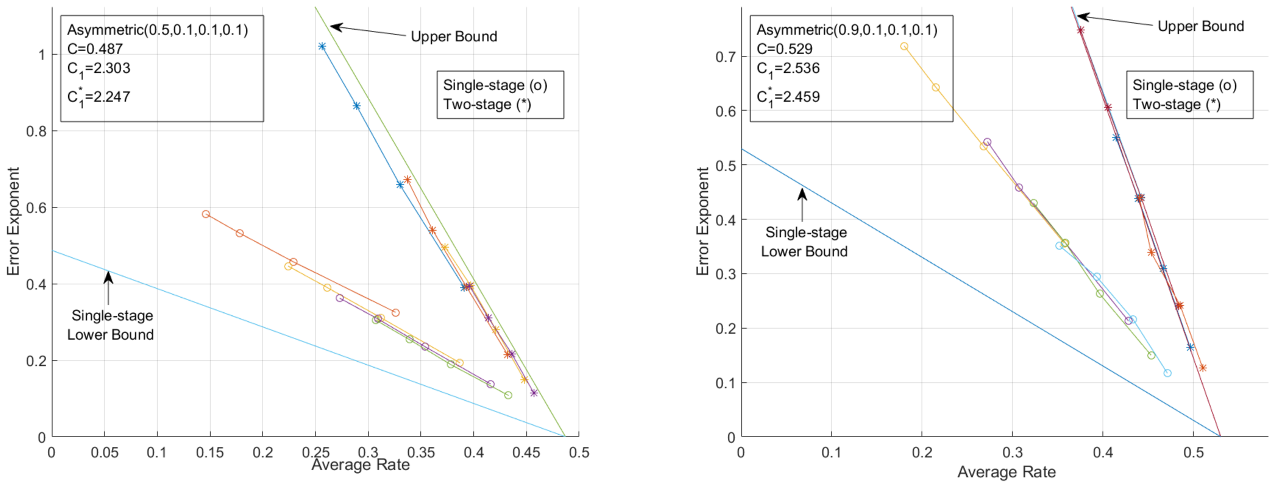

Fig. 4: Error exponent for the asymmetric unifilar channel D(0.5,0.1,0.1,0.1) and D(0.9,0.1,0.1,0.1).

VI. Conclusions

We propose two variable-length transmission schemes for unifilar channels with noiseless feedback. Their error exponent is analyzed by generalizing the techniques of Burnashev [3]. Both schemes have a posterior matching flavor; the latter does posterior matching even at the second stage where a binary hypothesis is resolved, which is a unique feature of this encoding scheme compared to its DMC counterpart. The analysis assumes the presence of some common randomness shared by the transmitter and receiver, and is based on the study of the log-likelihood ratio of the transmitted message posterior belief, and in particular on the study of its multi-step drift. The theoretical results hinge on an additional assumption we make about the induced Markov chain $(\underline{S}_t, \Pi_{t-1})$, i.e., that its first component converges fast to its steady state and the corresponding empirical distribution $\hat{B}_{t-1}$ has the same statistics (asymptotically) as those of the Markov chain $\{B_t\}$. Simulation results for the trapdoor, chemical, and other binary input/state/output unifilar channels provide strong support towards the validity of this assumption and show remarkable tightness compared to the upper bounds derived in the companion paper [16].

We conclude by noting that the techniques used in this work can be applied to channels with Markov states and inter-symbol interference (ISI) where the state is observed at the receiver and with unit delay at the transmitter.

Appendix

A. Proof of Lemma 1

$$\begin{aligned}
&E[L_t - L_{t-1}|W, Y^{t-1}, V^{t-1}, S_1] \\
&\quad = \log(1-\Pi_{t-1}(W)) + E[\log \frac{\Pi_t(W)}{(1-\Pi_t(W))\Pi_{t-1}(W)}|W, \mathcal{F}_{t-1}] \\
&\quad = \log(1-\Pi_{t-1}(W)) + E[\log \frac{Q(Y_t|X_t,S_t)}{P(Y_t|V_t,\mathcal{F}_{t-1}) - Q(Y_t|X_t,S_t)\Pi_{t-1}(W)}|W, \mathcal{F}_{t-1}].
\end{aligned} \tag{33}$$

We look into $P(Y_t = y|V_t, \mathcal{F}_{t-1})$

$$\begin{aligned}
P(Y_t = y|V_t, \mathcal{F}_{t-1}) &= \sum_{x,s} Q(y|x,s) \sum_{i=1}^{M} P(X_t = x, S_t = s, W = i|V_t, \mathcal{F}_{t-1}) \\
&= \sum_{x,s} Q(y|x,s) \sum_{i=1}^{M} \delta_{e(i,\underline{S}_t,\Pi_{t-1},V_t^{S_t^i})}(x)\delta_{S_t^i}(s)\Pi_{t-1}(i),
\end{aligned} \tag{34}$$

and after taking expectations

$$\begin{aligned}
&E[P(Y_t=y|V_t,\mathcal{F}_{t-1})|W,X_t,S_t,\mathcal{F}_{t-1}]\\
&\quad=E[\sum_{x,s}Q(y|x,s)\sum_{i=1}^{M}\delta_{e(i,\underline{S}_t,\Pi_{t-1},V_t^{S_t^i})}(x)\delta_{S_t^i}(s)\Pi_{t-1}(i)|W,X_t,S_t,V^{t-1},Y^{t-1},S_1]\\
&\quad=\sum_{x,s}Q(y|x,s)\sum_{i=1}^{M}E[\delta_{e(i,\underline{S}_t,\Pi_{t-1},V_t^{S_t^i})}(x)|W,X_t,S_t,\mathcal{F}_{t-1}]\delta_{S_t^i}(s)\Pi_{t-1}(i)\\
&\quad=\sum_{x,s}Q(y|x,s)\sum_{i=1}^{M}\overline{e}(x|i,\underline{S}_t,\Pi_{t-1})\delta_{S_t^i}(s)\Pi_{t-1}(i)\\
&\quad=\sum_{x,s}Q(y|x,s)(P_X(x|s,\hat{B}_{t-1})+\delta_{X_t}(x)\delta_{S_t}(s)\Pi_{t-1}(W)-\overline{e}(x|W,\underline{S}_t,\Pi_{t-1})\delta_{S_t}(s)\Pi_{t-1}(W))\\
&\quad=P(Y_t=y|\hat{B}_{t-1})+Q(y|X_t,S_t)\Pi_{t-1}(W)-\Pi_{t-1}(W)\sum_{x}Q(y|x,S_t)\overline{e}(x|W,\underline{S}_t,\Pi_{t-1}),
\end{aligned}\tag{35}$$

where $\overline{e}(x|i,\underline{S}_t,\Pi_{t-1})$ is given by

$$\overline{e}(x|i,\underline{S}_t,\Pi_{t-1})=E[\delta_{e(i,\underline{S}_t,\Pi_{t-1},V_t^{S_t^i})}(x)|W,X_t,S_t,V^{t-1},Y^{t-1},S_1].\tag{36}$$

Therefore, by convexity of $f(x)=\log\frac{A}{x-B}$,

$$\begin{aligned}
&E[L_t-L_{t-1}|W,\mathcal{F}_{t-1}]\\
&\quad\geqslant\log(1-\Pi_{t-1}(W))+E[\log\frac{Q(Y_t|X_t,S_t)}{P(Y_t|\hat{B}_{t-1})-\Pi_{t-1}(W)\sum_x Q(Y_t|x,S_t)\overline{e}(x|W,\Pi_{t-1},S_t)}|W,\mathcal{F}_{t-1}].
\end{aligned}\tag{37}$$

Looking further into the last term in the above inequality we get

$$\begin{aligned}
&E[\log\frac{Q(Y_t|X_t,S_t)}{P(Y_t|\hat{B}_{t-1})-\Pi_{t-1}(W)\sum_x Q(Y_t|x,S_t)\overline{e}(x|W,\Pi_{t-1},S_t)}|W,\mathcal{F}_{t-1}]\\
&\quad=E[\log\frac{Q(Y_t|X_t,S_t)}{P(Y_t|\hat{B}_{t-1})}+\log\frac{P(Y_t|\hat{B}_{t-1})}{P(Y_t|\hat{B}_{t-1})-\Pi_{t-1}(W)\sum_x Q(Y_t|x,S_t)\overline{e}(x|W,\Pi_{t-1},S_t)}|W,\mathcal{F}_{t-1}]\\
&\quad\geqslant E[\log\frac{Q(Y_t|X_t,S_t)}{P(Y_t|\hat{B}_{t-1})}|W,\mathcal{F}_{t-1}]+E[\log\frac{1}{1-\Pi_{t-1}(W)}|W,\mathcal{F}_{t-1}],
\end{aligned}\tag{38}$$

where the last equation is due to the convexity of $x\log\frac{1}{1-Ax}$. Combining (37) and (38), we get

$$\begin{aligned}
E[L_t-L_{t-1}|\mathcal{F}_{t-1}]&=E[E[L_t-L_{t-1}|W,\mathcal{F}_{t-1}]|\mathcal{F}_{t-1}]\\
&\geqslant E[\log\frac{Q(Y_t|X_t,S_t)}{P(Y_t|\hat{B}_{t-1})}|\mathcal{F}_{t-1}]\\
&=I(\hat{B}_{t-1}).
\end{aligned}\tag{39}$$

B. Proof of Proposition 1

Due to (20), there exists an $N=N(\epsilon)$, which is independent of K, such that

$$\sum_{t=1}^{N}I(\hat{B}_{t-1})\geqslant N(\tilde{C}-\epsilon)\quad\text{almost surely.}\tag{40}$$

Define a process $Z_t=L_{Nt}$ and filtration $\mathcal{F}'_t=\mathcal{F}_{Nt}$. For this process the following properties are satisfied

$$|Z_t-Z_{t-1}|\leqslant NC_2\tag{41a}$$

$$\begin{aligned}
E[Z_{t+1}-Z_t|\mathcal{F}'_t]&\geqslant E[\sum_{i=Nt+1}^{N(t+1)}I(\hat{B}_{t-1})|\mathcal{F}_{Nt}]\\
&\geqslant N(\tilde{C}-\epsilon).
\end{aligned}\tag{41b}$$

We now define a stopping time $\tilde{T}$ w.r.t to $\{\mathcal{F}'_t\}$ by $\tilde{T}=\min\{t|Z_t\geqslant\log\frac{1-Pe}{Pe}\}$. By definition we have $T\leqslant N\tilde{T}$ almost surely. Applying Fact 1 to $\{Z_t\}_{t\geqslant 0}$, we have

$$\frac{E[T]}{N}\leqslant E[\tilde{T}]\leqslant\frac{\log\frac{1-Pe}{Pe}-K+NC_2}{N(C-\epsilon)},\tag{42}$$

which after rearranging of terms becomes

$$-\frac{\log Pe}{T} \geqslant \tilde{C}(1-\frac{\overline{R}}{\tilde{C}}) - \epsilon + \frac{\log(1-Pe)}{K/\overline{R}} + \frac{NC_2}{K/\overline{R}}. \tag{43}$$

C. Proof of Lemma 2

$$\begin{aligned}
E[L_{t+1}-L_t|\mathcal{F}_t,H_0] &= E[\log\frac{\Pi_{t+1}(\hat{W})}{1-\Pi_{t+1}(\hat{W})} - \log\frac{\Pi_t(\hat{W})}{1-\Pi_t((\hat{W})}|\mathcal{F}_t,H_0] \\
&= E[\log\frac{Q(Y_{t+1}|e^0(S_{t+1}^{\hat{W}},\hat{B}_t^1),S_{t+1}^{\hat{W}})}{\sum_{i\neq\hat{W}} Q(Y_{t+1}|e^1(S_{t+1}^{\hat{W}},\hat{B}_t^1,i,S_{t+1}^i,V_{t+1}),S_{t+1}^i)\frac{\Pi_t(i)}{1-\Pi_t(\hat{W})}}|\mathcal{F}_t,H_0] \\
&= E[\log\frac{Q(Y_{t+1}|e^0(S_{t+1}^{\hat{W}},\hat{B}_t^1),S_{t+1}^{\hat{W}})}{\sum_{x,s} Q(Y_{t+1}|x,s)\sum_{i\neq\hat{W}} 1_{\{e^1(S_{t+1}^{\hat{W}},\hat{B}_t^1,i,S_{t+1}^i,V_{t+1})=x\}}\Pi_t^{1,s}(i)\hat{B}_t^1(s)}|\mathcal{F}_t,H_0] \\
&= E[E[\log\frac{Q(Y_{t+1}|e^0(S_{t+1}^{\hat{W}},\hat{B}_t^1),S_{t+1}^{\hat{W}})}{\sum_{x,s} Q(Y_{t+1}|x,s)\sum_{i\neq\hat{W}} 1_{\{e^1(S_{t+1}^{\hat{W}},\hat{B}_t^1,i,S_{t+1}^i,V_{t+1})=x\}}\Pi_t^{1,s}(i)\hat{B}_t^1(s)}|Y_{t+1},\mathcal{F}_t,H_0]|\mathcal{F}_t,H_0] \\
&\overset{(a)}{\geqslant} E[\log\frac{Q(Y_{t+1}|e^0(S_{t+1}^{\hat{W}},\hat{B}_t^1),S_{t+1}^{\hat{W}})}{\sum_{x,s} Q(Y_{t+1}|x,s)E[\sum_{i\neq\hat{W}} 1_{\{e^1(S_{t+1}^{\hat{W}},\hat{B}_t^1,i,S_{t+1}^i,V_{t+1})=x\}}\Pi_t^{1,s}(i)|Y_{t+1},\mathcal{F}_t,H_0]\hat{B}_t^1(s)}|\mathcal{F}_t,H_0] \\
&\overset{(b)}{=} E[\log\frac{Q(Y_{t+1}|X^0[S_{t+1}^{\hat{W}},\hat{B}_t^1],S_{t+1}^{\hat{W}})}{\sum_{x,s} Q(Y_{t+1}|x,s)X^1[S_{t+1}^{\hat{W}},\hat{B}_t^1](x|s)\hat{B}_t^1(s)}|\mathcal{F}_t,H_0] \\
&= E[\log\frac{Q(Y_{t+1}|X^0[S_{t+1}^{\hat{W}},\hat{B}_t^1],S_{t+1}^{\hat{W}})}{\sum_{x,s} Q(Y_{t+1}|x,s)X^1[S_{t+1}^{\hat{W}},\hat{B}_t^1](x|s)\hat{B}_t^1(s)}|\mathcal{F}_t,H_0] \\
&= R(S_{t+1}^{\hat{W}},\hat{B}_t^1,X_t^0[S_{t+1}^{\hat{W}},\hat{B}_t^1],X_{t+1}^1[S_{t+1}^{\hat{W}},\hat{B}_t^1]),
\end{aligned} \tag{44}$$

where (a) is due to the convexity of $f(x)=\frac{A}{x-B}$, (b) is due to our posterior matching scheme under H_1 hypothesis, and $R(s,b,x_0,x_1)$ is given by

$$R(s,b,x_0,x_1) = \sum_y Q(y|x_0,s)\log\frac{Q(y|x_0,s)}{\sum_{\tilde{x},\tilde{s}} Q(y|\tilde{x},\tilde{s})x_1(\tilde{x}|\tilde{s})b(\tilde{s})}. \tag{45}$$

We can think of this quantity as the instantaneous reward received by the Markov chain with state $(\underline{S}_{t+1},\Pi_t)$. Considering the N-step drift, we have

$$\begin{aligned}
E[L_{t+N}-L_t|\mathcal{F}_t,H_0] &= \sum_{i=t}^{i=t+N-1} E[L_{i+1}-L_i|\mathcal{F}_t,H_0] \\
&= \sum_{i=t}^{i=t+N-1} E[L_{i+1}-L_i|\mathcal{F}_t,H_0] \\
&= \sum_{i=t}^{i=t+N-1} E[E[L_{i+1}-L_i|\mathcal{F}_i,H_0]|\mathcal{F}_t,H_0] \\
&\overset{(a)}{\geqslant} \sum_{i=t}^{i=t+N-1} E[E[\log\frac{Q(Y_{i+1}|X_0[S_{i+1}^{\hat{W}},\hat{B}_i^1],S_{i+1}^{\hat{W}})}{\sum_{x,s} Q(Y_{i+1}|x,s)X^1[S_{i+1}^{\hat{W}},\hat{B}_i^1(s)])(x|s)\hat{B}_i^1(s)}|\mathcal{F}_i,H_0]|\mathcal{F}_t,H_0] \\
&= \sum_{i=t}^{i=t+N-1} E[R(S_{i+1}^{\hat{W}},\hat{B}_i^1,X_{i+1}^0[S_{i+1}^{\hat{W}},\hat{B}_i^1],X_{i+1}^1[S_{i+1}^{\hat{W}},\hat{B}_i^1])|\mathcal{F}_t,H_0],
\end{aligned} \tag{46}$$

where (a) is due to (44). Thus, the multi-step drift corresponds to the total average reward in the aforementioned Markov chain. This total reward relates to the MDP problem discussed in [16]. Thus, under the assumptions stated in the discussion after Lemma 1, the corresponding per-unit-time reward converges to $\tilde{C}_1$ almost surely as $N\to\infty$. Thus, given any $\epsilon>0$, there exist a $N_1=N_1(\epsilon)$ such that

$$E[L_{t+N}-L_t|F_t] \geqslant N_1(\tilde{C}_1-\epsilon). \tag{47}$$

Now we consider the drift of $\{L_t\}_{t\geqslant 0}$ w.r.t. $\{\mathcal{F}_t\}$ under H_1 hypothesis. Given any $N>0$,

$$\begin{aligned}&E[L_{t+N}-L_t|\mathcal{F}_t,H_1]\\&\quad=E[\log\frac{\prod_{j=t+1}^{t+N}Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,W,S_j^W,V_j),S_j^W)(1-\Pi_t(W))}{\sum_{i\neq W,\hat{W}}\prod_{j=t+1}^{t+N}Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,i,S_j^i,V_j),S_j^i)\Pi_t(i)+\prod_{j=t+1}^{t+N}Q(Y_j|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})\Pi_t(\hat{W})}|\mathcal{F}_t,H_1].\end{aligned}\tag{48}$$

As $\Pi_t(\hat{W})\to 1$, the right-hand side of (48) becomes

$$\begin{aligned}&E[\log\frac{\prod_{j=t+1}^{t+N}Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,W,S_j^W,V_j),S_j^W)}{\prod_{j=t+1}^{t+N}Q(Y_j|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}|\mathcal{F}_t,H_1]\\&=E[\sum_{j=t+1}^{t+N}\log\frac{Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,W,S_j^W,V_j),S_j^W)}{Q(Y_j|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}|\mathcal{F}_t,H_1]\\&=\sum_{j=t+1}^{t+N}E[\log\frac{Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,W,S_j^W,V_j),S_j^W)}{Q(Y_j|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}|\mathcal{F}_t,H_1]\\&=\sum_{j=t+1}^{t+N}E[E[\log\frac{Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,W,S_j^W,V_j),S_j^W)}{Q(Y_j|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}|\mathcal{F}_{j-1},H_1]|\mathcal{F}_t,H_1].\end{aligned}\tag{49}$$

Furthermore,

$$\begin{aligned}&E[\log\frac{Q(Y_j|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,W,S_j^W,V_j),S_j^W)}{Q(Y_j|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}|\mathcal{F}_{j-1},H_1]\\&=\sum_y\sum_{i\neq\hat{W}}\int_{V_j}P(dV_j)Q(y|e^1(S^{\hat{W}},\hat{B}_{j-1}^1,i,S_j^i,V_j),S_j^i)P(W=i|\mathcal{F}_{j-1},H_1)\log\frac{Q(y|e^1(S_j^{\hat{W}},\hat{B}_{j-1}^1,i,S_j^i,V_j),S_j^i)}{Q(y|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}\\&=\sum_{x,s}\sum_y\sum_{i\neq\hat{W}}\int_{V_j}P(dV_j)Q(y|x,s)1_{\{e^1(S^{\hat{W}},\hat{B}_{j-1}^1,i,S_j^i,V_j)=x\}}1_{\{S_j^i=s\}}\Pi_{j-1}^1(i)\log\frac{Q(y|x,s)}{Q(y|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}\\&=\sum_{x,s}\sum_y Q(y|x,s)\log\frac{Q(y|x,s)}{Q(y|e^0(S_j^{\hat{W}},\hat{B}_{j-1}^1),S_j^{\hat{W}})}\left[\sum_{i\neq\hat{W}}\int_{V_j}P(dV_j)1_{\{e^1(S^{\hat{W}},\hat{B}_{j-1}^1,i,S_j^i,V_j)=x\}}1_{\{S_j^i=s\}}\Pi_{j-1}^1(i)\right]\\&\overset{(a)}{=}\sum_{x,s}\left[\sum_y Q(y|x,s)\log\frac{Q(y|x,s)}{Q(y|X^0[S_j^{\hat{W}},\hat{B}_{j-1}^1],S_j^{\hat{W}})}\right]X^1[S_j^{\hat{W}},\hat{B}_{j-1}^1](x|s)\hat{B}_{j-1}^1(s)\\&=\sum_{x,s}X^1[S_j^{\hat{W}},\hat{B}_{j-1}^1](x|s)\hat{B}_{j-1}^1(s)D\left(Q(\cdot|x,s)\|Q(\cdot|X^0[S_j^{\hat{W}},\hat{B}_{j-1}^1],S_j^{\hat{W}})\right)\\&=R^*(S_j^{\hat{W}},\hat{B}_{j-1}^1,X_{j-1}^0[S_j^{\hat{W}},\hat{B}_{j-1}^1],X_j^1[S_j^{\hat{W}},\hat{B}_{j-1}^1]),\end{aligned}\tag{50}$$

where (a) is due to our posterior matching scheme, and $R^*(s,b,x_0,x_1)$ is given by

$$R^*(s,b,x_0,x_1)=\sum_{\tilde{x},\tilde{s}}x_1(\tilde{x}|\tilde{s})b(\tilde{s})\sum_y Q(y|\tilde{x},\tilde{s})\log\frac{Q(y|\tilde{x},\tilde{s})}{Q(y|x_0,s)}.\tag{51}$$

This quantity can be thought of as some instantaneous reward received by the Markov chain with state $(\underline{S}_{t+1},\Pi_t)$. Thus, under the assumptions stated in the discussion after Lemma 1, the corresponding per-unit-time reward converges to $\tilde{C}_1^*$ almost surely as $N\to\infty$. Therefore, under hypothesis H_1, for any $\epsilon>0$, there exists $N_2=N_2(\epsilon)$ such that

$$E[L_{t+N}-L_t|F_t]\geqslant N_2(\tilde{C}_1^*-\epsilon).\tag{52}$$

Therefore for any $\epsilon>0$, we have $N=\max(N_1(\epsilon),N_2(\epsilon))$ such that

$$E[L_{t+N}-L_t|\mathcal{F}_t]\geqslant N(\tilde{C}_1-\epsilon)\qquad\text{if } L_t\geqslant\log\frac{p_0}{1-p_0}\tag{53}$$

$$E[L_{t+N}-L_t|\mathcal{F}_t]\geqslant N(\tilde{C}_1^*-\epsilon)\qquad\text{if } L_t<\log\frac{p_0}{1-p_0}.\tag{54}$$

D. Proof of Proposition 2

Define a process $Z_t = L_{Nt}$ and filtration $\mathcal{F}'_t = \mathcal{F}_{Nt}$. We have

$$\begin{aligned} E[Z_{t+1} - Z_t|\mathcal{F}'_t] &\geqslant N(\tilde{C} - \epsilon) \\ E[Z_{t+1} - Z_t|\mathcal{F}'_t] &\geqslant N(\tilde{C}_1 - \epsilon) \qquad \text{if } Z_t > \log \frac{1-p_0}{p_0} \\ |Z_t - Z_{t-1}| &\leqslant NC_2. \end{aligned} \tag{55}$$

Define a stopping time $\tilde{T}$ w.r.t to $\{F'_t\}$ by

$$\tilde{T} = \min\{t|Z_t \geqslant \log \frac{1-Pe}{Pe}\}. \tag{56}$$

By definition we have $T \leqslant N\tilde{T}$ almost surely. Apply Fact 2 to $\{Z_t\}_{t\geqslant 0}$, we have an upper bound on the expectation of the stopping time $\tilde{T}$.

$$\frac{E[T]}{N} \leqslant E[\tilde{T}] \leqslant \frac{\log \frac{1-Pe}{Pe}}{\tilde{C}_1 - \epsilon} + \frac{K}{N(\tilde{C} - \epsilon)} + D(N\tilde{C}, NC_1, NC_2). \tag{57}$$

Rearranging the above inequality, we get

$$\frac{\log Pe}{E[T]} \geqslant \tilde{C}_1(1 - \frac{\overline{R}}{\tilde{C}}) + \frac{\tilde{C} - \tilde{C}_1}{\tilde{C}(\tilde{C} - \epsilon)}\epsilon - \frac{ND(N\tilde{C}, N\tilde{C}_1, NC_2)}{K/\overline{R}} - \frac{\log(1-Pe)}{K/\overline{R}} - \epsilon. \tag{58}$$

REFERENCES

[1] M. Horstein, “Sequential transmission using noiseless feedback,” *IEEE Trans. Inform. Theory*, vol. 9, no. 3, pp. 136–143, Jul 1963.

[2] J. Schalkwijk and T. Kailath, “A coding scheme for additive noise channels with feedback–I: No bandwidth constraint,” *IEEE Trans. Inform. Theory*, vol. 12, no. 2, pp. 172–182, Apr 1966.

[3] M. V. Burnashev, “Data transmission over a discrete channel with feedback. random transmission time,” *Problemy Peredachi Informatsii*, vol. 12, no. 4, pp. 10–30, Oct.-Dec. 1976.

[4] O. Shayevitz and M. Feder, “Optimal feedback communication via posterior matching,” *IEEE Trans. Information Theory*, vol. 57, no. 3, pp. 1186–1222, Mar. 2011.

[5] H. Yamamoto and K. Itoh, “Asymptotic performance of a modified schalkwijk-barron scheme for channels with noiseless feedback (corresp.),” *IEEE Transactions on Information Theory*, vol. 25, no. 6, pp. 729–733, Nov 1979.

[6] H. Viswanathan, “Capacity of Markov channels with receiver csi and delayed feedback,” *Information Theory, IEEE Transactions on*, vol. 45, no. 2, pp. 761–771, 1999.

[7] J. Chen and T. Berger, “The capacity of finite-state Markov channels with feedback,” *IEEE Transactions on Information Theory*, vol. 51, no. 3, pp. 780–798, 2005.

[8] S. Yang, A. Kavčić, and S. Tatikonda, “Feedback capacity of finite-state machine channels,” *Information Theory, IEEE Transactions on*, vol. 51, no. 3, pp. 799–810, 2005.

[9] H. Permuter, P. Cuff, B. V. Roy, and T. Weissman, “Capacity of the trapdoor channel with feedback,” *IEEE Trans. Information Theory*, vol. 54, no. 7, pp. 3150–3165, July 2008.

[10] S. Tatikonda and S. Mitter, “The capacity of channels with feedback,” *IEEE Trans. Information Theory*, vol. 55, no. 1, pp. 323–349, Jan. 2009.

[11] J. Wu and A. Anastasopoulos, “On the capacity of the chemical channel with feedback,” in *Proc. International Symposium on Information Theory*, Barcelona, Spain, July 2016.

[12] P. A. Stavrou, C. D. Charalambous, and C. K. Kourtellaris, “Sequential necessary and sufficient conditions for optimal channel input distributions of channels with memory and feedback,” in *2016 IEEE International Symposium on Information Theory (ISIT)*, July 2016, pp. 300–304.

[13] J. H. Bae and A. Anastasopoulos, “A posterior matching scheme for finite-state channels with feedback,” in *Proc. International Symposium on Information Theory*, Austin, TX, June 2010, pp. 2338–2342.

[14] A. Anastasopoulos, “A sequential transmission scheme for unifilar finite-state channels with feedback based on posterior matching,” in *Proc. International Symposium on Information Theory*, July 2012, pp. 2914–2918.

[15] G. Como, S. Yuksel, and S. Tatikonda, “The error exponent of variable-length codes over Markov channels with feedback,” *IEEE Trans. Information Theory*, vol. 55, no. 5, pp. 2139–2160, May 2009.

[16] A. Anastasopoulos and J. Wu, “Variable-length codes for channels with memory and feedback: error exponent upper bounds,” in *Proc. International Symposium on Information Theory*, Aachen, Germany, Jan. 2017, (submitted. Extended version available online on arxiv at https://arxiv.org/a/0000-0003-3795-1654 and at http://web.eecs.umich.edu/~anastas/preprints.html).

[17] ——, “Variable-length codes for channels with memory and feedback: error exponent lower bounds,” in *Proc. International Symposium on Information Theory*, Aachen, Germany, Jan. 2017, (submitted. Extended version available online on arxiv at https://arxiv.org/a/0000-0003-3795-1654 and at http://web.eecs.umich.edu/~anastas/preprints.html).

[18] M. V. Burnashev and K. S. Zingangirov, “An interval estimation problem for controlled observations,” *Problemy Peredachi Informatsii*, vol. 10, no. 3, pp. 51–61, Jul.-Sep. 1974.

[19] M. V. Burnashev and K. Zigangirov, “On one problem of observation control,” *Problemy Peredachi Informatsii*, vol. 11, no. 3, pp. 44–52, 1975.