On the capacity of the general trapdoor channel with feedback

Jui Wu and Achilleas Anastasopoulos
Electrical Engineering and Computer Science Department
University of Michigan
Ann Arbor, MI, 48109-2122
email: {juiwu, anastas}@umich.edu

***Abstract*—The trapdoor channel is a binary input/output/state channel with state changing deterministically as the modulo-2 sum of the current input, output and state. At each state, it behaves as one of two Z channels, each with crossover probability 1/2. Permuter et al. formulated the problem of finding the capacity of the trapdoor channel with feedback as a stochastic control problem. By solving the corresponding Bellman fixed-point equation, they showed that the capacity equals $C = \log \frac{1+\sqrt{5}}{2}$.**

In this paper, we consider a generalization of the trapdoor channel, whereby at each state the corresponding Z channel has crossover probability p. We characterize the capacity of this problem as the solution of a Bellman fixed-point equation corresponding to a Markov decision process (MDP). Numerical solution of this fixed-point equation reveals an unexpected behavior, that is, for a range of crossover probabilities, the capacity seems to be constant. To the authors knowledge, this is the first time that such behaviour is observed for the Shannon capacity of any channel. Our main contribution is to formalize and prove this observation. In particular, by explicitly solving the Bellman equation, we show the existence of an interval $[0.5, p^*]$ over which the capacity remains constant.

I. INTRODUCTION

Shannon [1] showed that feedback information does not increase the capacity of discrete memoryless channels (DMCs). However, the situation changes in channels with memory. There exists a rich literature on the feedback capacity of channels with memory. Viswanathan [2] derived the capacity of a finite-state channel (FSC) with receiver channel state information (CSI) and delayed output feedback in the absence of intersymbol interference (ISI). Chen and Berger [3] derived the feedback capacity of a FSC with ISI where current CSI is available at both the transmitter and the receiver. Yang et al [4] used a stochastic control methodology to derive the capacity of finite-state machine channels. Permuter et al. [5] derived a single-letter expression for the capacity of unifilar FSCs, which is a family of FSCs with deterministic evolution of the channel state. Later Tatikonda and Mitter [6] derived the capacity of finite-state Markov channels. In all the above works, the capacity is characterized as an optimization of a single-letter expression. However, this optimization problem is rarely solved analytically. A notable exception is [5] where the capacity of the trapdoor channel is analytically derived. The trapdoor channel is a special case of unifilar FSCs with binary input, output and channel state alphabets. For each state, the trapdoor channel behaves as a Z channel with crossover probability 0.5. The authors of [5] formulated the capacity as the optimal expected reward of a Markov decision process (MDP) and proved it to be equal to $C = \frac{1+\sqrt{5}}{2}$ by explicitly solving the Bellman fixed point equation.

In this paper, we consider a generalization of the trapdoor channel where we allow the crossover probability of each Z channel to be p instead of 0.5. We characterize the capacity of the channel though a Bellman fixed-point equation. Numerical evaluation of the capacity for different values of p reveals a surprising result: the capacity seems to remain constant over a range of p in the neighborhood of 0.5. To the authors' knowledge this is the first time such a phenomenon is observed for any kind of channel (with/without memory, with/without feedback, single-/multi-user, etc). To conclusively prove this conjecture, we explicitly solve the corresponding Bellman fixed-point equation. The proof follows the general technique of [5] for the special trapdoor channel. In particular, we define a initial function V_0 and the iterates $V_{k+1} = TV_k$ through an appropriate operator T. Subsequently we show existence of a limit V of $\{V_k\}_{k\geqslant 0}$ which satisfies the Bellman equation. As expected, several technical steps are more involved compared to the special case treated in [5]. These include the definition of the initial function V_0, as well as the proof of Lemma 2, where it is shown that parts of functions V_k remain "stationarity" for all k.

The remaining of this paper is organized as follows. In section II, we describe the channel model for the general trapdoor channel and formulate the capacity as the optimal expected reward for an MDP. In section III, we solve the corresponding Bellman fixed point equation for a range of the crossover probabilities and show that over this range the capacity is constant. Concluding remarks are discussed in section IV.

II. CHANNEL MODEL AND PRELIMINARIES

Consider a family of unifilar channels with inputs $X_t \in \mathcal{X}$, output $Y_t \in \mathcal{Y}$ and state $S_t \in \mathcal{S}$ at time t. The channel conditional probability is

$$\begin{aligned} &P(Y_t, S_{t+1}|X^t, Y^{t-1}, S^t) \\ &\quad = Q(Y_t|X_t, S_t)\delta_{h(S_t,X_t,Y_t)}(S_{t+1}), \end{aligned} \tag{1}$$

for a given stochastic kernel $Q \in \mathcal{X} \times \mathcal{S} \to \mathcal{P}(\mathcal{Y})$ and a deterministic kernel $h : \mathcal{S} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{S}$, where $\mathcal{P}(\mathcal{Y})$ denotes the space of all probability measure on $\mathcal{Y}$.

Permuter et al. [5] showed that the capacity of the unifilar finite-state channel with noiseless output feedback can be expressed as

$$C = \sup_{\{p(x_t|s_t,y^{t-1})\}_{t\geqslant 1}} \liminf_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} I(X_t,S_t;Y_t|Y^{t-1}). \quad (2)$$

Moreover, the capacity can be viewed as the optimal expected average reward of a Markov decision process (MDP) with state $B_{t-1} \in \mathcal{P}(\mathcal{S})$, action $U_t : \mathcal{S} \to \mathcal{P}(\mathcal{X})$ and instantaneous reward $g : \mathcal{P}(\mathcal{S}) \times (\mathcal{S} \to \mathcal{P}(\mathcal{X})) \to \mathbb{R}$ at time t defined by

$$B_{t-1}(s) = P(S_t = s|Y^{t-1}) \qquad \forall s \in \mathcal{S} \quad (3a)$$

$$U_t(x|s) = P(X_t = x|S_t = s, Y^{t-1}) \quad \forall x \in \mathcal{X}, s \in \mathcal{S} \quad (3b)$$

$$\begin{aligned} g(b,u) &= I(X_t,S_t;Y_t|B_{t-1}=b,U_t=u) \\ &= \sum_{x,s,y} \Big[Q(y|x,s)u(x|s)b(s) \\ &\qquad \cdot \log \frac{Q(y|x,s)}{\sum_{\tilde{x},\tilde{s}} Q(y|\tilde{x},\tilde{s})u(\tilde{x}|\tilde{s})b(\tilde{s})}\Big] \\ &\qquad \forall b \in \mathcal{P}(\mathcal{S}), u \in \mathcal{S} \to \mathcal{P}(\mathcal{X}). \end{aligned} \quad (3c)$$

Through Bayes law, the current state B_t can be recursively updated through

$$B_t(s) = \frac{\sum_{x,s'} \delta_{h(s',x,Y_t)}(s) Q(Y_t|x,s')U_t(x|s')B_{t-1}(s')}{\sum_{x,s'} Q(Y_t|x,s')U_t(x|s')B_{t-1}(s')} \qquad \forall s \in \mathcal{S}. \quad (4)$$

Let $\psi : (\mathcal{P}(\mathcal{S}) \times \mathcal{Y} \times (\mathcal{S} \to \mathcal{P}(\mathcal{X}))) \to \mathcal{P}(\mathcal{S})$ denote the updating formula, i.e., $B_t = \psi(B_{t-1}, Y_t, U_t)$.

The general trapdoor channel with parameter p (denoted GTC(p)) is a special family of unifilar FSCs with $\mathcal{X} = \mathcal{Y} = \mathcal{S} = \{0,1\}$, stochastic kernel Q defined in Tabel I, and deterministic kernel h defined as

$$h(s,x,y) = s \oplus x \oplus y, \quad (5)$$

where $\oplus$ denotes modulo-2 addition. We can interpret Q as

s_t	x_t	y_t	$Q(y_t\|x_t,s_t)$
0	0	0	1
0	0	1	0
0	1	0	p
0	1	1	$1-p$
1	0	0	$1-p$
1	0	1	p
1	1	0	0
1	1	1	1

TABLE I: The kernel Q of the general trapdoor channel with crossover probability p.

the kernel of two Z channels (see Fig. 1) with crossover probability p for different states.

Since the input and channel state alphabets are binary, the quantity $B_{t-1}(0)$ is equivalent to B_{t-1}, and similarly, $(U_t(0|0), U_t(0|1))$ is equivalent to U_t. Therefore, with a slight abuse of notation, from now on we will consider the equivalent MDP with state $B_{t-1} = B_{t-1}(0) \in [0,1]$, action $U_t = (U_{t,0}, U_{t,1}) = (U_t(0|0), U_t(0|1)) \in [0,1]^2$ and instantaneous reward g at time t defined as

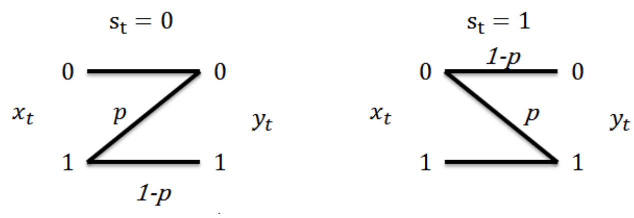

Fig. 1: The Z channels that compose the GTC(p).

$$\begin{aligned} g(b,u_0,u_1) &= H(bu_0 + pb(1-u_0) + (1-p)u_1(1-b)) \\ &\qquad - H(p)[b(1-u_0) + (1-b)u_1], \end{aligned} \quad (6a)$$

where $H(\cdot)$ is the binary entropy function.

The Bellman fixed point equation that defines the optimal input distribution and capacity is given by

$$\begin{aligned} C+V(b) = \\ \sup_{u_0,u_1\in[0,1]} &\Big\{H(y_0(b,u_0,u_1)) - H(p)[b(1-u_0)+(1-b)u_1] \\ &+ y_0(b,u_0,u_1)V(\phi_0(b,u_0,u_1)) \\ &+ y_1(b,u_0,u_1)V(\phi_1(b,u_0,u_1))\Big\} \qquad \forall b \in [0,1], \end{aligned} \quad (7)$$

where $y_0, y_1, \phi_0, \phi_1 : [0,1]^3 \to [0,1]$ are defined as

$$y_0(b,u_0,u_1) = bu_0 + pb(1-u_0) + (1-p)u_1(1-b) \quad (8a)$$

$$y_1(b,u_0,u_1) = 1 - y_0(b,u_0,u_1) \quad (8b)$$

$$\phi_0(b,u_0,u_1) = \frac{bu_0}{y_0(b,u_0,u_1)} \quad (8c)$$

$$\phi_1(b,u_0,u_1) = \frac{(1-p)(1-u_0)b + pu_1(1-b)}{y_1(b,u_0,u_1)}. \quad (8d)$$

Permuter et al. [5] showed that for the special case of GTC(0.5) the capacity is $C = \log\frac{1+\sqrt{5}}{2}$ by proving the existence of a function V that together with C satisfies the Bellman fixed point equation (7).

III. The Capacity of the General Trapdoor Channel

Numerical evaluation of the Bellman fixed point equation in (7) results in capacity values shown in Fig. 2. With the optimal input distribution obtained numerically, the steady-state distributions $\pi(b)$ of B_t can be evaluated through the equation

$$\begin{aligned} \pi(b) = \sum_{y,x}\int_0^1 &\Big[Q(y|x,1)P_{X|SB}(x|1,b')(1-b') \\ &+ Q(y|x,0)P_{X|SB}(x|0,b')b'\Big]\pi(b')db' \quad \forall b \in [0,1]. \end{aligned} \quad (9)$$

Fig. 3 depicts the support of $\pi(\cdot)$ for differnet values of p. In particular, for any p, the $*$ marks represent the points where $\pi(\cdot)$ is positive.

Based on Fig. 2, several comments on the capacity of GTC(p) can be made. First, GTC(0) is simply a noiseless channel and therefore achieves a rate of 1. Similarly, in

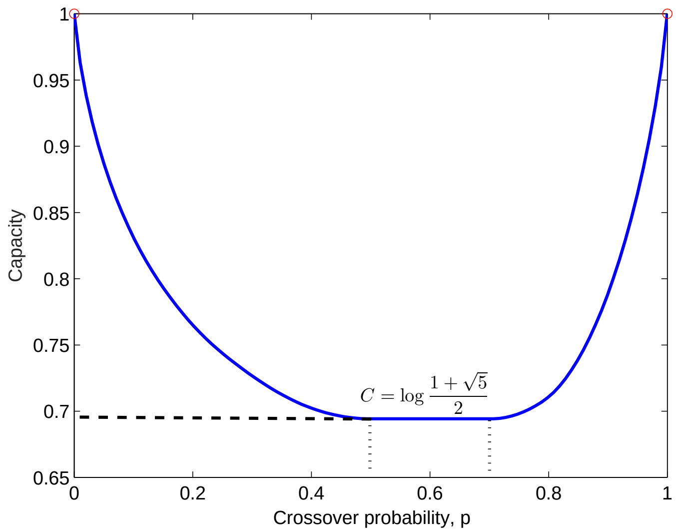

Fig. 2: Capacity for different values of p.

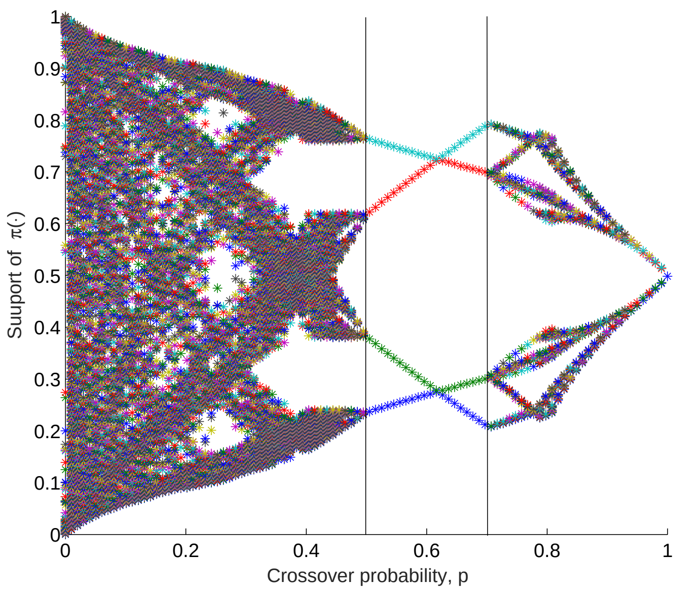

Fig. 3: Support of the steady-state distribution $\pi(\cdot)$ of B_t for different values of p.

GTC(1), $Y_t = S_t$ for all t, which results in $Y_{t+1} = X_t$. In other words, GTC(1) is a noiseless channel with unit-delayed output and therefore it also achieves rate 1.

A surprising finding from Fig. 2 is that for a range of values of p above $p = 0.5$ the capacity seems to remain constant. To the authors' knowledge this is the first time such a phenomenon is observed for any kind of channel (with/without memory, with/without feedback, single-/multi-user, etc). Further investigation of the steady-state distribution of B_t for this rage of values of p reveals (see Fig. 3) that the support of the steady-state distribution $\pi(\cdot)$ is finite, and in particular, it consists of exactly four values (as reported in [5] for the special case $p = 1/2$). It is natural to conjecture that the capacity of GTC(p) is constant for all p in the flat region. This is exactly what we set out to prove in the remaining of this section.

The sketch of the proof is as follows. We first define a special function $V_0 : [0,1] \to \mathbb{R}$ and the iterates $V_{k+1} = TV_k \quad \forall k \geqslant 0$ for an appropriately defined mapping T (depending on C). We then show that the resulting sequence $\{V_k\}_{k\geqslant 0}$ is monotonically nonincreasing, bounded, and therefore has a limit V. Moreover, the convergence is uniform and consequently V satisfies the fixed-point equation $V = TV$, due to continuity of T. This fixed-point equation is exactly the Bellman equation in (7), which proves that C is the capacity.

Define $T : ([0,1] \to \mathbb{R}) \to ([0,1] \to \mathbb{R})$ by

$$\begin{aligned}(Tv)(b) = & \\ \sup_{u_0,u_1\in[0,1]} & \Big\{ H(y_0(b,u_0,u_1)) - H(p)[b(1-u_0) + (1-b)u_1] \\ & + y_0(b,u_0,u_1)v(\phi_0(b,u_0,u_1)) \\ & + y_1(b,u_0,u_1)v(\phi_1(b,u_0,u_1)) \Big\} - C \\ & \qquad \forall v \in [0,1] \to \mathbb{R}. \qquad (10)\end{aligned}$$

Define two constants m, M by

$$M = \frac{(\frac{1-p}{p})^{2p-1}}{2^{-C+1}/(3-\sqrt{5}) + (\frac{1-p}{p})^{2p-1}} \qquad (11a)$$

$$m = 1 - \phi_0(M, 1, \frac{3-\sqrt{5}}{2}). \qquad (11b)$$

Finally, define $V_0 : [0,1] \to \mathbb{R}$ as the maximal concave function that satisfies

$$V_0(b) = \begin{cases} H(b) - Cb & , b \in [m, M] \\ H(b) - C(1-b) & , b \in [1-M, 1-m] \end{cases} \qquad (12)$$

To be more precise, for $b \in [0, m]$, V_0 is the tangent line of the function $H(b) - bC$ at $b = m$. Similarly, for $b \in [M, 0.5]$, V_0 is the tangent line of the function $H(b) - bC$ at $b = M$. Now the shape of V_0 is explicitly described for $b \in [0, 0.5]$. The shape of V_0 in $[0.5, 1]$ is the mirror image of that of V_0 in $[0, 0.5]$. Fig. 4 is the graph of V_0 for $p = 0.6$. The bold solid lines are values defined in (12) and the thin solid lines are the tangent lines which make V_0 the maximal concave function subject to (12).

We begin by proving the preservation of continuity and concavity under the mapping T. Since V_0 is continuous and concave by construction, this property of T will in turn establish the continuity and the concavity of $\{V_k\}_{k\geqslant 0}$.

Lemma 1. *If $v : [0,1] \to \mathbb{R}$ is continuous and concave, then Tv is continuous and concave.*

Proof. Since $y_0(\cdot)$, $y_1(\cdot)$, $y_0(\cdot)v(\phi_0(\cdot))$, and $y_1(\cdot)v(\phi_1(\cdot))$ are continuous w.r.t. (b, u_0, u_1) on the closed region $[0,1]^3$ and thus uniformly continuous, Tv is continuous on $[0,1]$.

To show the concavity of Tv, we introduce new variables $w = bu_0$ and $z = (1-b)u_1$ and define h by

$$\begin{aligned} h(b,w,z) = & H(\tilde{y}_0(b,w,z)) - H(p)(b - w + z) \\ & + \tilde{y}_0(b,w,z)v(\tilde{\phi}_0(b,w,z)) \\ & + \tilde{y}_1(b,w,z)v(\tilde{\phi}_1(b,w,z)), \qquad (13)\end{aligned}$$

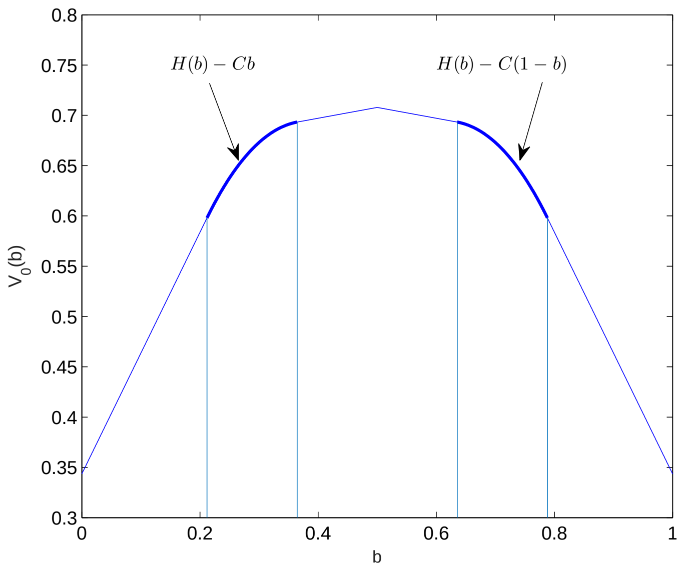

Fig. 4: Function $V_0(\cdot)$ for $p=0.6$.

where $\tilde{y}_0,\ \tilde{y}_1,\ \tilde{\phi}_0,\ \tilde{\phi}_1 : [0,1]^3 \to [0,1]$ are defined as

$$\tilde{y}_0(b,w,z) = pb + (1-p)w + (1-p)z \tag{14a}$$
$$\tilde{y}_1(b,w,z) = 1 - \tilde{y}_0(b,w,z) \tag{14b}$$
$$\tilde{\phi}_0(b,w,z) = \frac{w}{\tilde{y}_0(b,w,z)} \tag{14c}$$
$$\tilde{\phi}_1(b,w,z) = \frac{(1-p)(b-w)+pz}{\tilde{y}_1(b,w,z)}. \tag{14d}$$

With these definitions, Tv can be rewritten as

$$(Tv)(b) = \sup_{w\in[0,b],z\in[0,1-b]} h(b,w,z) - C \quad \forall b \in [0,1] \tag{15}$$

We will first show that h is concave in (b,w,z) and then show the concavity of Tv based on that of h. Consider the first term of h, which is $H(\tilde{y}_0(b,w,z))$. Since $\tilde{y}_0$ is affine in (b,w,z) and H is concave, $H(\tilde{y}_0(b,w,z))$ is concave in (b,w,z). The second term, $H(p)(b-w+z)$, is affine in (b,w,z). Recall the concavity of a function v implies that $tv(\frac{r}{t})$ is concave in (r,t). Since $\tilde{y}_0(b,w,z)$ is affine in (b,w,z), the third term

$$\tilde{y}_0(b,w,z)v(\tilde{\phi}_0(b,w,z)) = \tilde{y}_0(b,w,z)v(\frac{w}{\tilde{y}_0(b,w,z)}) \tag{16}$$

is concave in (b,w,z). Similarly, the fourth term, $\tilde{y}_1(b,w,z)v(\tilde{\phi}_1(b,w,z))$, is concave in (b,w,z). Subsequently, h is concave in (b,w,z). Now we are ready to prove the concavity of Tv.

Given any $\theta, b_1, b_2 \in [0,1]$,

$$\begin{aligned}
&(Tv)(\theta b_1 + \bar{\theta} b_2)\\
&= \sup_{\substack{w\in[0,\theta b_1+\bar{\theta}b_2]\\ z\in[0,1-(\theta b_1+\bar{\theta}b_2)]}} h(\theta b_1+\bar{\theta}b_2, w, z) - C\\
&= \sup_{\substack{w_1\in[0,b_1],z_1\in[0,1-b_1]\\ w_2\in[0,b_2],z_2\in[0,1-b_2]}} h(\theta b_1+\bar{\theta}b_2, \theta w_1+\bar{\theta}w_2, \theta z_1+\bar{\theta}z_2) - C\\
&\overset{(a)}{\geqslant} \sup_{\substack{w_1\in[0,b_1],z_1\in[0,1-b_1]\\ w_2\in[0,b_2],z_2\in[0,1-b_2]}} \theta h(b_1,w_1,z_1) + \bar{\theta} h(b_2,w_2,z_2) - C\\
&= \sup_{\substack{w_1\in[0,b_1]\\ z_1\in[0,1-b_1]}} \theta h(b_1,w_1,z_1) - \theta C\\
&\quad + \sup_{\substack{w_2\in[0,b_2]\\ z_2\in[0,1-b_2]}} \bar{\theta} h(b_2,w_2,z_2) - \bar{\theta} C\\
&= \theta (Tv)(b_1) + \bar{\theta}(Tv)(b_2),
\end{aligned} \tag{17}$$

where (a) is due to the concavity of v. □

Define p^* as the positive solution to the equation

$$p = (\frac{1-p}{p})^{2p-1}. \tag{18}$$

$p^* \approx 0.703506$ represents the upper bound on the flat region in Fig. 1, a fact that will be proved later. In the next lemma, we prove the stationarity of $\{V_k\}_{k\geqslant 0}$ in $[m,M]\cup[1-M,1-m]$.

Lemma 2. *For any $p \in [0.5, p^*]$, the values of functions $\{V_k\}_{k\geqslant 0}$ in $[m,M]\cup[1-M,1-m]$ are stationary. In other words,*

$$V_k(b) = V_0(b) \qquad \forall b \in [m,M]\cup[1-M,1-m]. \tag{19}$$

Proof. It is sufficient to prove that $V_1(b) = V_0(b) \quad \forall b \in [m,M]$. Define f by

$$\begin{aligned}
f(b,u_0,u_1) = {} & H(y_0(b,u_0,u_1)) - H(p)[b(1-u_0)+(1-b)u_1]\\
&+ y_0(b,u_0,u_1)V_0(\phi_0(b,u_0,u_1))\\
&+ y_1(b,u_0,u_1)V_0(\phi_1(b,u_0,u_1)).
\end{aligned} \tag{20}$$

Consider the optimization problem for a fixed $b \in [m,M]$

$$\sup_{u_0,u_1\in[0,1]} f(b,u_0,u_1). \tag{21}$$

Since f is concave in (u_0,u_1) due to the concavity of V_0 and the inequality constraints are convex, the KKT conditions are necessary and sufficient for optimality. In the following we will show that $(u_0^*, u_1^*) = (1, \frac{3-\sqrt{5}}{2})$ satisfy the KKT conditions, i.e., $\frac{\partial f}{\partial u_0}(b,u_0^*,u_1^*) \geqslant 0$ and $\frac{\partial f}{\partial u_1}(b,u_0^*,u_1^*) = 0$.

For $b = m$, we have

$$\begin{aligned}
\phi_0(m,1,\frac{3-\sqrt{5}}{2}) &\overset{(a)}{=} \phi_0(1-\phi_0(M,1,\frac{3-\sqrt{5}}{2}),1,\frac{3-\sqrt{5}}{2})\\
&\overset{(b)}{=} 1-M,
\end{aligned} \tag{22}$$

where (a) is due to the definition of m in (11) and (b) is due to the following property of ϕ_0

$$\phi_0(1-\phi_0(b,1,u_2),1,u_2) = 1-b \quad \forall b\in[0,1], u_2\in[0,1]. \tag{23}$$

Similarly, for $b = M$ we have form the definition of m in (11) that $\phi_0(M, 1, \frac{3-\sqrt{5}}{2}) = 1-m$. Since $\phi_0(b, 1, \frac{3-\sqrt{5}}{2})$ is increasing w.r.t. b, we conclude that $\phi_0(b, 1, \frac{3-\sqrt{5}}{2}) \in [1-M, 1-m]$ for all $b \in [m, M]$.

On the other hand, $\phi_1(b, 1, \frac{3-\sqrt{5}}{2}) = \frac{p}{\frac{1+\sqrt{5}}{2}+p}$, which is an increasing function of p and constant w.r.t. b. By observing that

$$\frac{p}{\frac{1+\sqrt{5}}{2}+p} = m \Leftrightarrow p = 0.5 \tag{24a}$$

$$\frac{p}{\frac{1+\sqrt{5}}{2}+p} = M \Leftrightarrow p = (\frac{1-p}{p})^{2p-1}, \tag{24b}$$

we deduce that $\phi_1(b, 1, \frac{3-\sqrt{5}}{2}) \in [m, M]$ whenever $p \in [0.5, p^*]$.

The above arguments result in that $\forall b \in [m, M]$, $\phi_0(b, 1, \frac{3-\sqrt{5}}{2})$ and $\phi_1(b, 1, \frac{3-\sqrt{5}}{2})$ are in $[m, M] \cup [1-M, 1-m]$ and therefore the closed-form expression in (12) can be used in calculating the values partial derivatives of f w.r.t. to u_0 and u_1 at $(b, 1, \frac{3-\sqrt{5}}{2})$ respectively. In particular, we have

$$\begin{aligned}
&\frac{\partial f}{\partial u_0}(b, 1, \frac{3-\sqrt{5}}{2}) \\
&\quad = b[(2p-1)\log\frac{1-p}{p} + \log\frac{1-b}{b} + C + \log\frac{3-\sqrt{5}}{2}] \\
&\quad \overset{(a)}{=} b\log(\frac{M}{1-M}\frac{1-b}{b}) \\
&\quad \geqslant 0 && (25a)\\
&\frac{\partial f}{\partial u_1}(b, 1, \frac{3-\sqrt{5}}{2}) = 0, && (25b)
\end{aligned}$$

where (a) is due to the definition of M in (11) and (25a) is due to $b \leqslant M$. Therfore, the KKT conditions are satisfied, so

$$\begin{aligned}
V_1(b) &= (TV_0)(b) \\
&= \sup_{u_0, u_1 \in [0,1]} f(b, u_0, u_1) - C \\
&= f(b, 1, \frac{3-\sqrt{5}}{2}) - C \\
&= H(b) - bC \\
&= V_0(b) \qquad \forall b \in [m, M]. && (26)
\end{aligned}$$

Due to symmetry, the stationarity also holds for $b \in [1-M, 1-m]$. Repeated application of this argument shows that $V_{k+1}(b) = V_k(b)$ for all $b \in [m, M] \cup [1-M, 1-m]$ and $\forall k \geqslant 1$. □

In anticipation of Lemma 3, we now define an order on $[0, 1] \to \mathbb{R}$ as follows. For any functions $v, v' : [0, 1] \to \mathbb{R}$, we say that v is greater or equal to v', denoted by $v \succeq v'$, if and only if

$$v(b) \geqslant v'(b) \qquad \forall b \in [0, 1]. \tag{27}$$

From the definition of T, we can easily verify that

$$Tv \succeq Tv' \qquad \text{if } v \succeq v'. \tag{28}$$

Lemma 3. *There exists a function $V : [0, 1] \to \mathbb{R}$ such that*

$$\lim_{k\to\infty} V_k = V, \tag{29}$$

where convergence is interpreted in the sup-norm.

Proof. Since V_0 is concave and satisfies (12), by Lemma 1 and Lemma 2, V_k is concave and satisfies (12) for all $k \geqslant 0$. Furthermore, $V_0 : [0, 1] \to \mathbb{R}$ is the maximal concave function that satisfies (12). This results in

$$V_k = T^k(V_0) \overset{(a)}{\succeq} T^k(V_1) = V_{k+1} \qquad \forall k \geqslant 0, \tag{30}$$

where (a) is due to (28). Therefore $\{V_k\}_{k\geqslant 0}$ is monotonically nonincreasing, and subsequently $\{V_k\}_{k\geqslant 0}$ has a pointwise limit V such that

$$V(b) \triangleq \lim_{k\to\infty} V_k(b). \tag{31}$$

To prove convergence of $\{V_k\}_{k\geqslant 0}$ to V in the sup-norm, it is equivalent to prove that $\{V_k(b)\}_{k\geqslant 0}$ converges to $V(b)$ uniformly on $[0, 1]$. By the concavity of V_k $\forall k \geqslant 0$, V is concave and thus is continuous on $(0, 1)$. Let $\widetilde{V}$ be the continuous extension of V from $(0, 1)$ to $[0, 1]$. $\widetilde{V} \succeq V$ due to the concavity of V. On the other hand, since $\{V_k\}_{k\geqslant 0}$ is monotonically nonincreasing, $V_k \succeq V$ and thus $V_k \succeq \widetilde{V}$ by the continuity of $\widetilde{V}$. Taking $k \to \infty$ in the above inequality we have $V \succeq \widetilde{V}$ and thus $V = \widetilde{V}$. Since $\{V_k\}_{k\geqslant 0}$ is a sequence of nonincreasing continuous functions and V is continuous, $\{V_k\}_{k\geqslant 0}$ converges to V uniformly by Dini's theorem [7]. □

With the convergence of $\{V_k\}_{k\geqslant 0}$ to V in the sup-norm, we are ready to present our main result.

Proposition 1. *For any $p \in [0.5, p^*]$, the capacity of the GTC(p) is equal to C.*

Proof. By Lemma 3, $\{V_k\}_{k\geqslant 0}$ converges to V in the sup-norm. Therefore we have

$$V = \lim_{k\to\infty} V_{k+1} = \lim_{k\to\infty} TV_k \overset{(a)}{=} T(\lim_{k\to\infty} V_k) = TV, \tag{32}$$

where (a) is due to that T is sup-norm continuous and the uniform convergence of $\{V_k\}_{k\geqslant 0}$ to V. The equation $TV = V$ implies that (C, V) satisfies the Bellman fixed point equation (7) and thus C is the capacity. □

IV. Conclusion

In this paper, we considered a generalization of the trapdoor channel and showed that for a specific region of crossover probabilities, the channel capacity is constant. This is a surprising and unique phenomenon, the consequences of which we have not yet fully understood.

An obvious research question that has not been addressed in this paper is whether it is possible to find a closed-form expression for the capacity for values of p outside the interval $[0.5, p^*]$. This seems to be a much harder problem since for this range of values we do not have a guess for the capacity expression as a function of p and therefore the techniques used in this work cannot be readily extended.

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