The feedback capacity of a class of finite state multiple access channels

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***Abstract*—For the discrete memoryless (DM) multiple access channel (MAC) with noiseless feedback Cover and Leung (CL) provided an achievable region that was later shown by Willems to be the capacity region for a special class of channels.**

In this paper, we investigate the generalization of the DM-MAC channel with state evolving as a Markov chain, with or without inter-symbol interference (ISI). The state is perfectly observed at the receiver and through noiseless feedback the state and output are also available at the transmitter with unit delay. For the case of no ISI, we provide an achievable region that can be thought of as the generalization of the CL region, and show that it is the capacity region of a special class of channels.

For the case where ISI is present such a single-letter achievable region is harder to find. To address this difficulty, we start from the general multi-letter capacity expression derived by Permuter, Weissman and Chen for MACs with state and feedback, and attempt a reduction towards a single-letter expression. This novel–in the context of MACs–approach is based on viewing the multi-letter optimization problem as a stochastic control problem and finding a sufficient "state" for control. The aforementioned reduction is not complete but it hints at a single-letter region that is proven to be achievable using typicality arguments.

I. INTRODUCTION

Output feedback was shown to provide a strict improvement in the capacity of multiple-access channels (MACs) by Gaarder and Wolf [1]. The well-known block Markov superposition coding scheme of Cover and Leung (CL) [2] for the discrete memoryless MAC (DM-MAC) with feedback was shown to be the actual capacity for a class of channels [3]. Improvements on the CL achievable region were reported in [4], [5]. A multi-letter expression based on directed information was found in [6] for the DM-MAC, while recently [7] provided similar expressions for MACs with feedback and memory. As of now, single-letter characterizations of achievable regions, let alone the capacity region for the MAC with feedback and state is still an open problem.

Recently, stochastic control techniques have been used successfully to simplify multi-letter capacity expressions for point-to-point finite-state channels (FSCs) with feedback [8], [9], [10], [11], [12], [13], [14] starting from the multi-letter directed information expressions. In [15] the authors attempted an interpretation of the CL region for the DM-MAC using a stochastic control framework. This interpretation is not complete and there is still a missing link between the CL expressions and the ones derived by stochastic control approaches. Incomplete as it is, this viewpoint can be extremely useful in providing clues for the capacity region of channels for which a single-letter expression is not yet available. In such a case, once the expression is hypothesized, more standard (typicality-based) arguments can be employed to prove achievability.

In this paper, we investigate the generalization of the DM-MAC channel with finite state evolving as a Markov chain, with or without inter-symbol interference (ISI). The state is perfectly observed at the receiver and through noiseless feedback the state and output are also available at the transmitter with unit delay. For the case of no ISI, we provide an achievable region that can be thought of as the generalization of the CL region. Not surprisingly, the expressions resemble those of [2] as well as the expression for the capacity of a point-to-point channels with finite state and no ISI [16], [13], [17]. We show that for the class of channels with inputs X, Y, state S, and output Z, having $H(X|Y,Z,S)=0$ the obtained achievable region is indeed the capacity region.

For the case where ISI is present, finding a single-letter expression for an achievable region (or the capacity in a special case) is a more difficult problem. This is due to the fact that it is not clear how the DM-MAC result of [2] and the point-to-point ISI result of [10], [13], [17] can be "merged" into the alleged capacity region expression. To address this problem we consider a novel (in the context of MACs) way that hints at a single-letter expression. The starting point for this reduction is the general multi-letter capacity expression derived by Permuter, Weissman and Chen [7] for MACs with state and feedback. Our approach is based on viewing the multi-letter optimization problem as an infinite-horizon stochastic control problem. Unfortunately, unlike the point-to-point case, this reduction is not complete and does not provide a readily useful expression as in [10], [13], [17]. It does however provide a good starting point for a "guess" of a single-letter expression for an achievable region. The presentation of this result is done in a reverse way in this paper. In particular, we first establish an achievable region using typicality arguments without appealing to the aforementioned control-theoretic framework. Subsequently, we discuss how this expression was hinted at starting from the multi-letter expression of [7] and performing the reduction technique mentioned above.

The rest of the paper is organized as follows. In Section II, the channel model and preliminaries are introduced. The achievable region for the case of no-ISI is developed in III, and it is shown to be the capacity for a special class of channels in Section IV. The achievable region for the case of ISI is developed in Section V. We formulate, simplify and discuss the stochastic control problem related to the ISI case in Section VI. Due to space limitations, most proofs are omitted and can be found in [18].

In the following, we denote random variables with capital letters $(X, Y, Z, ...)$, their realizations with small letters $(x, y, z, ...)$, and alphabets with caligraphic letters $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, ...)$. A sequence of random variables is denoted with $X^t = (X_1, ..., X_t)$. The space of probability distributions on the alphabet $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$.

II. Channel model and preliminaries

We consider a two-user finite-state MAC (FS-MAC) with inputs X_t, Y_t, output Z_t and state S_t at time t. Input, output and state alphabets are finite and of size $|\mathcal{X}|$, $|\mathcal{Y}|$, $|\mathcal{Z}|$, $|\mathcal{S}|$, respectively. The channel conditional probability is

$$\begin{aligned} P(Z_t, S_{t+1}|X^t, Y^t, S^t Z^{t-1}) &= \\ Q'(Z_t|X_t, Y_t, S_t)Q(S_{t+1}|S_t, X_t, Y_t), \end{aligned} \tag{1}$$

for given stochastic kernels $Q \in \mathcal{S}\times\mathcal{X}\times\mathcal{Y} \to \mathcal{P}(\mathcal{S})$ and $Q' \in \mathcal{S}\times\mathcal{X}\times\mathcal{Y} \to \mathcal{P}(\mathcal{Z})$, which implies that the considered channel states are affected by both nature and ISI. For the special case of no-ISI we have $P(S_{t+1}|S^t, X^t, Y^t, Z^t) = Q(S_{t+1}|S_t)$. At time t the receiver has access to the current channel output Z_t and state S_t, while the state S_t and output Z_t are fed back to the transmitters with unit delay through a noiseless channel. Encoders generate their channel inputs based on their private messages and past outputs/states. Thus

$$X_t = e^1_t(W_1, Z^{t-1}, S^{t-1}) \tag{2a}$$

$$Y_t = e^2_t(W_2, Z^{t-1}, S^{t-1}) \tag{2b}$$

The decoder estimates the messages W_1 and W_2 based on T channel outputs. Hence,

$$(\hat{W}_1, \hat{W}_2) = d(Z^T). \tag{3}$$

Throughout this paper we will assume that for all values of z, x, y, s, s' we have $Q'(z|x, y, s)$, $Q(s|s', x, y) \in (0, 1)$. This is a sufficient condition for channel ergodicity and directed information stability.

Unfortunately, a single-letter capacity expression is not known for this channel. A multi-letter capacity expression has been established in [7] and under the additional assumptions stated above, can be stated as follows.

Fact 1 (Theorem 5.1 in [7], [19])**.** *The capacity region of the FS-MAC with state and output feedback is*

$$\mathcal{C}^{FB} = \cup_{T\geq 1}\mathcal{C}_T \tag{4}$$

where $\mathcal{C}_T$ is defined as $\mathcal{C}_T = co\,(\mathcal{R}_T)$, where $co(A)$ denotes the convex hull of a set A, and

$$\mathcal{R}_T = \bigcup_{\mathcal{P}_T} \left\{ (R_1, R_2) : \begin{array}{lcl} R_1 & \leq & I_T(X \to Z, S||Y) \\ R_2 & \leq & I_T(Y \to Z, S||X) \\ R_1 + R_2 & \leq & I_T(X, Y \to Z, S) \end{array} \right\}, \tag{5}$$

where $I_T(A \to B||C) = \frac{1}{T}\sum_{t=1}^T I(A^t; B_t|C^t, B^{t-1})$. All information quantities are evaluated using the joint distribution

$$\begin{aligned} P(x^T, y^T, z^T, s^T) = \prod_{t=1}^{T} Q'(z_t|x_t, y_t, s_t)Q(s_{t+1}|s_t, x_t, y_t) \\ q_1(x_t|x^{t-1}, z^{t-1}, s^{t-1})q_2(y_t|y^{t-1}, z^{t-1}, s^{t-1}), \end{aligned} \tag{6}$$

and $\mathcal{P}_T$ contains all kernels q_1, q_2.

Furthermore [19], the region $\mathcal{C}_T$ can be expressed in the form

$$\begin{aligned} \mathcal{C}_T = \big\{(R_1, R_2) \geq \mathbf{0} : \forall\ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+, \\ \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3(R_1 + R_2) \leq C^{\boldsymbol{\lambda}}_T\big\}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} C^{\boldsymbol{\lambda}}_T \triangleq \sup_{\mathcal{P}_T} \{\lambda_1 I_T(X \to Z, S||Y) + \\ \lambda_2 I_T(Y \to Z, S||X) + \lambda_3 I_T(X, Y \to Z, S)\}. \end{aligned} \tag{8}$$

Similarly, $\mathcal{C}^{FB}$ can be expressed in the form

$$\begin{aligned} \mathcal{C}^{FB} = \big\{(R_1, R_2) \geq \mathbf{0} : \forall\ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+, \\ \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3(R_1 + R_2) \leq C^{\boldsymbol{\lambda}}\big\}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} C^{\boldsymbol{\lambda}} \triangleq \lim_{T\to\infty} \sup_{\mathcal{P}_T} \{\lambda_1 I_T(X \to Z, S||Y) + \\ \lambda_2 I_T(Y \to Z, S||X) + \lambda_3 I_T(X, Y \to Z, S)\}. \end{aligned} \tag{10}$$

We say that the channel is in the family $\mathcal{C}_{\mathcal{Y}\to\mathcal{X}}$ when the second user can perfectly determine the first user's channel inputs based on its own inputs and the channel output and state. In other words,

$$\mathcal{C}_{\mathcal{Y}\to\mathcal{X}} \triangleq \{Q' : H(X|Y, Z, S) = 0\}. \tag{11}$$

Finally we define the class $\mathcal{C}_{\mathcal{Y}\leftrightarrow\mathcal{X}} \triangleq \mathcal{C}_{\mathcal{Y}\to\mathcal{X}} \cap \mathcal{C}_{\mathcal{X}\to\mathcal{Y}}$

III. An achievable region for the FS-MAC with no ISI

In this section we provide a single-letter expression for an achievable region for the FS-MAC with state and output feedback and no ISI which can be thought of as the generalization of the Cover and Leung region [2].

Proposition 1. *For the no-ISI FS-MAC with state and output feedback an achievable region is given by the set of inequalities*

$$R_1 \leq I(X; Z|Y, S, V) \tag{12a}$$

$$R_2 \leq I(Y; Z|X, S, V) \tag{12b}$$

$$R_1 + R_2 \leq I(X, Y; Z|S, S'), \tag{12c}$$

where all information quantities are evaluated using the joint distribution

$$\begin{aligned} P_{VSS'XYZ}(v, s, s', x, y, z) = \pi(s')P_{V|S'}(v|s')\times \\ P_{X|V}(x|v)P_{Y|V}(y|v)Q(s|s')Q'(z|x, y, s), \end{aligned} \tag{13}$$

the distribution π is the steady-state distribution of the channel, and the cardinality of the alphabet $\mathcal{V}$ may be limited to $|\mathcal{V}| \leq |\mathcal{X}||\mathcal{Y}|$.

Proof. We extend the block Markov coding technique of [2] whereby at the begining of block b the two transmitters know each others messages from block $b-1$, and the receiver knows part of that message. During block b, the two transmitters cooperate in order to resolve the remaing uncertainty W_0 at the receiver about this message through the transmission of a codeword $v^n = (v_t(W_0; s_{t-1}))_{t=1}^n$. In addition, they superimpose (in a causal fashion) on v^n codewords $x^n = (x_t(W_1; v_t))_{t=1}^n$ and $y^n = (y_t(W_2; v_t))_{t=1}^n$, respectively, in order to encode their messages W_1, W_2 from block b. Note, that in this scheme we utilize a "causal" superposition strategy using only $|\mathcal{V}|$ pairs of codebooks, as opposed to the block superposition strategy of [2] that uses one pair of codebooks for each codeword v^n. At the end of block b, transmitter one performs typical decoding to extract message $\hat{W}_2$ based on $(v^n, x^n, y^n(\tilde{W}_2; v^n), z^n, s^n)$, and similarly for transmitter two. The receiver performs a two stage decoding. First it extracts $\hat{W}_0$ by typical decoding based on $(v^n(\tilde{W}_0; s^n), z^n, s^n)$ and obtains $\hat{v}^n = v^n(\hat{W}_0; s^n)$. At the second stage it performs list typical decoding for messages W_1, W_2 based on $(\hat{v}^n, x^n(\tilde{W}_1; \hat{v}^n), y^n(\tilde{W}_2; \hat{v}^n), z^n, s^n)$ producing a list $\mathcal{L}$.

The complete proof can be found in the extended version [18] □

IV. The capacity region for the FS-MAC with no ISI in the class $\mathcal{C}_{\mathcal{Y}\to\mathcal{X}}$

For the special case of FS-MAC with no ISI in the class $\mathcal{C}_{\mathcal{Y}\to\mathcal{X}}$ it can be shown the the above achievable region is also the capacity region. This result is formally stated below.

Proposition 2. *For the no-ISI FS-MAC in the class $\mathcal{C}_{\mathcal{Y}\to\mathcal{X}}$ with state and output feedback the capacity region is the set $\mathcal{C}_{\mathcal{Y}\to\mathcal{X}}^{FB} = co(\mathcal{R}_{\mathcal{Y}\to\mathcal{X}}^{FB})$, where*

$$\mathcal{R}_{\mathcal{Y}\to\mathcal{X}}^{FB} = \bigcup_{\mathcal{P}} \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \leq & I(X; Z|Y, S, V) \\ R_2 & \leq & I(Y; Z|X, S, V) \\ R_1 + R_2 & \leq & I(X, Y; Z|S, S') \end{array} \right\}, \quad (14)$$

where all information quantities are evaluated using joint distributions of the form

$$P_{VSS'XYZ}(v, s, s', x, y, z) = \pi(s') P_{V|S'}(v|s') \times P_{X|V}(x|v) P_{Y|V}(y|v) Q(s|s') Q'(z|x, y, s), \quad (15)$$

the distribution π is the steady-state distribution of the channel, the set $\mathcal{P}$ contains all distributions $(P_{V|S'}, P_{X|V}, P_{Y|V})$, with $|\mathcal{V}| \leq |\mathcal{X}||\mathcal{Y}|$.

Proof. The proof follows that of [3] by establishing an outer bound for the rate region under the stated assumptions and showing that this outer bound coincides with that of Proposition 1.

The complete proof can be found in the extended version [18] □

V. An Achievable Rate Region for MAC with finite states and ISI

Here we introduce an auxiliary random variable $V_t \in \mathcal{V}$. Define the random quantity $\Phi_t \in P(\mathcal{X} \times \mathcal{Y} \times \mathcal{V})$. For now this quantity is not given any meaning, but it can be generated by a recursive function of the form $\Phi_t = \Psi(\Phi_{t-1}, Z_t, Z_{t-1}, S_t, S_{t-1})$, where the mapping Ψ is implicitly defined through the recursion

$$\Phi_t(x_t, y_t, v_t) = \frac{P(x_t, y_t, v_t, Z_t, S_t|S^{t-1}, Z^{t-1})}{\sum_{\tilde{x}_t, \tilde{y}_t, \tilde{v}_t} P(x_t, y_t, v_t, Z_t, S_t|S^{t-1}, Z^{t-1})}, \quad (16)$$

with

$$P(x_t, y_t, v_t, Z_t, S_t|S^{t-1}, Z^{t-1}) = \sum_{x_{t-1}, y_{t-1}, v_{t-1}} Q'(Z_t|x_t, y_t, S_t) Q(S_t|x_{t-1}, y_{t-1}, S_{t-1}) P_{X|V}(x_t|v_t) P_{Y|V}(y_t|v_t) P_{V|V'Z'S'\Phi}(v_t|v_{t-1}, Z_{t-1}, S_{t-1}, \Phi_{t-1}) \Phi_{t-1}(x_{t-1}, y_{t-1}, v_{t-1}). \quad (17)$$

Proposition 3. *For the ISI FS-MAC with state and output feedback an achievable region is given by the set of inequalities*

$$R_1 \leq I(X, X'; Z, S|Y, Y', V, V', Z', S', \Phi) \quad (18a)$$
$$R_2 \leq I(Y, Y'; Z, S|X, X', V, V', Z', S', \Phi) \quad (18b)$$
$$R_1 + R_2 \leq I(X, X', Y, Y'; Z, S|Z', S', \Phi), \quad (18c)$$

where all information quantities are evaluated using the joint distribution

$$P_{Z'S'\Phi V'X'Y'VXYSZ}(z', s', \phi, v', x', y', v, x, y, s, z) = \pi(z', s', \phi) \phi(x', y', v') P_{V|V'Z'S'\Phi}(v|v', z', s', \phi) P_{X|V}(x|v) P_{Y|V}(y|v) Q(s|x', y', s') Q(z|x, y, s). \quad (19)$$

The distribution π is the steady-state distribution of Markov chain with the transition kernel

$$P(z_t, s_t, \phi_t|z_{t-1}, s_{t-1}, \phi_{t-1}) = \sum_{x_{t-1}, y_{t-1}, v_{t-1}} \phi_{t-1}(x_{t-1}, y_{t-1}, v_{t-1}) \sum_{v_t, x_t, y_t} P_{V|V'Z'S'\Phi}(v_t|v_{t-1}, z_{t-1}, s_{t-1}, \phi_{t-1}) P_{X|V}(x_t|v_t) P_{Y|V}(y_t|v_t) \delta_{\Psi(\phi_{t-1}, z_t, z_{t-1}, s_t, s_{t-1})}(\phi_t) Q(s_t|x_{t-1}, y_{t-1}, s_{t-1}) Q'(z_t|x_t, y_t, s_t), \quad (20)$$

and the cadinality of $\mathcal{V}$ may be limited to with $|\mathcal{V}| \leq |\mathcal{X}||\mathcal{Y}|$.

Proof. The transmission scheme is similar to that of CL except for replacing (V^n, X^n, Y^n) with encoding functions (F_0^n, F_1^n, F_2^n), respectively. At the beginning of block b the two transmitters know each others' messages from block $b-1$, and the receiver knows part of that message. During block b, the two transmitters cooperate in order to resolve the remaining uncertainty W_0 at the receiver about this message through the transmission of a codeword $f_0^n[W_0]$. In addition, they superimpose on $f_0^n[W_0]$ codewords $f_1^n[W_0, W_1]$ and

$f_2^n[W_0, W_2]$, respectively, in order to encode their messages W_1, W_2 from block b. At the end of block b, transmitter one performs typical decoding to extract message $\hat{W}_2$ based on $(f_0^n[W_0], f_1^n[W_0, W_1], f_2^n(W_0, \tilde{W}_2), z^n, s^n)$, and similarly for transmitter two. The receiver performs a two stage decoding. First it extracts $\hat{W}_0$ by typical decoding based on $(f_0^n[\tilde{W}_0], z^n, s^n)$ and obtains $\hat{f}_0^n = f_0^n[\hat{W}_0]$. At the second stage it performs list typical decoding for messages W_1, W_2 based on $(\hat{f}_0^n, f_1^n[\hat{W}_0, \tilde{W}_1], f_2^n[\hat{W}_0, \tilde{W}_2], z^n, s^n)$ producing a list $\mathcal{L}$.

The complete proof can be found in the extended version [18]. □

VI. Deriving the achievable region for the FS-MAC with ISI

In this section, we would like to show how the achievable region in Proposition 3 was hypothesized in the first place. For this we restrict attention to the FS-MAC with ISI in the class $\mathcal{C}_{\mathcal{Y}\leftrightarrow\mathcal{X}}$.

Lemma 1. *For the class $\mathcal{C}_{\mathcal{Y}\leftrightarrow\mathcal{X}}$, the quantity $C_T^{\boldsymbol{\lambda}}$ in (8) can be simplified as*

$$\begin{aligned} C_T^{\boldsymbol{\lambda}} = \sup_{\mathcal{P}'_T} \Bigg\{ \frac{1}{T}\sum_{t=1}^{T} & \\ & \lambda_1 I(X_{t-1}^t; Z_t, S_t | Y_t, X^{t-1}, Y^{t-1}, Z^{t-1}, S^{t-1}) + \\ & \lambda_2 I(Y_{t-1}^t; Z_t, S_t | X_t, X^{t-1}, Y^{t-1}, Z^{t-1}, S^{t-1}) + \\ & \lambda_3 I(X_{t-1}^t, Y_{t-1}^t; Z_t, S_t | Z^{t-1}, S^{t-1}) \Bigg\}, \end{aligned} \tag{21}$$

where all information quantities are evaluated using the joint distribution

$$\begin{aligned} P(x^T, y^T, z^T, s^T) = \prod_{t=1}^{T} Q'(z_t|x_t, y_t, s_t) Q(s_{t+1}|s_t, x_t, y_t) \\ q_1(x_t|x^{t-1}, y^{t-1}, z^{t-1}, s^{t-1}) q_2(y_t|x^{t-1}, y^{t-1}, z^{t-1}, s^{t-1}), \end{aligned} \tag{22}$$

and $\mathcal{P}'_T$ contains all kernels q_1, q_2.

Proof. The proof can be found in the extended version [18]. □

The above lemma implies that we can think of this problem as an optimization problem involving a single agent who chooses both distributions q_1 on x_t and q_2 on y_t after observing the information $x^{t-1}, y^{t-1}, z^{t-1}, s^{t-1}$. We now proceed to formulate an equivalent centralized stochastic control problem in order to further simplify the capacity region expression. Towards this end we introduce the following dynamic system.

- **state** at time t: $(X^{t-1}, Y^{t-1}, Z^{t-1}, S^{t-1}) \in \mathcal{X}^{t-1} \times \mathcal{Y}^{t-1} \times \mathcal{Z}^{t-1} \times \mathcal{S}^{t-1}$
- **observation** at time t: $(Z_{t-1}, S_{t-1}) \in \mathcal{Z} \times \mathcal{S}$
- **action** at time t: $U_t = \left(U_t^1, U_t^2\right) : \mathcal{X}^{t-1} \times \mathcal{Y}^{t-1} \to \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$. Actions at time t can depend on the observations up to time t and the interpretation is

$$\begin{aligned} & u_t^1[z^{t-1}, s^{t-1}](x_t|x^{t-1}, y^{t-1}) = \\ & \quad q_1(x_t|x^{t-1}, y^{t-1}, z^{t-1}, s^{t-1}), \\ & u_t^2[z^{t-1}, s^{t-1}](y_t|x^{t-1}, y^{t-1}) = \\ & \quad q_2(y_t|x^{t-1}, y^{t-1}, z^{t-1}, s^{t-1}) \end{aligned} \tag{23}$$

- **instantaneous reward** at time t (given $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$):

$$\begin{aligned} R_t^{\boldsymbol{\lambda}} = & \lambda_1 \log \frac{Q'(Z_t|X_t, Y_t, S_t) Q(S_t|S_{t-1}, X_{t-1}, Y_{t-1})}{P(Z_t, S_t|Y_t, X^{t-1}, Y^{t-1}, Z^{t-1}, S^{t-1})} \\ & + \lambda_2 \log \frac{Q'(Z_t|X_t, Y_t, S_t) Q(S_t|S_{t-1}, X_{t-1}, Y_{t-1})}{P(Z_t, S_t|X_t, X^{t-1}, Y^{t-1}, Z^{t-1}, S^{t-1})} \\ & + \lambda_3 \log \frac{Q'(Z_t|X_t, Y_t, S_t) Q(S_t|S_{t-1}, X_{t-1}, Y_{t-1})}{P(Z_t, S_t|Z^{t-1}, S^{t-1})}. \end{aligned} \tag{24}$$

The control problem is to determine the optimal policy $g = \{g_t\}_{t=1}^T$ (such that $u_t = g_t[z^{t-1}, s^{t-1}]$) that maximizes the average reward per unit time $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E^g\{R_t^{\boldsymbol{\lambda}}\}$. The main difficulty with the above dynamical system is that the expressions in the denominators of the instantaneous rewards do not consist of known kernels.

The next step towards simplifying the capacity expression in (10) utilizes Markov decision process (MDP) theory in order to provide a characterization of the solution to the optimal stochastic control problem mentioned above. Towards this end let us define a random variable $\Theta_t \in \mathcal{P}(\mathcal{X}^t \times \mathcal{Y}^t)$ to be a probability distribution on (x^t, y^t) conditioned on Z^t, S^t, U^t, i.e., $\Theta_t(x^t, y^t) \triangleq P(x^t y^t | Z^t, S^t, U^t), \forall x^t \in \mathcal{X}^t$. The following lemma establishes important properties of the random quantity Θ_t.

Lemma 2. *There exists a mapping Ψ such that θ_t can be recursively generated as $\theta_t = \Psi(\theta_{t-1}, u_t, z_t, s_t, s_{t-1})$. Furthermore, $(\Theta_t, S_t)_t$ is a controlled Markov chain with control u_t, i.e., $P(\theta_t, s_t|\theta^{t-1}, s^{t-1}, u^t) = P(\theta_t, s_t|\theta_{t-1}, s_{t-1}, u_t)$. Finally, an instantaneous reward can be defined with the property $\Xi^{\boldsymbol{\lambda}}(\theta_{t-1}, s_{t-1}, u_t) \stackrel{def}{=} E\{R_t^{\boldsymbol{\lambda}}|\theta^{t-1}, s^{t-1}, u^t\} = E\{R_t^{\boldsymbol{\lambda}}|\theta_{t-1}, s_{t-1}, u_t\}$.*

Proof. The complete proof can be found in the extended version [18]. □

Based on the above lemma we are now ready to state the following result.

Lemma 3. *For the class $\mathcal{C}_{\mathcal{Y}\leftrightarrow\mathcal{X}}$, the quantity $C_T^{\boldsymbol{\lambda}}$ in (8) can be simplified as*

$$\begin{aligned} C_T^{\boldsymbol{\lambda}} = \sup_{\bar{\mathcal{P}}} \Bigg\{ \frac{1}{T}\sum_{t=1}^{T} & \\ & \lambda_1 I(X_{t-1}^t; Z_t, S_t | Y_t, S_{t-1}, X^{t-1}, Y^{t-1}, \Theta_{t-1}) \\ & + \lambda_2 I(Y_{t-1}^t; Z_t, S_t | X_t, S_{t-1}, X^{t-1}, Y^{t-1}, \Theta_{t-1}) \\ & + \lambda_3 I(X_{t-1}^t, Y_{t-1}^t; Z_t, S_t | S_{t-1}, \Theta_{t-1}) \Bigg\}. \end{aligned} \tag{25}$$

where all information quantities are evaluated using the joint *distribution*

$$P(x^T, y^T, z^T, s^T, \theta^T) =$$
$$\prod_{t=1}^{T} Q'(z_t|x_t, y_t, s_t) Q(s_t|s_{t-1}x_{t-1}y_{t-1}) \times$$
$$\overline{q}_1(x_t|x^{t-1}, y^{t-1}, s_{t-1}, \theta_{t-1}) \overline{q}_2(y_t|x^{t-1}, y^{t-1}, s_{t-1}, \theta_{t-1})$$
$$\theta_{t-1}(x^{t-1}y^{t-1}) \delta_{\Psi(\theta_{t-1}, \overline{q}, z_t, s_t, s_{t-1})}(\theta_t), \quad (26)$$

and the $\bar{\mathcal{P}}$ *contains all input joint distributions on* x_t, y_t *that are conditionally factorizable as*

$$P(x_t, y_t|x^{t-1}, y^{t-1}, z^{t-1}, s^{t-1}) =$$
$$\overline{q}_1(x_t|x^{t-1}, y^{t-1}, s_{t-1}, \theta_{t-1}) \overline{q}_2(y_t|x^{t-1}, y^{t-1}, s_{t-1}, \theta_{t-1}) \in \bar{\mathcal{P}}. \quad (27)$$

Proof. Due to Lemma 2 and using standard MDP results, the optimal strategy g_t is a Markov policy (i.e., it is only a function of the state (θ_{t-1}, s_{t-1})). This implies that the optimizing distributions can be of the form $\overline{q}_1(x_t|x^{t-1}, y^{t-1}, s_{t-1}, \theta_{t-1}) \overline{q}_2(y_t|x^{t-1}, y^{t-1}, s_{t-1}, \theta_{t-1})$. Furthermore, using an inductive argument similar to the one used in the alternative proof for the no-ISI case, we can show the equivalence between the stochastic control problem and the optimization problem in (21), which concludes the proof. □

Comparing the expressions in (25) with those of Proposition 3 we see that they are not the same. Although the reduction followed above was not complete it was sufficient to hint at the latter expression.

VII. Conclusion

The MAC with Markov state and feedback is consider in this paper. The analysis is divided into the no-ISI and the ISI case. For the former we prove achievability for a region that can be considered an extension to the CL region. For the special class of channels in $\mathcal{C}_{\mathcal{X} \to \mathcal{Y}}$ it is shown that this region is indeed the channel capacity. For the ISI case the problem is much more difficult: it is hard to envision/imagine the structure of a rate region given in single-letter form that will serve as an inner/outer bound (hopefully matching for certain channel classes). For this reason we develop a novel–in the context of MACs–methodology that starts with the multi-letter expression for the capacity region of the MAC with state and feedback and try to simplify it in the hope that the simplified expression will be single-letter and will hint at a capacity region. This is done by reformulating the optimization problem as an average cost per unit time stochastic control problem and identify a sufficient statistic for control. Completing this reduction from the multi-letter to single-letter expressions using stochastic control, as well as providing outer bounds for the ISI capacity region is currently pursued by the authors.

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