A Posterior Matching Scheme for Finite-State Channels with Feedback

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Abstract

For a memoryless channel, although feedback cannot increase capacity, it can reduce the complexity and/or improve the error performance of a communication system. Recently, Shayevitz and Feder proposed the posterior matching scheme (PMS) which is a simple recursive transmission scheme that achieves the capacity of memoryless channels with feedback. Furthermore, Coleman provided a Lyapunov function approach to prove capacity achievability of the PMS.

In this paper, we investigate a capacity-achieving PMS for the case of finite-state channels (FSCs). We first derive a single-letter expression for the capacity of the FSC with delayed output and state feedback by formulating the problem in a stochastic control framework. The resulting capacity expression can be evaluated using dynamic programming. We then propose a simple recursive PMS-like transmission scheme. To prove capacity achievability of the proposed PMS, we identify an appropriate Markov chain induced by the PMS, and modify Coleman's Lyapunov function approach to prove the chain is positive Harris-recurrent, which eventually leads to capacity achievability.

I. Introduction

Communication in the presence of feedback has been a long studied problem which dates back to Shannon's early work [1], where he proved that feedback cannot increase the capacity of memoryless channels. Feedback, however, can improve the error performance and/or can simplify the transmission scheme. Horstein [2] proposed a simple sequential transmission scheme which is capacity-achieving and provides larger error exponents than traditional fixed-length block-coding. Similarly, Schalkwijk and Kailath [3] showed that capacity and a double exponentially decreasing error probability can be achieved by a simple sequential transmission scheme for the additive white Gaussian noise channel (AWGNC) with average power constraint. Recently, Shayevitz and Feder [4], [5] identified an underlying principle shared by the aforementioned Horstein and Schalkwijk-Kailath schemes and introduced a simple encoding scheme, namely the posterior matching scheme (PMS) for general memoryless channels. Furthermore, they showed that the PMS achieves the capacity of general DMCs. Subsequently, Coleman [6] revisited the PMS and provided a proof of capacity achievability by reformulating the problem in a stochastic control framework.

In this paper, we generalize the PMS for finite state channels (FSC) where the channel state is affected both by nature and by the input sequence (thus introducing inter-symbol interference (ISI)), the channel state information (CSI) and output are available at the receiver, and the CSI and output are available at the transmitter with unit delay through noiseless feedback. We show that the proposed scheme admits a simple sequential structure and achieves capacity. To analyze our transmission scheme, we adopt and extend the analytical method provided by Coleman [6] for the original PMS. The starting point of our investigation and one of the contributions of this paper is the derivation of a single-letter capacity expression for this channel.

One of the first capacity results for FSCs was by Viswanathan [7], who found the capacity of a FSC with receiver CSI and delayed feedback where there is no ISI. Later, Chen and Berger [8] found the capacity of a FSC with ISI where current CSI is available at the transmitter and the receiver. Yang et. al. [9] used a stochastic control method to find the capacity of the ISI channel. Recently, Tatikonda and Mitter [10] provided a general stochastic control framework for evaluating the capacity of the FSC with feedback. In that paper, the capacity was characterized as the solution of a dynamic programming average cost optimality equation (ACOE). Como et. al. [11] used an approach similar to [10] to find the capacity of the FSC when current CSI is available at the transmitter and the receiver. An upper bound on the capacity of the FSC without ISI and CSI was found using dynamic programming by Huang et. al. [12]. We point out that although the channel considered in this paper is indeed a special case of the one considered in [10], our approach in deriving a single-letter capacity expression is somewhat different and the resulting capacity expression is significantly simpler.

The remainder of the paper is organized as follows. In Section II, the channel model and the general form of the capacity are introduced. We derive a simplified single-letter expression of the capacity in Section III. In Section IV, the generalized PMS is presented, and the capacity achievability of the PMS is proven. Section V concludes the paper.

II. Channel model and preliminaries

We consider channels with input X_t, output Y_t and state S_t at time t. The corresponding input, output and state random processes are denoted by $(X_t)_{t=1}^{\infty}$, $(Y_t)_{t=1}^{\infty}$, $(S_t)_{t=1}^{\infty}$, respectively. Input, output and state alphabets are finite and of size $|\mathcal{X}| = K_x$, $|\mathcal{Y}| = K_y$, $|\mathcal{S}| = K_s$, respectively. At time t the receiver has access to the current channel output y_t and state s_t. The state s_t and output y_t are fed back to the transmitter with unit delay. Let $P(x^T, s^T, y^T)$ be the joint probability mass function (pmf) of X^T, Y^T, and s^T, where v^T denotes the length-T vector $(v_1, \ldots, v_T)$. Then,

$$P(x^T, s^T, y^T) = Q'(y_1|x_1, s_1)Q(s_1)P(x_1)\prod_{t=2}^{T} Q'(y_t|x_t, s_t)Q(s_t|s_{t-1}, x_{t-1})P(x_t|x^{t-1}, s^{t-1}, y^{t-1}). \tag{1}$$

Implicit in (1), is the fact that the considered channel states are affected by both nature and ISI.

A sequence of joint measures $\{P(x^T, s^T, y^T)\}_{T=1}^{\infty}$ is *directed information stable* if

$$\lim_{T\to\infty} P(|\frac{\overrightarrow{i}(X^T; S^T, Y^T)}{I(X^T \to S^T, Y^T)} - 1| > \epsilon) = 0, \qquad \forall \epsilon > 0, \tag{2}$$

where $\overrightarrow{i}(X^T; S^T, Y^T) = \log \frac{P(x^T|s^T, y^T)}{\prod_{t=1}^{T} P(x^t|s^{t-1}, y^{t-1})}$ and $I(X^T \to S^T, Y^T) = \sum_{t=1}^{T} I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1})$. Throughout the paper we assume directed information stability.

In [10] the authors have developed a capacity expression for the general class of such channels in the form of

$$C = \sup_{\{\{P(x_t|x^{t-1}, s^{t-1}, y^{t-1})\}_{t=1}^{T}\}_{T=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}). \tag{3}$$

This expression was further simplified in [10] to

$$C = \sup_{\{\{P(x_t|\pi_t, \gamma_t, s_{t-1})\}_{t=1}^{T}\}_{T=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} I(X_t, \Pi_t; S_t, Y_t|S_{t-1}, \Gamma_t) \tag{4}$$

$$= \sup_{\{P(X|\Pi, \Gamma, S')\}_{\Pi, \Gamma, S'}} I(X, \Pi; S, Y|S', \Gamma) \tag{5}$$

where $\Pi_t \in \mathcal{P}(\mathcal{S})$ defined as $\Pi_t(s_t) \stackrel{\text{def}}{=} P(s_t|X^{t-1}, S^{t-1}, Y^{t-1})$, and $\Gamma_t \in \mathcal{P}(\mathcal{P}(\mathcal{S}))$ defined as $\Gamma_t(\pi_t) \stackrel{\text{def}}{=} P(\pi_t|S^{t-1}, Y^{t-1})$. In the above, the notation $\mathcal{P}(\mathcal{S})$ is used to denote the set of probability measures on the set S. It was further shown in [10] that the capacity expression can in principle be evaluated as the solution of an appropriate ACOE [13, Th. 6.2., Th. 6.3.].

Since the above expressions are developed for general feedback patterns, and since we are only interested in the special case where both state and output are fed back to the transmitter with unit delay, it may be possible to further simplify the capacity expression in (4). We will pursue this direction in the following section.

III. A SIMPLIFIED SINGLE-LETTER CAPACITY EXPRESSION

Consider the term $I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1})$ with the channel input distribution $P(x_t|x^{t-1}, s^{t-1}, y^{t-1})$ in (3). The following theorem proves that the form of the optimal channel input distribution can be simplified.

Lemma 1. *For every T,*

$$\sup_{\{P(x_t|x^{t-1},s^{t-1},y^{t-1})\}_{t=1}^{T}} \frac{1}{T}\sum_{t=1}^{T} I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}) = \sup_{\{P(x_t|x_{t-1},s^{t-1},y^{t-1})\}_{t=1}^{T}} \frac{1}{T}\sum_{t=1}^{T} I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1}) \tag{6}$$

Proof: First, note that for every t

$$I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}) = I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1}) + I(X^{t-2}; S_t, Y_t|S^{t-1}, Y^{t-1}, X_{t-1}^t) \tag{7a}$$

$$\stackrel{(a)}{=} I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1}). \tag{7b}$$

where (a) is due to the fact that S_t, Y_t is independent of X^{t-2} conditioned on $S^{t-1}, Y^{t-1}, X_{t-1}^t$.

Each of the terms $I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1})$ in the summation is evaluated based on the joint distribution $P(x_{t-1}^t, s^t, y^t)$. We now proceed by induction to prove that the sequence of measures $\{P(x_{t-1}^t, s^t, y^t)\}_{t=1}^T$ induced by the sequence of channel input distributions $\{P(x_t|x^{t-1}, s^{t-1}, y^{t-1})\}_{t=1}^T$ equals to the sequence of measures $\{P_1(x_{t-1}^t, s^t, y^t)\}_{t=1}^T$ induced by an appropriately defined sequence of channel input distributions of the form $\{P_1(x_t|x_{t-1}, s^{t-1}, y^{t-1})\}_{t=1}^T$.

For $t = 1$ we set $P_1(x_1) = P(x)$ and have

$$P_1(x_1, s_1, y_1) = Q'(y_1|s_1, x_1)Q(s_1)P_1(x_1) = Q'(y_1|s_1, x_1)Q(s_1)P(x_1) = P(x_1, s_1, y_1). \tag{8}$$

Now for $t+1$ we set $P_1(x_{t+1}|x_t,s^t,y^t) = P(x_{t+1}|x_t,s^t,y^t) = \frac{\sum_{x^{t-1}} P(x_{t+1}|x^t,s^t,y^t)P(x^t,s^t,y^t)}{\sum_{x^{t-1}} P(x^t,s^t,y^t)}$ and have

$$P_1(x_t^{t+1},s^{t+1},y^{t+1}) = Q'(y_{t+1}|s_{t+1},x_{t+1})Q(s_{t+1}|s_t,x_t)P_1(x_{t+1}|x_t,s^t,y^t)\sum_{x_{t-1}} P_1(x_{t-1}^t,s^t,y^t) \tag{9a}$$

$$\stackrel{(a)}{=} Q'(y_{t+1}|s_{t+1},x_{t+1})Q(s_{t+1}|s_t,x_t)P(x_{t+1}|x_t,s^t,y^t)\sum_{x_{t-1}} P(x_{t-1}^t,s^t,y^t) \tag{9b}$$

$$= P(x^{t+1},s^{t+1},y^{t+1}), \tag{9c}$$

where (a) is due to the construction of $P_1(x_{t+1}|x_t,s^t,y^t)$, and the induction hypothesis. ■

The above lemma shows that in order to achieve capacity it is sufficient to restrict the channel input distributions to be the form of $P(x_t|x_{t-1},s^{t-1},y^{t-1})$, i.e., the capacity expression becomes

$$C = \sup_{\{\{P(x_t|x_{t-1},s^{t-1},y^{t-1})\}_{t=1}^T\}_{T=1}^\infty} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^T I(X^t;S_t,Y_t|S^{t-1},Y^{t-1}). \tag{10}$$

To further simplify the capacity expression, we will formulate a control problem which is equivalent to the problem of computing capacity. Towards this end, let $(x_{t-1},s_{t-1},y_{t-1})$ be the system state at time t, and (s_{t-1},y_{t-1}) be the controller observation at time t. Let the control action at time t be $U_t : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined as $u_t(x_t|x_{t-1}) \stackrel{\text{def}}{=} P(x_t|x_{t-1},s^{t-1},y^{t-1})$, i.e., the control action at time t can be a function of all observations s^{t-1},y^{t-1} up to time t. Further, define the instantaneous reward at time t to be $R_t = \log\frac{P(S_t,Y_t|S^{t-1},Y^{t-1},X_{t-1}^t)}{P(S_t,Y_t|S^{t-1},Y^{t-1})}$. The control problem is to determine the optimal policy $g = \{g_t\}_{t=1}^\infty$ (such that $u_t = g_t(s^{t-1},y^{t-1})$) that maximizes the average expected reward $\liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^T E^g[R_t]$.

First we need to prove that the above control problem is equivalent to the problem of computing capacity as stated in (10). In view of the fact that $E^g[R_t] = I(X^t;S_t,Y_t|S^{t-1},Y^{t-1})$, this equivalence is established in the following lemma.

Lemma 2. *For every sequence of channel input distributions $\{P(x_t|x_{t-1},s^t,y^t)\}_{t=1}^\infty$ with resulting sequence of joint measures $\{P(x_{t-1}^t,s^t,y^t)\}_{t=1}^\infty$ there exists a policy g with resulting sequence of joint measures $\{P^g(x_{t-1}^t,s^t,y^t)\}_{t=1}^\infty$ such that for each t: $P^g(x_{t-1}^t,s^t,y^t) = P(x_{t-1}^t,s^t,y^t)$. Conversely, for every policy g with resulting sequence of joint measures $\{P^g(x_{t-1}^t,s^t,y^t)\}_{t=1}^\infty$*

there exists a sequence of channel input distributions $\{P(x_t|x_{t-1}, s^t, y^t)\}_{t=1}^{\infty}$ *with resulting sequence of joint measures* $\{P(x_{t-1}^t, s^t, y^t)\}_{t=1}^{\infty}$ *such that for each* t*:* $P(x_{t-1}^t, s^t, y^t) = P^g(x_{t-1}^t, s^t, y^t)$.

Proof: We will use the notation $u_t = g_t(s^{t-1}, y^{t-1})$ and for convenience, we will write $u_t(x_t|x_{t-1}) = g_t[s^{t-1}, y^{t-1}](x_t|x_{t-1})$.

For the direct part, for each t we choose a policy g_t as

$$g_t[s^{t-1}, y^{t-1}](x_t|x_{t-1}) = P(x_t|x_{t-1}, s^{t-1}, y^{t-1}), \tag{11}$$

and proceed by induction.

For $t = 1$ we have

$$P^g(x_1, s_1, y_1) = Q'(y_1|s_1, x_1)Q(s_1)g_1(x_1) = Q'(y_1|s_1, x_1)Q(s_1)P(x_1) = P(x_1, s_1, y_1). \tag{12}$$

Now for $t+1$ we have

$$P^g(x_t^{t+1}, s^{t+1}, y^{t+1}) = Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)g_{t+1}[s^t, y^t](x_{t+1}|x_t)\sum_{x_{t-1}} P^g(x_{t-1}^t, s^t, y^t) \tag{13a}$$

$$\stackrel{(a)}{=} Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)P(x_{t+1}|x_t, s^t, y^t)\sum_{x_{t-1}} P(x_{t-1}^t, s^t, y^t) \tag{13b}$$

$$= P(x_t^{t+1}, s^{t+1}, y^{t+1}), \tag{13c}$$

where (a) is due to the choice of the policy g_{t+1} and the induction hypothesis.

For the converse, for each t we choose a channel input distribution as

$$P(x_t|x_{t-1}, s^{t-1}, y^{t-1}) = P^g(x_t|x_{t-1}, s^{t-1}, y^{t-1}) = g_t[s^{t-1}, y^{t-1}](x_t|x_{t-1}). \tag{14}$$

Then, for $t = 1$ we have

$$P(x_1, s_1, y_1) = Q'(y_1|s_1, x_1)Q(s_1)P(x_1) = Q'(y_1|s_1, x_1)Q(s_1)g_1(x_1) = P^g(x_1, s_1, y_1). \tag{15}$$

Now for $t+1$ we have

$$P(x_t^{t+1}, s^{t+1}, y^{t+1}) = Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)P(x_{t+1}|x_t, s^t, y^t)\sum_{x_{t-1}} P(x_{t-1}^t, s^t, y^t) \tag{16a}$$

$$\overset{(a)}{=} Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)P^g(x_{t+1}|x_t, s^t, y^t)\sum_{x_{t-1}} P^g(x_{t-1}^t, s^t, y^t) \tag{16b}$$

$$= P^g(x_t^{t+1}, s^{t+1}, y^{t+1}) \tag{16c}$$

where (a) is due to the construction of the channel input distributions and the induction hypothesis. ■

We are now ready to state and prove the main result of this section.

Proposition 1. *The capacity of the finite-state channel with feedback defined in Section II is*

$$C = \sup_{\{P(X|X',S',\Theta)\}_{X',S',\Theta}} I(X, X'; S, Y|S', \Theta) \tag{17}$$

where $\Theta \in \mathcal{P}(\mathcal{X})$, and the mutual information is evaluated using the joint measure

$$P(Y, S, X, X', S', d\Theta) = Q'(Y|S, X)Q(S|S', X')P(X|X', S', \Theta)\Theta(X')P(S', d\Theta). \tag{18a}$$

The distribution $P(S, d\Theta)$ is the solution of the equation

$$P(S, d\Theta') = \int_{S',\Theta} P(S', d\Theta)\sum_Y \delta_{\phi(\Theta, P(X|X',S',\Theta), Y, S, S')}(\Theta')\times \sum_X Q'(Y|X, S)\sum_{X'} Q(S|S', X')P(X|X', S', \Theta)\Theta(X), \tag{18b}$$

where the function $\phi(\cdot)$ is defined in the following proof.

Proof: We consider the stochastic control problem described above. The system state $(x_{t-1}, s_{t-1}, y_{t-1})$ evolves as a controlled Markov chain with control action u_t because

$$P(x_t, s_t, y_t|x^{t-1}, s^{t-1}, y^{t-1}, u^t) = Q'(y_t|s_t, x_t)Q(s_t|x_{t-1}, s_{t-1})u_t(x_t|x_{t-1}) \tag{19a}$$

$$= P(x_t, s_t, y_t|x_{t-1}, s_{t-1}, y_{t-1}, u_t) \tag{19b}$$

Define the information state $\Theta_t \in \mathcal{P}(\mathcal{X})$ with $\Theta_t(x_t) \stackrel{\text{def}}{=} P(x_t|S^t, Y^t)$ as the posterior belief on the unobserved part of the system state. Then,

$$\theta_t(x_t) = P(x_t|s^t, y^t) \tag{20a}$$
$$= \frac{P(x_t, s_t, y_t|s^{t-1}, y^{t-1})}{P(s_t, y_t|s^{t-1}, y^{t-1})} \tag{20b}$$
$$= \frac{\sum_{x_{t-1}} Q'(y_t|x_t, s_t)Q(s_t|x_{t-1}, s_{t-1})u_t(x_t|x_{t-1})P(x_{t-1}|s^{t-1}, y^{t-1})}{\sum_{x_t} P(x_t, s_t, y_t|s^{t-1}, y^{t-1})} \tag{20c}$$
$$= \frac{\sum_{x_{t-1}} Q'(y_t|x_t, s_t)Q(s_t|x_{t-1}, s_{t-1})u_t(x_t|x_{t-1})\theta_{t-1}(x_{t-1})}{\sum_{x_t}\sum_{x_{t-1}} Q'(y_t|x_t, s_t)Q(s_t|x_{t-1}, s_{t-1})u_t(x_t|x_{t-1})\theta_{t-1}(x_{t-1})}, \tag{20d}$$

which implies that θ_t can be recursively updated as $\theta_t = \phi(\theta_{t-1}, u_t, y_t, s_t, s_{t-1})$.

We now show that $\{(S_{t-1}, \Theta_{t-1})\}_t$ is a controlled Markov chain with control U_t. Indeed,

$$P(s_t, d\theta_t|s^{t-1}, \theta^{t-1}, u^t) = \sum_{y_t} P(d\theta_t|y_t, s^t, \theta^{t-1}, u^t)P(s_t, y_t|s^{t-1}, \theta^{t-1}, u^t) \tag{21a}$$
$$= \sum_{y_t} \delta_{\phi(\theta_{t-1}, u_t, y_t, s_t, s_{t-1})}(\theta_t) \sum_{x_t} Q'(y_t|x_t, s_t) \sum_{x_{t-1}} Q(s_t|s_{t-1}, x_{t-1})u_t(x_t|x_{t-1})\theta_{t-1}(x_{t-1}) \tag{21b}$$
$$= P(s_t, d\theta_t|s_{t-1}, \theta_{t-1}, u_t). \tag{21c}$$

Furthermore, the instantaneous reward r_t can be written as

$$r_t = \log \frac{P(s_t, y_t|s^{t-1}, y^{t-1}, x^t_{t-1})}{P(s_t, y_t|s^{t-1}, y^{t-1})} \tag{22a}$$
$$= \log \frac{Q'(y_t|x_t, s_t)Q(s_t|s_{t-1}, x_{t-1})}{\sum_{x_t, x_{t-1}} Q'(y_t|x_t, s_t)Q(s_t|s_{t-1}, x_{t-1})P(x_t|x_{t-1}, s^{t-1}, y^{t-1})P(x_{t-1}|s^{t-1}, y^{t-1})} \tag{22b}$$
$$= \log \frac{Q'(y_t|x_t, s_t)Q(s_t|s_{t-1}, x_{t-1})}{\sum_{x_t, x_{t-1}} Q'(y_t|x_t, s_t)Q(s_t|s_{t-1}, x_{t-1})u_t(x_t|x_{t-1})\theta_{t-1}(x_{t-1})}. \tag{22c}$$

The expected reward at time t conditioned on the states and control actions up to time t is

$$
\begin{aligned}
&E\left[R_t|s^{t-1},\theta^{t-1},u^t\right]\\
&\quad= E\left[\log\frac{Q'(Y_t|X_t,S_t)Q(S_t|s_{t-1},X_{t-1})}{\sum_{x'_t,x'_{t-1}}Q'(Y_t|x'_t,S_t)Q(S_t|s_{t-1},x'_{t-1})u_t(x'_t|x'_{t-1})\theta_{t-1}(x'_{t-1})}\mid s^{t-1},\theta^{t-1},u^t\right] &\text{(23a)}\\
&\quad= E\Bigg[\sum_{y_t,x_t,s_t,x_{t-1}}Q'(y_t|x_t,s_t)Q(s_t|s_{t-1},x_{t-1})u_t(x_t|x_{t-1})\theta_{t-1}(x_{t-1})\\
&\qquad\qquad\times\log\frac{Q'(y_t|x_t,s_t)Q(s_t|s_{t-1},x_{t-1})}{\sum_{x'_t,x'_{t-1}}Q'(y_t|x'_t,s_t)Q(s_t|s_{t-1},x'_{t-1})u_t(x'_t|x'_{t-1})\theta_{t-1}(x'_{t-1})}\Bigg] &\text{(23b)}\\
&\quad= \bar{r}(s_{t-1},\theta_{t-1},u_t), &\text{(23c)}
\end{aligned}
$$

which is only a function of the observed part of the state s_{t-1}, the belief on the unobserved part of the state θ_{t-1}, and the action u_t.

We can now deduce from standard results in Markov decision processes that the optimal policy at time t need only be a function of (s_{t-1},θ_{t-1}), i.e., the optimizing distributions in (10) need only be of the form $P(x_t|x_{t-1},s_{t-1},\theta_{t-1})$ and the capacity expression becomes

$$
C=\sup_{\{P(x_t|x_{t-1},\theta_{t-1},s_{t-1})\}_t}\liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}I(X_{t-1}^t;S_t,Y_t|S_{t-1},\Theta_{t-1}). \tag{24}
$$

Note that above described controlled Markov chain is time-homogenous, and hence the optimal channel input distribution is time-invariant, and consequently the capacity expression reduces to the one in (17). ■

To find the capacity using (17), we need to identify the stationary distribution of S and Θ for each choice of $P(X|X',S',\Theta)$ using (18b), and evaluate the mutual information by using the joint measure specified in (18a). Alternatively, we may use dynamic programming to find the capacity. In other words, the optimal value in (17) can be obtained by the solution of the following ACOE with some bounded function $\eta:\mathcal{S}\times\Theta\to\mathcal{R}$ [13, Th. 6.2., Th. 6.3.]

$$
C+\eta(s,\theta)=\sup_u\left\{\bar{r}(s,\theta,u)+\sum_{s',y'}\eta(s',\phi(\theta,u,y',s',s))\sum_{x'}Q'(y'|s',x')\sum_{x}Q(s'|s,x)u(x'|x)\theta(x)\right\}. \tag{25}
$$

IV. A CAPACITY-ACHIEVING POSTERIOR MATCHING SCHEME

In this section we will describe a transmission scheme which achieves the capacity of the finite-state channel. Let W be a random message point uniformly distributed over the unit interval. A transmission scheme is a sequence of functions $\{\tilde{g}_t : [0,1) \times \mathcal{Y}^{t-1} \times \mathcal{S}^{t-1} \to \mathcal{X}\}_{t=1}^{\infty}$ such that

$$X_t = \tilde{g}_t(W, Y^{t-1}, S^{t-1}). \tag{26}$$

Let $\hat{W}_t$ be the message point estimate at the receiver at time t. Then, a transmission scheme achieves rate R if

$$\lim_{t\to\infty} P(|W - \hat{W}_t| > 2^{-tR}) = 0. \tag{27}$$

We will now describe a simple sequential transmission scheme. We assume that the capacity achieving distributions $\{\hat{P}(X'|X, S, \Theta)\}_{X,S,\Theta}$ in (17) have been found for all values of (X, S, Θ) and the corresponding steady-state distribution on X, $\hat{P}(X)$ has been evaluated. Define the random variable F_t to be the a-posteriori cdf of W conditioned on Y^t, S^t, i.e., $F_t(w) \stackrel{\text{def}}{=} F(w|Y^t, S^t)$, $\Theta_t \in \mathcal{P}(\mathcal{X})$ with $\Theta_t(x_t) \stackrel{\text{def}}{=} P(x_t|S^t, Y^t)$, and the function $H_t : \mathcal{W} \to \mathcal{X}$. Note that X_t is perfectly determined by S^t, Y^t, W. Then, at time $t = 1$ the transmitter updates the corresponding quantities as

$$F_1 = \phi'(F_0, Y_1, S_1), \qquad F_0 = Uniform[0,1) \tag{28a}$$

$$\Theta_1 = \phi(\hat{P}(\cdot), Y_1, S_1), \tag{28b}$$

$$h_1 = \phi''(F_0), \tag{28c}$$

and for $t \geq 2$,

$$F_t = \phi'(F_{t-1}, Y_t, S_t, S_{t-1}, \Theta_{t-1}, H_{t-1}), \tag{28d}$$

$$\Theta_t = \phi(\Theta_{t-1}, \hat{P}(\cdot|\cdot, S_{t-1}, \Theta_{t-1}), Y_t, S_t, S_{t-1}), \tag{28e}$$

$$H_t = \phi''(H_{t-1}, S_{t-1}, F_{t-1}, \Theta_{t-1}), \tag{28f}$$

where ϕ' is given in (32), ϕ is given in (20), and ϕ'' is given in (34).The channel input X_t is generated as

$$X_t = F^{-1}_{\hat{P}(x_t|X_{t-1},S_{t-1},\Theta_{t-1})}(F_{t-1}(W)) \stackrel{\text{def}}{=} g(F_{t-1}(W), X_{t-1}, S_{t-1}, \Theta_{t-1}). \tag{29}$$

Assuming that the channel and state transition probabilities $Q'(y|x,s)$ and $Q(s'|s,x)$ are non-zero for all x, y, s, s', the recursion (32) guarantees that for every realization of the random variables of interest, F_t will always have a pdf; in addition the pdf will be non-zero everywhere in $[0,1]$.

At the receiver, the quantities F_t, Θ_t, and H_t are updated according to (28), and the message estimate is obtained as

$$\hat{W}_t = \arg\max_{w}\{F_t(w + 2^{-Rt}/2) - F_t(w - 2^{-Rt}/2)\}. \tag{30}$$

This can be thought as a finite-state channel counterpart of the PMS proposed in [4], [5]. We will prove that the above scheme achieves the capacity of the finite-state channel with feedback.

Lemma 3. $(F_t, \Theta_t, H_t, S_t, W)_t$ *is a Markov chain.*

Proof: Define $H_t(\cdot) = g(F_{t-1}(\cdot), X_{t-1}, S_{t-1}, \Theta_{t-1})$, and consider the following recursive expression for the realization f_{t+1} of F_{t+1}

$$df_{t+1}(w) = \frac{P(y^{t+1}, s^{t+1}|w)dw}{P(y^{t+1}, s^{t+1})} \tag{31a}$$

$$= \frac{Q'(y_{t+1}|s_{t+1}, g(f_t(w), h_t(w), s_t, \theta_t))Q(s_{t+1}|s_t, h_t(w))P(y^t, s^t|w)dw}{P(y_{t+1}, s_{t+1}|y^t, s^t)P(y^t, s^t)} \tag{31b}$$

$$= \frac{Q'(y_{t+1}|s_{t+1}, g(f_t(w), h_t(w), s_t, \theta_t))Q(s_{t+1}|s_t, h_t(w))df_t(w)}{P(y_{t+1}, s_{t+1}|y^t, s^t)} \tag{31c}$$

$$= \frac{Q'(y_{t+1}|s_{t+1}, g(f_t(w), h_t(w), s_t, \theta_t))Q(s_{t+1}|s_t, h_t(w))df_t(w)}{\int_{w'} Q'(y_{t+1}|s_{t+1}, g(f_t(w'), h_t(w'), s_t, \theta_t))Q(s_{t+1}|s_t, h_t(w'))df_t(w')}, \tag{31d}$$

which implies that f_t can be recursively updated as

$$f_{t+1} = \phi'(f_t, y_{t+1}, s_{t+1}, s_t, \theta_t, h_t). \tag{32}$$

The following is true for h_{t+1} evaluated at w.

$$h_{t+1}(w) = g(f_t(w), h_t(w), s_t, \theta_t), \tag{33a}$$

which implies that h_t can be recursively updated as

$$h_{t+1} = \phi''(h_t, s_t, f_t, \theta_t). \tag{34}$$

Using above relationships, we can identify the one-step transition probability of the Markov chain as follows.

$$\begin{aligned} &P(df_{t+1}, d\theta_{t+1}, dh_{t+1}, s_{t+1}, dw|f^t, \theta^t, h^t, s^t, w') \\ &\quad = \sum_{y_{t+1}} \delta_{\phi'(f_t, y_{t+1}, s_{t+1}, s_t, \theta_t, h_t)}(df_{t+1}) \delta_{\phi(\theta_t, \hat{P}(\cdot|\cdot, s_t, \theta_t), y_{t+1}, s_{t+1}, s_t)}(d\theta_{t+1}) \\ &\qquad\qquad \times \delta_{\phi''(h_t, s_t, f_t, \theta_t)}(dh_{t+1}) Q'(y_{t+1}|s_{t+1}, g(f_t(w'), h_t(w'), s_t, \theta_t)) Q(s_{t+1}|s_t, h_t(w')) \delta_{w'}(dw) \end{aligned} \tag{35a}$$

$$\quad = P(df_{t+1}, d\theta_{t+1}, dh_{t+1}, s_{t+1}, dw|f_t, \theta_t, h_t, s_t, w'). \tag{35b}$$

■

Lemma 4. *For every realization* y^t, s^t *we have* $I(W; Y_{t+1}, S_{t+1}|Y^t = y^t, S^t = s^t) = E\{\log \frac{F'_{t+1}(W)}{F'_t(W)}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$*, where* f_t, h_t, θ_t *correspond to* y^t, s^t.

Proof:

$$\begin{aligned} &I(W; Y_{t+1}, S_{t+1}|Y^t = y^t, S^t = s^t) \\ &\quad = H(Y_{t+1}, S_{t+1}|Y^t = y^t, S^t = s^t) - H(Y_{t+1}, S_{t+1}|W, Y^t = y^t, S^t = s^t) && \text{(36a)} \\ &\quad = E\left[\log \frac{1}{P(Y_{t+1}, S_{t+1}|Y^t = y^t, S^t = s^t)}\right] && \text{(36b)} \\ &\quad - E\left[\log \frac{1}{Q'(Y_{t+1}|S_{t+1}, g(f_t(W), h_t(W), S_t = s_t, \theta_t))Q(S_{t+1}|S_t = s_t, h_t(W))}\right] && \text{(36c)} \\ &\quad = E\left[\log \frac{Q'(Y_{t+1}|S_{t+1}, g(f_t(W), h_t(W), S_t = s_t, \theta_t))Q(S_{t+1}|S_t = s_t, h_t(W))}{P(Y_{t+1}, S_{t+1}|Y^t = y^t, S^t = s^t)}\right] && \text{(36d)} \\ &\quad = E\left[\log \frac{F'_{t+1}(W)}{F'_t(W)}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\right], && \text{(36e)} \end{aligned}$$

where the last equality is due to (31c). ■

Let $C(s_{t-1}, \theta_{t-1}) \stackrel{\text{def}}{=} I(X_{t-1}^t; S_t, Y_t|s_{t-1}, \theta_{t-1})$ where $I(X_{t-1}^t; S_t, Y_t|s_{t-1}, \theta_{t-1})$ is evaluated using (18a) with the optimal channel input distribution.

Lemma 5. *For every realization y^{t-1}, s^{t-1} and corresponding θ_{t-1} we have*

$$I(W; Y_t, S_t | Y^{t-1} = y^{t-1}, S^{t-1} = s^{t-1}) = C(s_{t-1}, \theta_{t-1}) \tag{37}$$

if the PMS is used as a transmission scheme.

Proof: Note that for a given realization y^{t-1}, s^{t-1}, and thus for a corresponding realization of f_{t-1}, the random variable $f_{t-1}(W)$ is uniformly distributed, and thus $X_t = F^{-1}_{\hat{P}(x_t|X_{t-1}, S_{t-1}, \Theta_{t-1})}(f_{t-1}(W))$ is $\hat{P}(\cdot|X_{t-1}, S_{t-1}, \Theta_t)$ distributed and hence is independent of Y^{t-1}, S^{t-2}. Then,

$$\begin{aligned}
&I(W; Y_t, S_t | Y^{t-1} = y^{t-1}, S^{t-1} = s^{t-1}) \\
&\quad = H(Y_t, S_t | Y^{t-1} = y^{t-1}, S^{t-1} = s^{t-1}) - H(Y_t, S_t | Y^{t-1} = y^{t-1}, S^{t-1} = s^{t-1}, W) && \text{(38a)} \\
&\quad \overset{(a)}{=} H(Y_t, S_t | Y^{t-1} = y^t, S^{t-1} = s^{t-1}, \theta_{t-1}) - H(Y_t, S_t | Y^{t-1} = y^{t-1}, S^{t-1} = s^{t-1}, W, X^t_{t-1}, \theta_{t-1}) && \text{(38b)} \\
&\quad \overset{(b)}{=} H(Y_t, S_t | S_{t-1} = s_{t-1}, \theta_{t-1}) - H(Y_t, S_t | S_{t-1} = s_{t-1}, X^t_{t-1}, \theta_{t-1}) && \text{(38c)} \\
&\quad = I(X^t_{t-1}; S_t, Y_t | S_{t-1} = s_{t-1}, \theta_{t-1}) && \text{(38d)} \\
&\quad = C(s_{t-1}, \theta_{t-1}), && \text{(38e)}
\end{aligned}$$

where (a) comes from the fact that Θ_{t-1} is a function of Y^{t-1}, S^{t-1} and X^t_{t-1} is a function of Y^{t-1}, S^{t-1}, W; (b) comes from the channel property and the fact that X_t is independent of S^{t-2}, Y^{t-1} given S_{t-1}, Θ_{t-1}; and the last equation is due to the fact that the channel input sequence for the PMS has distribution which is capacity-achieving. ■

Using previous Lemmas, we can show that $\lim_{t\to\infty} F_t = F^*_W$ in total variation.

Proposition 2. $\lim_{t\to\infty}(F_t, W) = (F^*_W, W)$ *in total variation for* $R < C$*.*

Proof: Let $\mathcal{F} \times \mathcal{P}(\mathcal{X}) \times \mathcal{W} \times \mathcal{X} \times \mathcal{S} \times \mathcal{W}$ be the state space of the Markov chain $(F_t, \Theta_t, H_t, S_t, W)_t$. For a cdf f on $[0, 1)$, define $< f >_t$ to be the pmf corresponding to "quantizing" f' with resolution 2^{-tR}:

$$< f >_t (i) = \int_{(i-1)2^{-tR}}^{i2^{-tR}} df(w), \qquad 1 \le i \le 2^{tR}. \tag{39}$$

Consider now a sequence of functions $V_t : \mathcal{F} \times \mathcal{P}(\mathcal{X}) \times \mathcal{W} \times \mathcal{X} \times \mathcal{S} \times \mathcal{W} \to \mathcal{R}_+$,

$$V_t(f,p,h,s,w) \stackrel{\text{def}}{=} V_t(f) \stackrel{\text{def}}{=} \int_{w'=0}^{1} D(< F^*_{w'} >_t \,\|\, < f >_t) f'(w') = \sum_{i=1}^{2^{tR}} < f >_{t,i} \log \frac{1}{< f >_{t,i}} \tag{40}$$

$$= H(W| < f >_t), \tag{41}$$

where $D(\cdot\|\cdot)$ is KL-distance, F^*_w is a step function with jump at w. Similarly, define V_∞ : $\mathcal{F} \times \mathcal{P}(\mathcal{X}) \times \mathcal{W} \times \mathcal{X} \times \mathcal{S} \times \mathcal{W} \to \mathcal{R}_+$,

$$V_\infty(f,p,h,s,w) \stackrel{\text{def}}{=} V_\infty(f) \stackrel{\text{def}}{=} \int_{w'=0}^{1} D(F^*_{w'} \| f_t) f'(w') = H(W|f_t). \tag{42}$$

Then, for every realization of f_t we have

$$V_t(f_t) = \sum_{i=1}^{2^{tR}} < f_t >_{t,i} \log \frac{1}{< f_t >_{t,i}} = E\{\log \frac{1}{< F_t >_{t,[W2^{tR}]}} | F_t = f_t\}. \tag{43a}$$

In addition,

$$E\{V_t(F_{t+1}) | F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$$

$$= E\{E\{V_t(F_{t+1}) | F_t = f_t, F_{t+1} = f_{t+1}\} | F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \tag{44a}$$

$$= E\{\sum_{i=1}^{2^{tR}} < F_{t+1} >_{t,i} \log \frac{1}{< F_{t+1} >_{t,i}} | F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \tag{44b}$$

$$= E\{\log \frac{1}{< F_{t+1} >_{t,[W2^{tR}]}} | F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}, \tag{44c}$$

and similarly

$$E\{V_{t+1}(F_{t+1}) | F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$$

$$= E\{\log \frac{1}{< F_{t+1} >_{t+1,[W2^{(t+1)R}]}} | F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}. \tag{45}$$

Thus,

$$
\begin{aligned}
&E\{V_{t+1}(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - V_t(f_t) \\
&\quad = E\{V_{t+1}(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - E\{V_t(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \\
&\qquad + E\{V_t(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - V_t(f_t) \qquad \text{(46a)} \\
&\quad = E\{\log \frac{< F_{t+1} >_{t,[W2^{tR}]}}{< F_{t+1} >_{t+1,[W2^{(t+1)R}]}}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \\
&\qquad - E\{\log \frac{< F_{t+1} >_{t,[W2^{tR}]}}{< F_t >_{t,[W2^{tR}]}}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}. \qquad \text{(46b)}
\end{aligned}
$$

The first term in the last equation is upper bounded by R as follows.

$$
\begin{aligned}
&E\{\log \frac{< F_{t+1} >_{t,[W2^{tR}]}}{< F_{t+1} >_{t+1,[W2^{(t+1)R}]}}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \\
&\quad = E\{\sum_{i=1}^{t} < F_{t+1} >_{t,2^{iR}} \sum_{j=1}^{2^R} \frac{< F_{t+1} >_{t+1,[j2^{iR}]}}{< F_{t+1} >_{t,2^{iR}}} \log \frac{< F_{t+1} >_{t,2^{iR}}}{< F_{t+1} >_{t+1,[j2^{iR}]}}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \\
&\qquad \text{(47a)} \\
&\quad \overset{(a)}{\leq} E\{R \sum_{i=1}^{t} < F_{t+1} >_{t,2^{iR}} |F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \qquad \text{(47b)} \\
&\quad = R, \qquad \text{(47c)}
\end{aligned}
$$

where (a) is due to the fact that the second sum inside the expectation equals to the entropy of a random variable with cardinality 2^R.

Note that for two cdf's F and F' [14, Corollary 5.2.3]

$$
D(< F >_t \, || < F' >_t) \uparrow D(F||F'). \qquad \text{(48)}
$$

Then,

$$\lim_{t\to\infty} E\{D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$$

$$\stackrel{(a)}{=} E\{\lim_{t\to\infty} D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \quad (49a)$$

$$= E\{D(F_{t+1}||F_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \quad (49b)$$

$$= E\{\log \frac{F'_{t+1}(W)}{F'_t(W)}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \quad (49c)$$

$$\stackrel{(b)}{=} C(s_t, \theta_t)1_{\{V_\infty(f_t)>0\}} \quad (49d)$$

where (a) is due to the dominated convergence theorem, and (b) is due to Lemmas 4 and 5.

Note that $C(s_t, \theta_t)1_{\{V_t(f_t)>0\}} - E\{D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \downarrow 0$. Assume now that $E\{D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$ is continuous in $f_t \in \mathcal{F}$ such that $V_t(f_t) > 0$. Then $C(s_t, \theta_t)1_{\{V_t(f_t)>0\}} - E\{D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$ is continuous in $f_t \in \mathcal{F}$ such that $V_t(f_t) > 0$. Hence, by Dini's theorem [15], $C(s_t, \theta_t)1_{\{V_t(f_t)>0\}} - E\{D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \downarrow 0$ uniformly in $f_t \in \mathcal{F}$ such that $V_t(f_t) > 0$.

Therefore, for any $\epsilon > 0$ there exists t_0 such that the following is true for all $t \geq t_0$.

$$E\{\log \frac{< F_{t+1} >_{t,[W2^{tR}]}}{< F_t >_{t,[W2^{tR}]}}|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\}$$

$$= E\{D(< F_{t+1} >_t \,||\, < F_t >_t)|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} \quad (50a)$$

$$\geq (C(s_t, \theta_t) - \epsilon)1_{\{V_t(f_t)>0\}}. \quad (50b)$$

(47) and (50) implies

$$E\{V_{t+1}(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - V_t(f_t) \leq -(C(s_t, \theta_t) - R - \epsilon)1_{\{V_t(f_t)>0\}}. \quad (51)$$

Let $A_t \stackrel{\text{def}}{=} \{f \in \mathcal{F}|V_t(f) > 0\}$. Note that $A_{t_1} \subset A_{t_2}$ for $t_1 \leq t_2$. Furthermore, it is true that $\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} C(s_i, \theta_i) = C$ by the law of large numbers of Markov chains [16, Th. 17.0.1]. Hence there exists t_1 such that $\frac{1}{t}\sum_{i=1}^{t} C(s_i, \theta_i) \geq C - \delta$ for $t \geq t_1$.

Then, by Dynkin's formula [16, Th. 11.3.1], we have

$$-E\{\sum_{t=1}^{\tau_{A_1}} (E\{V_{t+1}(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - V_t(f_t))\,|f_1, s_1, h_1, \theta_1\} \leq V_1(f_1) \tag{52}$$

where $\tau_A \stackrel{\text{def}}{=} \min\{t \geq 2 | f_t \in A\}$. Let $t_2 \stackrel{\text{def}}{=} \max\{t_0, t_1\}$. Then, by using (51) we get for $f_1 \notin A_1$

$$V_1(f_1) \geq -\sum_{i=1}^{t_2-1} P(\tau_{A_1} = i | f_1, s_1, h_1, \theta_1) \sum_{t=1}^{i} (E\{V_{t+1}(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - V_t(f_t))$$
$$- \sum_{i=t_2} P(\tau_{A_1} = i | f_1, s_1, h_1, \theta_1) \sum_{t=1}^{i} (E\{V_{t+1}(F_{t+1})|F_t = f_t, S_t = s_t, H_t = h_t, \Theta_t = \theta_t\} - V_t(f_t)) \tag{53a}$$

$$\geq -N_1 + \sum_{i=t_2} P(\tau_{A_1} = i | f_1, s_1, h_1, \theta_1)(\sum_{t=1}^{i}(C - R - \epsilon - \delta)), \tag{53b}$$

where N_1 is some finite positive number. Consequently, for $f_1 \notin A_1$ and $R < C - \epsilon - \delta$

$$E\{\tau_{A_1} | f_1, s_1, h_1, \theta_1\} = \sum_{i=1}^{t_2-1} P(\tau_{A_1} = i | f_1, s_1, h_1, \theta_1) i + \sum_{i=t_2} P(\tau_{A_1} = i | f_1, s_1, h_1, \theta_1) i \tag{54a}$$

$$\leq N_2 + N_3(V_1(f_1) + N_1) \tag{54b}$$

$$< \infty, \tag{54c}$$

where N_1, N_2, N_3 are some finite positive numbers.

Therefore, for any $R < C$, by [16, Th. 10.4.10], $\{(F_t, \Theta_t, H_t, S_t, W)\}_t$ is positive Harris-recurrent, and there exists a unique stationary distribution. Let π be the stationary distribution on the Markov chain $(F_t, \Theta_t, H_t, S_t, W)_t$. Then,

$$\pi(df_{t+1}, d\theta_{t+1}, dh_{t+1}, s_{t+1}, dw)$$
$$= \int_{f_t, \theta_t, h_t, w'} \sum_{s_t} P(df_{t+1}, d\theta_{t+1}, dh_{t+1}, s_{t+1}, w | f_t, \theta_t, h_t, s_t, w') \pi(df_t, d\theta_t, dh_t, s_t, dw'). \tag{55}$$

Set $\pi(df_t, d\theta_t, dh_t, s_t, dw') = \pi(d\theta_t, dh_t, s_t | f_t, w')\delta_{F^*_{w'}}(df_t)Uniform(dw')$. Then,

$$
\begin{aligned}
&\pi(df_{t+1}, dw) \\
&\quad = \int_{\theta_{t+1}, h_{t+1}} \sum_{s_{t+1}} \pi(df_{t+1}, d\theta_{t+1}, dh_{t+1}, s_{t+1}, dw) && \text{(56a)} \\
&\quad = \int_{\theta_{t+1}, h_{t+1}} \sum_{s_{t+1}} \int_{f_t, \theta_t, h_t, w'} \sum_{s_t} P(df_{t+1}, d\theta_{t+1}, dh_{t+1}, s_{t+1}, dw | f_t, \theta_t, h_t, s_t, w')\pi(df_t, d\theta_t, dh_t, s_t, dw') && \text{(56b)} \\
&\quad = \sum_{s_{t+1}} \int_{f_t, \theta_t, h_t, w'} \sum_{s_t, y_{t+1}} \delta_{\phi'(f_t, y_{t+1}, s_{t+1}, s_t, \theta_t, h_t)}(df_{t+1}) Q'(y_{t+1} | s_{t+1}, g(f_t(w'), h_t(w'), s_t, \theta_t)) \\
&\qquad\qquad \times Q(s_{t+1} | s_t, h_t(w'))\delta_{w'}(w)\pi(d\theta_t, dh_t, s_t | f_t, w')\delta_{F^*_{w'}}(df_t)Uniform(dw') && \text{(56c)} \\
&\quad = \sum_{s_{t+1}} \int_{\theta_t, h_t} \sum_{s_t, y_{t+1}} \delta_{\phi'(F^*_w, y_{t+1}, s_{t+1}, s_t, \theta_t, h_t)}(df_{t+1}) Q'(y_{t+1} | s_{t+1}, g(F^*_w, h_t(w'), s_t, \theta_t)) \\
&\qquad\qquad \times Q(s_{t+1} | s_t, h_t(w'))\pi(d\theta_t, dh_t, s_t | F^*_w, w)Uniform(dw). && \text{(56d)}
\end{aligned}
$$

f_{t+1} is determined as follows by ϕ'.

$$
\begin{aligned}
df_{t+1}(\tilde{w}) &= \frac{Q'(y_{t+1} | s_{t+1}, g(F^*_w(\tilde{w}), h_t(\tilde{w}), s_t, \theta_t))Q(s_{t+1} | s_t, h_t(\tilde{w}))dF^*_w(\tilde{w})}{\int_{w'} Q'(y_{t+1} | s_{t+1}, g(F^*_w(\tilde{w}'), h_t(\tilde{w}'), s_t, \theta_t))Q(s_{t+1} | s_t, h_t(\tilde{w}'))dF^*_w(\tilde{w}')} && \text{(57a)} \\
&= \frac{Q'(y_{t+1} | s_{t+1}, g(F^*_w(\tilde{w}), h_t(\tilde{w}), s_t, \theta_t))Q(s_{t+1} | s_t, h_t(\tilde{w}))dF^*_w(\tilde{w})}{Q'(y_{t+1} | s_{t+1}, g(F^*_w(\tilde{w}), h_t(\tilde{w}), s_t, \theta_t))Q(s_{t+1} | s_t, h_t(\tilde{w}))} && \text{(57b)} \\
&= dF^*_w(\tilde{w}). && \text{(57c)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\pi(df_{t+1}, dw) \\
&\quad = \delta_{F^*_w}(df_{t+1})Uniform(dw) \\
&\qquad \times \sum_{s_{t+1}} \int_{\theta_t, h_t} \sum_{s_t, y_{t+1}} Q'(y_{t+1} | s_{t+1}, g(F^*_w, h_t(w), s_t, \theta_t))Q(s_{t+1} | s_t, h_t(w))\pi(d\theta_t, dh_t, s_t | F^*_w, w) && \text{(58a)} \\
&\quad = \delta_{F^*_w}(df_{t+1})Uniform(dw). && \text{(58b)}
\end{aligned}
$$

Hence, $\lim_{t\to\infty}(F_t, W) = (F^*_W, W)$ in total variation [16, Th. 13.0.1]. ■

Using above result, we can prove the following proposition which is the main result of this paper.

Proposition 3. *The PMS is capacity-achieving. In other words,* $\lim_{t\to\infty} P(|W - \hat{W}_t| > 2^{-tR}) = 0$ *for* $R < C$.

Proof: In the following we fix $W = 1/2$ and perform the analysis. Let us also denote

$$E_t = \{f : \quad |1/2 - \arg\max_{\tilde{w}} f(\tilde{w} + 2^{-tR}/2) - f(\tilde{w} - 2^{-tR}/2)| > 2^{-tR}\}. \tag{59}$$

We need to show that $\lim_{t\to\infty} \pi_t(E_t) = 0$ for all $R < C$.

Consider the sets

$$A_t = \{f : \quad ||f - F^*_{1/2}||_L < 2^{-tC}\}, \tag{60}$$

where $||\cdot||_L$ denotes the Levy distance. We have

$$\pi_t(E_t) = \pi_t(E_t \cap A_t) + \pi_t(E_t \cap A_t^c) \tag{61a}$$

$$\leq \pi_t(E_t \cap A_t) + \pi_t(A_t^c) \tag{61b}$$

$$\leq \pi_t(E_t \cap A_t) + |\pi_t(A_t^c) - \pi(A_t^c)| + \pi(A_t^c). \tag{61c}$$

In the last expression, $\pi(A_t^c) = 0$, since π puts all the probability mass on $F^*_{1/2}$ and $F^*_{1/2} \notin A_t^c$. Also, from Proposition 2, we know that for every $\epsilon > 0$ there exists $t_0(\epsilon)$, such that for every $t > t_0(\epsilon)$, $|\pi_t(A_t^c) - \pi(A_t^c)| < \epsilon$.

It remains to bound the first term $\pi_t(E_t \cap A_t)$. We will actually show that for sufficiently large t, the set $E_t \cap A_t$ is empty. Indeed, consider all cdf's in A_t. They are sufficiently "close" to $F^*_{1/2}$, which means that the maximum of $f(\tilde{w} + 2^{-tR}/2) - f(\tilde{w} - 2^{-tR}/2)$ is at $\hat{w} \in [1/2 - 2^{-tC} - 2^{-tR}/2, 1/2 + 2^{-tC} + 2^{-tR}/2]$ for sufficiently large t, and thus $|1/2 - \hat{w}| < 2^{-tC} + 2^{-tR}/2 \approx 2^{-tR}/2$ (for any $R < C$ and sufficiently large t). This further implies that $f \notin E_t$.

Hence, $P(|W - \hat{W}_t| > 2^{-tR}) \to 0$ as $t \to \infty$ for $R < C$. ■

V. Conclusion

A single-letter expression for the capacity of the FSC with delayed feedback was derived. The methodology was based on reformulating the capacity problem as a stochastic control problem.

This methodology is quite general and will likely be useful in finding single-letter expressions for the capacity of other channels, such as the multiple-access channel with feedback. A simple recursive transmission scheme was then presented that can be thought of as the generalization of the PMS. We identified an underlying Markov chain, and showed that the proposed scheme achieves the capacity of FSCs.

An interesting future research direction is the investigation of the existence of a PMS-like transmission scheme for multi-user channels and in particular for the multiple-access channel with feedback. Another future research direction is the investigation of a simple sequential transmission scheme akin to the PMS that achieves the channel reliability function.

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