A sequential transmission scheme for the multiple access channel with noiseless feedback

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Abstract

In this paper we study transmission of information over a multiple access channel (MAC) with noiseless feedback. We formulate this problem as a decentralized stochastic control problem, the three controllers being the decoder and the two encoders who, in the presence of limited information about each other, decide what to transmit at each time instance, in order to jointly achieve a common goal.

Our contribution is two-fold. First, we identify structural properties of the optimal communication system that result in considerable simplification of the encoding/decoding process. The derived structural properties make it possible to consider transmission schemes that are akin to the posterior-matching scheme (PMS) for the point-to-point channel. Since the optimal communication system has this structure, we need only restrict attention to the study of those simplified systems, even when the optimal one is not known. Second, the aforementioned structural results allow us to view the original MAC system as an equivalent point-to-point communication system over a Markov channel with perfect state observation and delayed state feedback. Based on this equivalence, we derive a single-letter expression for the capacity of the original channel.

I. Introduction

It is well known that feedback can improve the performance of communication systems. In particular, feedback cannot increase the capacity of single-user discrete memoryless channels [1] but can improve the corresponding error exponent [2] and/or simplify the transmission scheme [3]–[5]. For the case of multi-user channels the improvement is more dramatic, since capacity increases are also possible. In particular, it has been demonstrated that noiseless feedback can enlarge the capacity region of the multiple access channel (MAC) [6]–[10]. Interestingly enough, for the case of the MAC with noiseless feedback (NLF-MAC) the capacity is not known as a single-letter expression. A multi-letter expression for the NLF-MAC capacity was developed in [11] utilizing multi-letter directed information expressions.

In their recent works [12], [13], the authors generalized the techniques of [3], [4] for arbitrary memoryless channels with noiseless feedback thus deriving a simple capacity-achieving transmission scheme for this class of channels, namely the posterior matching scheme (PMS). Following this work, [14] restated the PMS scheme as the solution of an appropriately formulated stochastic control problem, thus providing a natural connection between communication system design in the presence of noiseless feedback and optimal control. However such a connection is still absent for the case of multi-user channels.

In this paper we formulate the problem of information transmission over a NLF-MAC as a decentralized stochastic control problem, the three controllers being the decoder and the two encoders that, in the presence of limited information about each other, decide what to transmit at each time instance. Although the two users have limited information about each-other, they have a common objective, i.e., to minimize the average of some increasing function of their error performance (and/or the average of some decreasing function of their information rate). In this respect the problem can be classified as a dynamic team problem with non-classical information structure. The decentralized aspect of this control problem makes it fundamentally different from the corresponding single-user counterparts. As a result, the problem under consideration cannot be solved applying directly the tools from Markov decision processes (MDPs) and partially observed MDPs (POMDPs). Decentralized stochastic control problems with non-classical information structure similar to the one considered here have been studied in [15], [16] and recently in [17]–[21].

Our contribution is twofold. First, we identify structural properties of the optimal transmission strategies for the two users so that the domain of the optimal strategies is not increasing with time. In general, at time t, each user can base its decision on everything that he has seen and done, i.e., on the individual messages, previous transmitted symbols and received feedback. Obviously such an increasing domain is not practical for implementation and it also generates conceptual difficulties for the infinite-horizon, as well as hindering the evaluation of the capacity as a single-letter expression. Based on the above structural properties, we can identify the optimal strategies as the solution of a fixed point equation. They also allow us to consider (possibly suboptimal) transmission techniques that are akin to the PMS for the single-user channels. Second, the aforementioned structural results allow us to view the original NLF-MAC system as an equivalent point-to-point communication system over a Markov channel with perfect state observation at the receiver and delayed state feedback. Based on this equivalence, we derive a single-letter expression for the capacity of the original channel.

The remaining of this paper is structured as follows. The model of the studied communication system is presented in Section II. Structural properties of the optimal strategies and the optimal solution are developed in Section III. A characterization of the capacity of the NLF-MAc based on the derived structural results is presented in Section IV.

II. System model and problem statement

We consider a multiple access communication system consisting of two transmitters and one receiver. The transmitters want to communicate independent, uniformly distributed messages $W^i \in \mathcal{W}^i \subset [0,1)$, $i = 1,2$ by transmitting a sequence of symbols $X_t^i \in \mathcal{X}^i$, $t = 1, \ldots$. The encoded sequence is transmitted through a discrete-memoryless multiple-access channel (DM-MAC) producing a sequence of output symbols $Y_t \in \mathcal{Y}$, $t = 1, \ldots$ at the receiver. The probabilistic description[1] of the messages is $Pr(w^1, w^2) = P^1(w^1)P^2(w^2)$, and of the channel is $Pr(y_t|x^{1,t}, x^{2,t}, y^{t-1}, w^1, w^2) = Q(y_t|x_t^1, x_t^2)$. The channel output is also available at both

[1]We are using the notation $Pr(a|b)$ in a casual manner to denote conditional probability mass functions for discrete random variables and the measure $P(da|b)$ for continuous random quantities in order to simplify our notation. In the following we will also use sums in place of integrals. All results can be expressed more rigorously using the latter notation.

transmitters through a noiseless feedback channel with unit delay. The receiver produces an estimate of the pair of messages $(\hat{W}_t^1, \hat{W}_t^2) \in \mathcal{W}^1 \times \mathcal{W}^2$ at each time t by performing decoding of the form[2] $(\hat{W}_t^1, \hat{W}_t^2) = d_t(Y^t)$, where $d_t : \mathcal{Y}^t \to \mathcal{W}^1 \times \mathcal{W}^2$. We further define a sequence of cost functions of the form $\rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2)$, $t = 1, \ldots,$ that quantify the receiver performance. Because of the noiseless feedback assumption, at each time t each encoder produces the transmitted symbol through the encoding function[3] $X_t^i = f_t^i(W^i, Y^{i,t-1})$, where $f_t^i : \mathcal{W}^i \times \mathcal{Y}^{t-1} \to \mathcal{X}^i$. A block diagram of the communication system is depicted in Fig. 1.

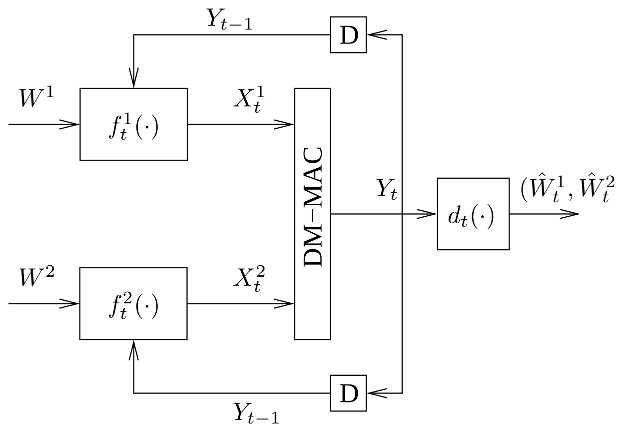

Fig. 1. The discrete memoryless multiple access channel with noiseless feedback.

For the case of fixed-length block coding, given a block length n, a design $\mathcal{D}_n \stackrel{\text{def}}{=} (f_t^1, f_t^2, d_t)_{t=1}^n$, is a sequence of encoding and decoding functions. Associated with a design $\mathcal{D}_n$ there is an average cost of the form

$$J_n(\mathcal{D}_n) = \mathbf{E}^{\mathcal{D}_n}\{\frac{1}{n}\sum_{t=1}^{n} \rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2)\}, \tag{1}$$

where the notation $\mathbf{E}^{\mathcal{D}_n}\{\cdot\}$ is used to emphasize that expectation is taken after the design $\mathcal{D}_n$ is fixed. Observe that associated with each such system, there is a pair of rates (R_n^1, R_n^2) defined as $R_n^i = H(W^i)/n$, $i = 1, 2$ (it is implicitly assumed that for each block length n, the a-priori message distributions $P^i(\cdot)$ are different, and can be thought of as uniform over a discrete variable that takes $2^{nR_n^i}$ values in $\mathcal{W}^i$).

In the following we will also be interested in the infinite-horizon average-cost version of this problem, whereby the associated cost is of the form

$$J(\mathcal{D}) = \liminf_{n\to\infty} \mathbf{E}^{\mathcal{D}}\{\frac{1}{n}\sum_{t=1}^{n} \rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2)\}, \tag{2}$$

with $\mathcal{D} \stackrel{\text{def}}{=} (f_t^1, f_t^2, d_t)_{t=1}^\infty$ (in this setup we assume that the messages are continuous random variables, uniformly distributed in $[0, 1)$).

Note that the above formulation is general enough to address several problems of information theoretic interest. In particular:

[2]We consider this more generic receiver, although in the case of fixed-length block coding only a single estimate at time n is considered.

[3]In this work we only consider deterministic encoders; it is known that stochastic encoders may enlarge the MAC capacity region.

1) In the finite-horizon case (1), if we set $\rho_t(\cdot) = \mathbf{1}_{\{(W^1,W^2)\neq(\hat{W}_t^1,\hat{W}_t^2)\}}$, then we say that a sequence of designs $(\mathcal{D}_n)_n$ achieves a rate pair (R^1, R^2) if $\liminf_{n\to\infty} R_n^1 \geq R^1$, $\liminf_{n\to\infty} R_n^2 \geq R^2$, and $\liminf_{n\to\infty} J_n(\mathcal{D}_n) = 0$ Similarly, we say that a sequence of designs $(\mathcal{D}_n)_n$ achieves a rate pair (R^1, R^2) and an error exponent E if $\liminf_{n\to\infty} R_n^1 \geq R^1$, $\liminf_{n\to\infty} R_n^2 \geq R^2$ and $\liminf_{n\to\infty} \frac{1}{n}\log J_n(\mathcal{D}_n) \leq -E$.
2) In the infinite-horizon case (2), if we set $\rho_t(\cdot) = \mathbf{1}_{\{|\hat{W}_t^1 - W^1| \geq \frac{1}{2}2^{-tR^1} \text{ or } |\hat{W}_t^2 - W^2| \geq \frac{1}{2}2^{-tR^2}\}}$, then we say that a design $\mathcal{D}$ achieves a rate pair (R^1, R^2) if $J(\mathcal{D}) = 0$.

In addition, with some appropriate modifications, we can formulate problems relating to variable-length block coding (see [13]). Returning to the fixed-length coding problem, the optimization problems corresponding to (1), and (2) are

Problem 1. *Find $\mathcal{D}_n^*$ such that* $J_n^* \stackrel{def}{=} J_n(\mathcal{D}_n^*) = \inf_{\mathcal{D}_n} J_n(\mathcal{D}_n)$

and

Problem 2. *Find $\mathcal{D}^*$ such that* $J^* \stackrel{def}{=} J(\mathcal{D}^*) = \inf_{\mathcal{D}} J(\mathcal{D})$,

respectively.

Note that the above cost functions are only examples of how the information transmission problem can be stated. For instance, one can introduce a cost function of a form similar to the one used in [14] (i.e., the reduction of uncertainty on the messages W^i from time $t-1$ to time t) in order to make a more direct connection with channel capacity. In the following we do not focus on the particular problem formulation, neither are we interested in the particular form of objective function. Instead we focus on structural properties of the communication system that are common regardless of these choices.

III. The NLF-MAC as a decentralized control problem

We now address the problem stated in the previous section in several steps. First we will show that the users' strategies $f_t^i(W^i, Y^{t-1})$ can be restricted without loosing optimality (as defined in problems 1, 2). Then, by looking at the problem from the viewpoint of an agent that observes the common information y^{t-1} available to both encoders, we will provide an equivalence between the original system and a single-user communication scheme. Finally, we will show how this equivalent description can be used to provide a single-letter expression for the capacity of the original channel and a PMS-like scheme.

In the following we denote by $\Delta(S)$ the space of probability distributions over the (possibly uncountably infinite) set S. We will also denote the space of functions with domain A and range B as $B^{|A|}$, i.e., $f \in B^{|A|}$ for $f : A \to B$.

A. Structural results

One of the difficulties with the general encoding strategies $x_t^i = f_t^i(W^i, Y^{i,t-1})$ is that their domain increases with time. From the analytical viewpoint this is undesirable since the structure of the encoder has a time-varying domain. In addition, this complicates the solution of the problem in the infinite horizon case. Finally, from the practical viewpoint, it requires essentially infinite memory at the transmitters if the optimal solution is to be implemented. In the following we show that we can restrict ourselves to strategies with finite domains without loss of optimality.

To address this problem we consider the following equivalent encoder description. Each user's transmission $x_t^i = f_t^i(w^i, y^{i,t-1})$ can be though of as a two-stage process. In the first stage, a pair of pre-encoding functions $e_t^i : \mathcal{W}^i \to \mathcal{X}^i$, $i = 1, 2$ are generated based on the available feedback y^{t-1}, through the mapping $(e_t^1, e_t^2) = g_t(y^{t-1})$, where $g_t : \mathcal{Y}^{t-1} \to \mathcal{X}^{|\mathcal{W}^1|} \times \mathcal{X}^{|\mathcal{W}^2|}$. In the second step the two transmitted messages are generated by evaluating the pre-encoding functions at w^1 and w^2, respectively, i.e., $x_t^i = e_t^i(w^i)$. This transformed system is shown in Fig. 2 and it is clearly in one-to-one correspondence with the original one. Observe that selecting the pair (f_t^1, f_t^2)

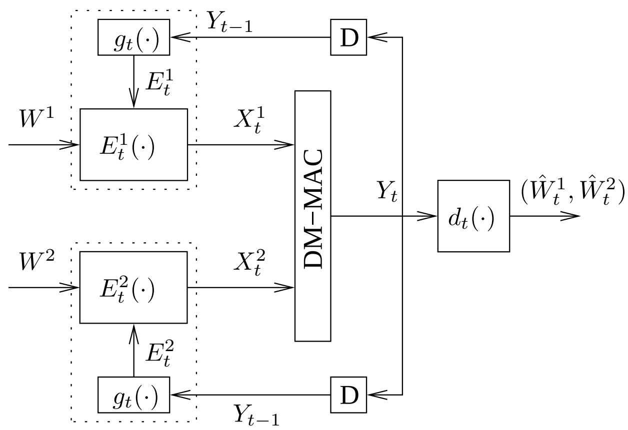

Fig. 2. Equivalent system with pre-encoder functions.

is now equivalent to selecting the mapping g_t. In addition, the decentralization of information in the two encoders is maintained by design even though the two pre-encoding functions (e_t^1, e_t^2) are selected jointly. This is so because g_t only uses the common information y^{t-1} available to both encoders and can essentially be replicated locally at the two encoders. This also implies that although encoder 1 does not know what the transmitted symbols $(x_t^2)_t$ are, it does know how they are generated, i.e., it knows causally all the pre-encoding functions $(e_t^2)_t$.

The above transformation still suffers from the problem of the increasing domain $\mathcal{Y}^{t-1}$ of the functions g_t. We now address this problem as follows. Define the random quantities $\Pi_t \in \Delta(\mathcal{W}^1 \times \mathcal{W}^2)$, $t = 1, \dots$ as $\Pi_t(w^1, w^2) \stackrel{\text{def}}{=} Pr(w^1, w^2 | Y^t, E^{1,t}, E^{2,t})$.[4] The next lemma shows how Π_t evolves in time.

Lemma 1. *The following statements are true:*

1) *The quantity π_t can be recursively generated as $\pi_0 \stackrel{def}{=} Unif(\mathcal{W}^1 \times \mathcal{W}^2)$, $\pi_t = \Phi(\pi_{t-1}, e_t^1, e_t^2, y_t)$ for $t = 1, 2, \dots$, where Φ is a known function.*
2) *The above implies that $(\Pi_t)_t$ is a controlled Markov chain with control (e_t^1, e_t^2), i.e., $Pr(\pi_t | \pi^{t-1}, e^{1,t}, e^{2,t}) = Pr(\pi_t | \pi_{t-1}, e_t^1, e_t^2)$.*
3) *The optimal decoder function at time t is only a function of π_t. In addition, the instantaneous average cost $\mathbf{E}\{\rho_t(\cdot)\}$ at time t can be expressed as the average of a known a function of the Markov state π_{t-1} and the action (e_t^1, e_t^2).*

[4]Throughout this paper, each realization π_t of the random quantity Π_t can be thought of as the conditional joint cdf of W^1, W^2 conditioned on $Y^t, E^{1,t}, E^{2,t}$.

Proof:

1) For every w^1, w^2 we have

$$\pi_t(w^1,w^2) = Pr(w^1,w^2|y^t,e^{1,t},e^{2,t}) \tag{3}$$
$$= \frac{Pr(w^1,w^2,y_t|y^{t-1},e^{1,t},e^{2,t})}{Pr(y_t|y^{t-1},e^{1,t},e^{2,t})} \tag{4}$$
$$= \frac{Pr(y_t|w^1,w^2,y^{t-1},e^{1,t},e^{2,t}))Pr(w^1,w^2|y^{t-1},e^{1,t},e^{2,t}))}{Pr(y_t|y^{t-1},e^{1,t},e^{2,t}))} \tag{5}$$
$$= \frac{Q(y_t|e_t^1(w^1),e_t^2(w^2))\pi_{t-1}(w^1,w^2)}{\sum_{w'^1,w'^2} Q(y_t|e_t^1(w'^1),e_t^2(w'^2))\pi_{t-1}(w'^1,w'^2)}, \tag{6}$$

thus establishing the recursion $\pi_t = \Phi(\pi_{t-1}, e_t^1, e_t^2, y_t)$.

2) The above recursion implies that $Pr(\pi_t|\pi^{t-1},e^{1,t},e^{2,t}) = Pr(\pi_t|\pi_{t-1},e_t^1,e_t^2)$. Indeed

$$Pr(\pi_t|\pi^{t-1},e^{1,t},e^{2,t}) = \sum_{y^t}\sum_{w^1,w^2} Pr(\pi_t|\pi^{t-1},e^{1,t},e^{2,t},y^t,w^1,w^2)Pr(y_t|\pi^{t-1},e^{1,t},e^{2,t},y^{t-1},w^1,w^2)$$
$$Pr(w^1,w^2|\pi^{t-1},e^{1,t},e^{2,t},y^{t-1})Pr(y^{t-1}|\pi^{t-1},e^{1,t},e^{2,t}) \tag{7}$$
$$= \sum_{y_t} \delta_{\Phi(\pi_{t-1},e_t^1,e_t^2,y_t)}(\pi_t) \sum_{w^1,w^2} Q(y_t|e_t^1(w^1),e_t^2(w^2))\pi_{t-1}(w^1,w^2) \tag{8}$$
$$= Pr(\pi_t|\pi_{t-1},e_t^1,e_t^2), \tag{9}$$

which establishes that the process $(\Pi_t)_t$ is a control Markov chain with control (e_t^1, e_t^2).

3) Consider the average cost at time t

$$\mathbf{E}\{\rho_t(W^1,W^2,\hat{W}_t^1,\hat{W}_t^2)\} = \mathbf{E}\{\mathbf{E}\{\rho_t(W^1,W^2,\hat{W}_t^1,\hat{W}_t^2)|\Pi^t,Y^t\}\} \tag{10}$$
$$= \mathbf{E}\{\sum_{w^1,w^2} \rho_t(w^1,w^2,\hat{W}_t^1,\hat{W}_t^2)Pr(w^1,w^2|\Pi^t,Y^t)\} \tag{11}$$
$$= \mathbf{E}\{\sum_{w^1,w^2} \rho_t(w^1,w^2,\hat{W}_t^1,\hat{W}_t^2)\Pi_t(w^1,w^2)\}. \tag{12}$$

Now the expression inside the expectation can be minimized by selecting $(\hat{W}_t^1, \hat{W}_t^2)$ as follows

$$(\hat{W}_t^1,\hat{W}_t^2) = \arg\min_{\hat{w}_t^1,\hat{w}_t^2} \sum_{w^1,w^2} \rho_t(w^1,w^2,\hat{w}_t^1,\hat{w}_t^2)\Pi_t(w^1,w^2) \stackrel{\text{def}}{=} \tilde{d}_t(\Pi_t), \tag{13}$$

and substituting to the previous expression we get

$$\mathbf{E}\{\rho_t(W^1,W^2,\hat{W}_t^1,\hat{W}_t^2)\} = \mathbf{E}\{\min_{\hat{w}_t^1,\hat{w}_t^2} \sum_{w^1,w^2} \rho_t(w^1,w^2,\hat{w}_t^1,\hat{w}_t^2)\Pi_t(w^1,w^2)\} = \mathbf{E}\{\hat{\rho}_t(\Pi_t)\} \tag{14}$$

where $\hat{\rho}_t(\cdot)$ is a known function depending on the original cost $\rho_t(\cdot)$. We can further express this average as

$$\mathbf{E}\{\hat{\rho}_t(\Pi_t)\} = \mathbf{E}\{\mathbf{E}\{\hat{\rho}_t(\Phi(\Pi_{t-1},E_t^1,E_t^2,Y_t))|\Pi^{t-1},E^{1,t},E^{2,t},Y^{t-1}\}\}, \tag{15}$$

where the quantity inside the expectation can be reduced to

$$\mathbf{E}\{\hat{\rho}_t(\Phi(\Pi_{t-1}, E_t^1, E_t^2, Y_t))|\Pi^{t-1}, E^{1,t}, E^{2,t}, Y^{t-1}\}$$
$$= \sum_{y_t} \hat{\rho}_t(\Phi(\Pi_{t-1}, E_t^1, E_t^2, y_t)) \sum_{w^1,w^2} Pr(y_t|\Pi^{t-1}, E^{1,t}, E^{2,t}, Y^{t-1}, w^1, w^2) Pr(w^1, w^2|\Pi^{t-1}, E^{1,t}, E^{2,t}, Y^{t-1}) \tag{16}$$
$$= \sum_{y_t} \hat{\rho}_t(\Phi(\Pi_{t-1}, E_t^1, E_t^2, y_t)) \sum_{w^1,w^2} Q(y_t|E_t^1(w^1), E_t^2(w^2))\Pi_{t-1}(w^1, w^2) \tag{17}$$
$$= \tilde{\rho}_t(\Pi_{t-1}, E_t^1, E_t^2), \tag{18}$$

where $\tilde{\rho}_t(\cdot)$ is a known function depending on the original cost $\rho_t(\cdot)$. ■

The above lemma establishes that from the viewpoint of the designer of the communication system, designing an optimal system is equivalent to a perfectly observed Markov decision process (MDP) problem with state $\pi_{t-1} \in \Delta(\mathcal{W}^1 \times \mathcal{W}^2)$, actions $(e_t^1, e_t^2) \in \mathcal{X}^{|\mathcal{W}^1|} \times \mathcal{X}^{|\mathcal{W}^2|}$, and instantaneous cost $\tilde{\rho}_t(\pi_{t-1}, e_t^1, e_t^2)$. This further implies that the optimal strategies for choosing the actions (e_t^1, e_t^2) are Markov strategies [22], i.e., they need only depend on the current state π_{t-1} of the Markov process, and not on (π^{t-1}, y^{t-1}). In other words, we can restrict attention to pre-encoders of the form $(e_t^1, e_t^2) = \tilde{g}_t(\pi_{t-1})$, where $\tilde{g}_t : \Delta(\mathcal{W}^1 \times \mathcal{W}^2) \to \mathcal{X}^{|\mathcal{W}^1|} \times \mathcal{X}^{|\mathcal{W}^2|}$ which has a time invariant domain! We note that the actual optimal pre-encoding mappings $\tilde{g}_t^*$, $t = 1, \ldots$ can be found by standard methods in MDP problems, i.e., by solving the backwards recursive Bellman equations for the finite-horizon problem in 1. In the infinite-horizon case (**??**), if the cost functions $\tilde{\rho}_t(\cdot)$ are time-invariant, the optimal mapping will also be time-invariant, i.e., $\tilde{g}_t^* = \tilde{g}^*$ and can be obtained by solving a fixed point equation.

Theorem 1. *The optimal communication system (in the sense of the optimization problems 1, 2) for the DM-MAC channel with noiseless feedback consists of*

1) *Encoders of the form* $x_t^i = e_t^i(w^i)$, $i = 1, 2$ *where* $(e_t^1, e_t^2) = \tilde{g}_t(\pi_{t-1})$, *for some choice of the mappings* $\tilde{g}_t$ *that is determined off-line (i.e., they do not depend on the realization of the random variables). Furthermore, the sufficient statistic* π_t *can be updated recursively through* $\pi_t = \Phi(\pi_{t-1}, e_t^1, e_t^2, y_t) = \Phi(\pi_{t-1}, \tilde{g}_t(\pi_{t-1}), y_t)$, *where* Φ *is a known function.*
2) *A receiver that generates estimates of the messages as* $(\hat{w}_t^1, \hat{w}_t^2) = \tilde{d}_t(\pi_t)$, *where* $\tilde{d}_t$ *is a known function.*

Proof: Follows from Lemma 1, the above discussion, and standard results in Markov decision theory [22]. ■

Since the optimal strategies are of that form we can restrict our attention to this class of communication systems without loss of optimality. In other words, even if it is not possible to solve for the optimal mappings $\tilde{g}_t$, we know that there is no loss in optimality by restricting attention to communication systems of this form. This communication system is depicted in Fig. 3.

IV. A SINGLE-LETTER CAPACITY EXPRESSION

We now show how the results of the previous section can be used for analysis purposes and in particular for deriving an expression for the capacity of the NLF-MAC. We restrict ourselves to the sum-capacity of this channel

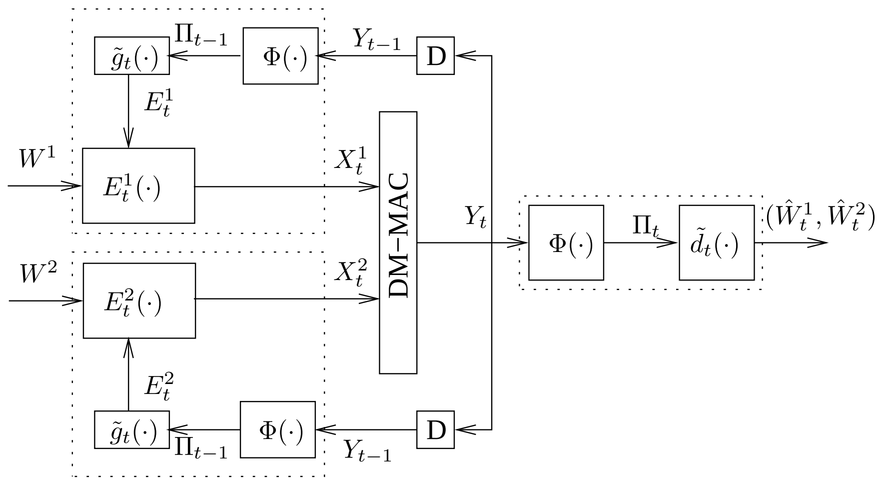

Fig. 3. Equivalent system with sufficient statistic π_t.

but the methodology is similar for obtaining the entire capacity region. Indeed, based on the results of the previous section, we can think of this channel as an equivalent point-to-point channel with input (E_t^1, E_t^2), state Π_{t-1} (perfectly observed at the transmitters), and output Π_t, Y_t. This equivalent channel is described by

$$\hat{Q}(Y, \Pi | E^1, E^2, \Pi') = \delta_{\Phi(\Pi', E^1, E^2, Y)}(\Pi) \sum_{w^1, w^2} Q(Y | E^1(w^1), E^2(w^2)) \Pi'(w^1, w^2). \tag{19}$$

Furthermore, we showed that we can restrict ourselves to encoders of the form $(e_t^1, e_t^2) = \tilde{g}_t(\pi_{t-1})$. However this is a Markov channel with feedback which has been studied in [23], [24] apart form the fact that we have an uncountably infinite state, π_t. Following the development in [23], [24] and under some technical assumptions that guarantee informational stability, we can express the sum-capacity of this channel as

$$C_{sum} = \sup_L I(E^1, E^2; \Pi, Y | \Pi'), \tag{20}$$

where $L : \Delta(\mathcal{W}^1 \times \mathcal{W}^2) \to \Delta(\mathcal{X}^{|\mathcal{W}^1 \times \mathcal{W}^2|})$, i.e., $L(\cdot|\pi')$ is a measure on the pair of functions (e^1, e^2) from $\mathcal{W}^1 \times \mathcal{W}^2 \to \mathcal{X}$. The mutual information expression is evaluated using the joint measure

$$Pr(Y, \Pi, E^1, E^2, \Pi') = \hat{Q}(Y, \Pi | E^1, E^2, \Pi') L(E^1, E^2 | \Pi') \mu_L(\Pi'). \tag{21}$$

and μ_L is the steady-state distribution of Π_t, given by the solution of the equation

$$\mu_L(\pi) = \int_{\pi'} \left[\sum_{e^1, e^2} \sum_y \hat{Q}(y, \pi | \pi', e^1, e^2) L(e^1, e^2 | \pi') \right] d\mu_L(\pi'), \tag{22}$$

We note that a significant simplification of this expression is possible that will also introduce in the expression the variables X^1, X^2. This is so due to the distribution on X^1, X^2 induced by the selection of the pre-encoders E^1, E^2 and the uniformly distributed messages. Due to space limitations we do not elaborate on this simplified expression.

We conclude by saying that there are several technical points that need to be clarified and they have to do with the uncountably infinite state space $\Delta(\mathcal{W}^1 \times \mathcal{W}^2)$ of π_t; with the fact that the supremizing distribution of the directed information expression $I(E^{1,n}, E^{2,n} \to \Pi^n, Y^n)$ is time invariant and independent of n; and the properties of the induced Markov processes $(\Pi_t, E_t^1, E_t^2)_t$ and $(\Pi_t)_t$.

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