Sparse single-system multiple-output (SSMO) using hybrid input-output (HIO) algorithm

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Abstract—**The single-system multiple-output (SSMO) sparse solution problem is to solve the greatly underdetermined linear systems of equations Y=HX where Y is M-by-L, H is M-by-N, X is N-by-L, and K rows of X are not all-zero (NAZ). We map this problem to a slightly-underdetermined dual SSMO problem W=GZ where G is N–(K–L)-by-N, Z is N-by-(M–K), W is N–(K–L)-by-(M–K), and N–K rows of Z are NAZ. The all-zero rows of Z correspond to the NAZ rows of X. For K=M–1, this dual problem is single-channel, with N unknowns, while the original problem has NL unknowns. The dual problem is then solved by adapting the Hybrid-Input-Output (HIO) algorithm used for phase retrieval. A Matlab program is also included. Fax: 734-763-1503. Email: aey@eecs.umich.edu**

I. INTRODUCTION

A. Problem Statement

The single-system multiple-output (SSMO) sparse reconstruction problem is to solve the multichannel underdetermined linear systems of equations

$$Y = HX \tag{1}$$

where
- Y is an $M \times L$ matrix having full rank.
- H is an $M \times N$ matrix having full rank.
- X is an $N \times L$ matrix having full rank.
- X has only K rows that are not all zero (NAZ).

But the location of these NAZ rows is unknown.
- $L < K < M < N$ for a well-defined problem.

The SSMO problem is a multichannel version of the usual sparse reconstruction problem of solving the underdetermined linear system of equations $y = Hx$ where y is an M-vector, K elements of x are nonzero, but the location of the nonzero elements of N-vector x is unknown. This is the single-system single-output (SSSO) reconstruction problem.

Sparse reconstruction is currently of great interest in compressed sensing, since many real-world signals and images have sparse (mostly zero) representations in an appropriate basis, such as a set of wavelet or curvelet basis functions. The number of observations necessary to reconstruct the signal is therefore greatly reduced from the size of the signal. The SSMO problem arises in several applications in which multiple snapshots over time of a signal whose characteristics are varying over time are available.

B. Problem Background

Many approaches to solving the SSSO problem are known. The most common approach is basis pursuit, in which linear programming is used to find the solution to $y = Hx$ which has the minimum ℓ_1 norm (sum of absolute values). Another approach is matching pursuit, in which columns of H most highly correlated with the residual y–$H\hat{x}_i$ are successively chosen to minimize the residual y–$H\hat{x}_{i+1}$. Iterative thresholding, in which the Landweber iteration is applied to $y = Hx$ and some elements of $\hat{x}_i$ are thresholded to zero at each iteration. We will not attempt to summarize the many variations on these themes.

Extending these approaches to the SSMO problem has proven to be difficult. The most common approach is to find the solution to $Y{=}HX$ that minimizes a mixed-norm criterion like

$$\underset{x_{ij}}{\text{MIN}} \sum_{i=1}^{N} \sqrt{\sum_{j=1}^{L} x_{ij}^2}. \tag{2}$$

This is the ℓ_1 norm over columns i of the ℓ_2 norm of each row. The ℓ_2 norm of a row is zero if and only if all elements in that row are zero, and positive otherwise. The ℓ_1 norm is then just the sum of these, and so can be expected to maximize the number of all-zero rows of X, as desired.

C. Contribution of This Paper

This paper maps the greatly-underdetermined (by N–M) SSMO problem $Y{=}HX$ to a slightly-underdetermined (by K–L) so-called dual SSMO problem $W{=}GZ$ where

- G is a known $(N\text{–}(K\text{–}L)) \times N$ matrix.
- W is a known $(N\text{–}(K\text{–}L)) \times (M\text{–}K)$ matrix.
- Z is an unknown $N \times (M\text{–}K)$ matrix.
- Z has $(N\text{–}K)$ not-all-zero (NAZ) rows.

But the location of these NAZ rows is unknown.
- The location of all-zero rows of Z correspond to, and are an indicator function for, NAZ rows of X.
- Since number of NAZ rows$(N\text{–}K) < (N\text{–}(K\text{–}L))$ =number of equations, the sparse solution is unique.

Solving the slightly-underdetermined dual problem effectively solves the greatly-underdetermined original problem, since once the location of NAZ rows of X in the original problem is known, X can be computed quickly. Z is an indicator function for NAZ rows of X. Comparing matrix sizes in $Y{=}HX$ and $W{=}GZ$ shows why the latter is a dual problem.

The next section presents the mapping from original to dual SSMO problems. However, following sections will consider only the case $K{=}M{-}1$. This is the maximum number of NAZ rows for which the original SSMO problem has a unique solution. For $K{=}M$, any choice of NAZ rows leads to a solution; for $K > M$, any choice of NAZ rows leads to an infinite number of solutions. $K{=}M{-}1$ means the number of observations (rows of Y) is minimum.

When $K{=}M{-}1$, the dual problem becomes SSSO (single-channel), since then $M{-}K{=}1$. The number of unknows in the dual problem is then N, while the original problem had NL unknowns. Since the dual problem is only slightly underdetermined, the actual number of unknowns is $K{-}L$. If the number of channels is not less than the number of rows, the problem is trivial to solve. Otherwise, we adapt the Hybrid-Input-Output (HIO) algorithm used in phase retrieval to the dual problem.

II. MAPPING ORIGINAL TO DUAL

We map the original SSMO problem to the dual SSMO problem. For completeness, this section does not yet assume $K{=}M{-}1$.

A. Left and Right Nullspaces

The original problem, repeated as

$$M\{\underbrace{\boxed{\mathrm{Y}}}_{L} = M\{\underbrace{\boxed{\mathrm{H}}}_{N} N\{\underbrace{\boxed{\mathrm{X}}}_{L}. \tag{3}$$

can be rewritten as

$$M\{\underbrace{\boxed{\mathrm{H}}}_{N}\underbrace{\boxed{\mathrm{Y}}}_{L}\begin{bmatrix} N\{\boxed{\mathrm{X}} \\ L\{\underbrace{\boxed{\text{-I}}}_{L}\end{bmatrix} = M\{\underbrace{\boxed{0}}_{L}. \tag{4}$$

Let $\{i_1, i_2 \dots i_K\}$ be the indices of the NAZ rows of X, and define the $M \times K$ submatrix of H

$$\tilde{H} = [H_{i_1}|H_{i_2}|\dots|H_{i_K}] \tag{5}$$

whose columns multiply the NAZ rows X_{i_k} of X, so

$$M\{\underbrace{\boxed{H_{i_1 \dots i_K}}}_{K}\underbrace{\boxed{\mathrm{Y}}}_{L}\begin{bmatrix} \boxed{X_{i_k}} \\ \underbrace{\boxed{\text{-I}}}_{L}\end{bmatrix} = M\{\underbrace{\boxed{0}}_{L}. \tag{6}$$

Suppose momentarily that $K{+}L \geq M$. The right nullspace of an $M \times (K+L)$ matrix nominally has size $K{+}L{-}M$. Since the right nullspace of $[\tilde{H}|Y]$ has size L, $[\tilde{H}|Y]$ is rank-deficient by $M{-}K$. So $[\tilde{H}|Y]$ has a left nullspace of size $M{-}K$, and there exists an $(M{-}K) \times M$ matrix D' such that

$$M-K\{\underbrace{\boxed{D'}}_{M} M\{\underbrace{\boxed{\tilde{H}}}_{K}\underbrace{\boxed{\mathrm{Y}}}_{L} = M-K\{\underbrace{\boxed{0\dots0}}_{K+L}. \tag{7}$$

A similar argument can be applied if $K{+}L \leq M$. Now define the $(M{-}K) \times N$ matrix Z' as

$$M-K\{\underbrace{\boxed{Z'}}_{N} = M-K\{\underbrace{\boxed{D'}}_{M} M\{\underbrace{\boxed{H}}_{N}. \tag{8}$$

Then $Z'_{i_1 \dots i_K}{=}[0]$ and other columns of Z' are NAZ. Z acts as an indicator function for NAZ rows of X.

B. Partitioning

The next step is to reduce the size of the problem by eliminating some columns of the matrix D, using $D'Y{=}[0]$. Partition $D'Y{=}[0]$ as follows:

$$M-K\{\underbrace{\boxed{D'_1}}_{M-L}\underbrace{\boxed{D'_2}}_{L}\begin{bmatrix} M-L\{\boxed{Y_1} \\ L-0\{\underbrace{\boxed{Y_2}}_{L}\end{bmatrix} = \underbrace{\boxed{0}}_{L}. \tag{9}$$

Solving for D'_2 gives

$$D'_2 = -D'_1 Y_1 Y_2^{-1}. \tag{10}$$

Now partition (8) similarly as

$$M-K\{\underbrace{\boxed{Z'}}_{N} = \underbrace{\boxed{D'_1}}_{M-L}\underbrace{\boxed{D'_2}}_{L}\begin{bmatrix} M-L\{\boxed{H_1} \\ L-0\{\underbrace{\boxed{H_2}}_{N}\end{bmatrix}. \tag{11}$$

Defining

$$H'_3 = H_1 - Y_1 Y_2^{-1} H_2, \tag{12}$$

this becomes

$$M-K\{\underbrace{\boxed{Z'}}_{N} = M-K\{\underbrace{\boxed{D'_1}}_{M-L} M-L\{\underbrace{\boxed{H'_3}}_{N}. \tag{13}$$

and taking a transpose gives

$$Z = H_3 D_1. \tag{14}$$

Partition H_3 and D_1 as

$$N\{\underbrace{\boxed{Z}}_{M-K} = N\{\underbrace{\boxed{H_{31}}}_{K-L}\underbrace{\boxed{H_{32}}}_{M-K}\begin{bmatrix} K-L\{\boxed{D_{11}} \\ M-K\{\underbrace{\boxed{D_{12}}}_{M-K}\end{bmatrix}. \tag{15}$$

Now define the two quantities

$$\begin{aligned} D_3 &= D_{11}D_{12}^{-1} \\ \tilde{Z} &= ZD_{12}^{-1} \end{aligned} \qquad (16)$$

and postmultiply (15) by D_{12}^{-1} to get

$$\begin{aligned} \tilde{Z} &= ZD_{12}^{-1} \\ &= (H_{31}D_{11} + H_{32}D_{12})D_{12}^{-1} \\ &= H_{31}D_3 + H_{32}. \end{aligned} \qquad (17)$$

Note that $\tilde{Z}$, like Z, has all-zero rows $\{i_1 \dots i_K\}$ since postmultiplying a zero row by D_{12}^{-1} gives a zero row.

The dual SSMO equation we use in the sequel is

$$N\{\underbrace{\boxed{\tilde{Z}}}_{M-K} = N\{\underbrace{\boxed{H_{31}}}_{K-L}K-L\{\underbrace{\boxed{D_3}}_{M-K} + N\{\underbrace{\boxed{H_{32}}}_{M-K}. \qquad (18)$$

For K=M–1, this becomes a SSSO problem.

C. Formulation of Dual Problem

Although we will not use it, we can put (18) into the usual form of an SSMO problem. Let G be the $(N–(K–L)) \times N$ left null matrix of H_{31}. Then

$$N-(K-L)\{\underbrace{\boxed{G}}_{N} N\{\underbrace{\boxed{Z}}_{M-K} = GH_{32} = W \qquad (19)$$

which is the dual problem noted above. Note that

- Number of equations: N–(K–L).
- Number of nonzero: N–K < N–(K–L).

So the sparse solution is unique.

III. SOLUTION OF DUAL PROBLEM

A. Introduction

In the sequel, we consider only K=M–1, so that *the original problem has the maximum number of NAZ rows that allow a unique solution.* Then the dual problem becomes an SSSO problem, since the number of channels is M–K=1.

This still leaves the problem of how to solve the dual problem. The usual procedure (ℓ_1-norm minimization) does not work seem to work here. This is not surprising; ℓ_1-norm minimization can only be expected to work if the unknown vector is almost entirely sparse (zero-valued), and Z is only (N–K) sparse. Another approach is needed.

The Hybrid-Input-Output (HIO) algorithm has been used with great success in phase retrieval, and it has also been applied to some other combinatorial-type problems. It does not seem to have been applied to sparse reconstruction problems, since ℓ_1-norm minimization works so well.

B. Hybrid Input-Output (HIO)

The HIO algorithm, adapted to solution of (18), is

$$Z^{k+1} = Z^k + Q(2P(Z^k) - Z^k) - P(Z^k) \qquad (20)$$

where projections $P(Z)$ and $Q(Z)$ are the vectors closest to Z (in the least-squares sense) such that:

- $P(Z)$ has the form $H_{31}d$+H_{32} for some vector d,
- $Q(Z)$ has K zero values (out of N), and we define
- Z^k is the solution after k iterations.

$Q(Z)$ sets the K smallest elements of Z to 0, and leaves others unaltered. $P(Z)$ is computed using

$$\begin{aligned} w &= H'_{31}(Z - H_{32}) \\ d &= (H'_{31}H_{31})^{-1}w \\ Z &= H_{32} + H_{31}d \end{aligned} \qquad (21)$$
$$(22)$$

since w and d are (K–L)-vectors and their lengths (K–L) << N. $(H'_{31}H_{31})^{-1}$ may be precomputed.

IV. MATLAB PROGRAM

The following program illustrates the algorithm. The output `KHAT` is the indices of NAZ rows of X. Computation of X is then straightforward.

The algorithm requires the number of channels L to be large. However, it should be recalled that the number of NAZ rows of X is M–1, the maximum possible for a unique solution, so the number M of observations is minimum. The algorithm trades off the number of observations for number of channels.

```
clear;N=100;M=51;L=50;
H=randn(M,N);X(N,L)=0;
X(1:N/(M-1):N,:)=randn(M-1,L);
Y=H*X;%GOAL:Compute X from Y and H.
Y1=Y(1:M-L,:);Y2=Y(M-L+1:M,:);
H1=H(1:M-L,:);H2=H(M-L+1:M,:);
H3=(H1-Y1*inv(Y2)*H2)';
H31=H3(:,1:M-L-1);H32=H3(:,M-L:M-L);
%Hybrid Input-Output algorithm:
HH=pinv(H31);%HH=inv(H31'*H31)*H31';
Z=randn(N,1);for I=1:1000;
DHAT=HH*(Z-H32);
PZ=H31*DHAT+H32;W=2*PZ-Z;
W1=sort(abs(W));W(abs(W)<W1(N-M+2))=0;
Z=Z+W-PZ;end
[V,J]=sort(abs(Z));KHAT=sort(J(1:M-1))
```